Study of a Loaded Continuous Beam with the Yielding Supports

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Abstract. The paper presents an iterative method for the computation of the displacements of the supports for movable control surfaces of an airplane (elevator, rudder, aileron), which are attached at *N*-points to an elastic structure. The reactions that act in these points are computed using the finite element method and the materials strength computing techniques that relies on the interaction between the rigidity of the supporting and the movable structures.

Key-Words: Yielding supports, trapezoidal loading, finite elements method, slope deflection method, reactions.

1 Introduction

The movable control surfaces of an airplane, namely the elevator, the rudder and the aileron are attached to *N*-points to stabilizer, fin and wing, respectively. These supporting surfaces are usually cantilevers with variable moment of inertia, carrying the air loads that determine the deflections of the structure and hence, the displacements of the supporting points of movable structure. With the help of an iterative method we find the reaction forces in the attachment points. In the step 1 we find the displacements of the cantilever by the finite element method and then, using the techniques of the strength materials, we find the reactions in the supporting points of the movable structure subjected to external loads.

For a correct structural response we determine by the step 2, the deflections of the supporting surfaces under the combined action of air loads, reaction forces in the attachment points obtained by the step 1. With the new settling of supports, we repeat the calculation of the reactions as to the step 1.

Numerical example shows that this iterative method is rapidly converging.

2 Modelling and formulation

Step 1

Supporting structure

The primary function of any structure is to support and to transfer externally applied loads to the reactions points when is subjected to some specified constraints. In the matrix structural analysis an important step is the formulation of a discrete-element mathematical model equivalent to the actual continuous structure.

Let us consider the following the general assumptions:

- displacements of the structural element are not very large and the geometry of the system is well defined before the analysis is attempt;
- displacements and stains of the loaded structure are small and hence, linear elasticity theory applies.

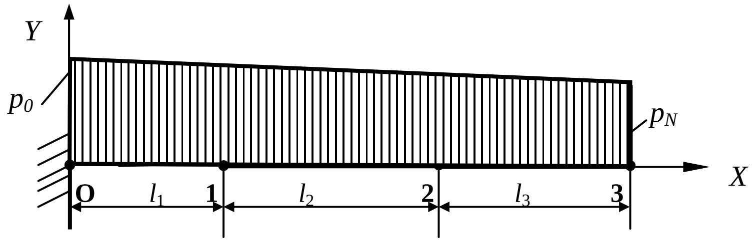

Fig. 1

A cantilever beam with the nodes 0,1,2,3,4 idealizes the support structure. The node zero corresponds to the fixed point; the points 1,2, and 3 correspond to the support points of the movable structure. The applied loading consists of the transverse forces, which are equivalent with the trapezoidal loading from the Fig.1. Since are applied only the transversal forces, we have a two dimensional problem. The beam element will be assumed to be a straight bar of uniform cross section capable of resisting axial forces S_1 and S_4; shearing forces S_2 and S_5; and bending moments S_3 and S_6.

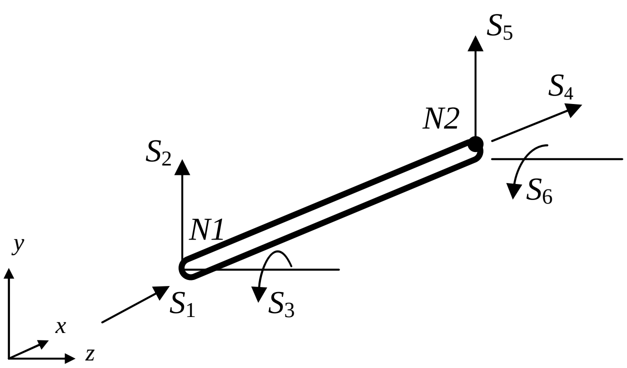

Fig.2

Let us consider that the moment of inertia I_z and the cross-sectional area A of the beam element of the length l are constantly. Within each element (i), the stresses are equilibrated by a set of element forces S_k in the direction of the element displacements u_k

In the local coordinate system (Fig. 2), the stiffness matrix for a beam element (i) bounded by the nodes **N1**, **N2** is of the form

$$k^{(i)} = E \cdot \begin{bmatrix} \frac{A}{l} & & & & & \\ 0 & \frac{12I}{l^3} & & symmetric & & \\ 0 & \frac{6I}{l^2} & \frac{4I}{l} & & & \\ -\frac{A}{l} & 0 & 0 & \frac{A}{l} & & \\ 0 & -\frac{12I}{l^3} & -\frac{6I}{l^2} & 0 & \frac{12I}{l^3} & \\ 0 & \frac{6I}{l^2} & \frac{2I}{l} & 0 & -\frac{6I}{l^2} & \frac{4I}{l} \end{bmatrix}$$

where E is the Young's modulus.

The rigidity matrix $k^{(i)}$ may be rewritten in the following form

$$k^{(i)} = \begin{bmatrix} k_{i1} & k_{i2} \\ k_{i3} & k_{i4} \end{bmatrix}$$

where each matrix k_{ij} is defined by a 3×3 block. For an element (i), the local forces from the Fig. 2 are denoted by $S_k^{(i)}$ and the corresponding displacements by $u_k^{(i)}$. These are related by the matrix equation

$$S^{(i)} = k^{(i)} u^{(i)} \quad (1)$$

for each element separately. For the complete structure, all these equations can be combined into a single matrix equation of the form

$$S = ku \quad (2)$$

where $S = \left\{S^{(1)}, S^{(2)}, S^{(3)}\right\}$, $u = \left\{u^{(1)}, u^{(2)}, u^{(3)}\right\}$ and $k = \left\{k^{(1)}, k^{(2)}, k^{(3)}\right\}$ for the elements: 0-1; 1-2 and 2-3 (Fig.1). Let us now define a matrix of displacements for the assembled structure

$$U = \left\{O \ \tilde{U}_1 \ \tilde{U}_2 \ \tilde{U}_3\right\} \quad (3)$$

where each block $\tilde{U}_k = \left\{U_k \ V_k \ \Theta_k\right\}$ with

U_k the displacement corresponding to the forces S_1;
V_k the displacement corresponding to the forces S_2;
Θ_k the displacement corresponding to the forces S_3.

These components are defined with respect to the datum coordinate system *XOY* and the block *O* corresponds to the fixed node **O.**

If $\tilde{u}_1^{(i)} = \{u_1^{(i)}\ v_1^{(i)}\ \theta_1^{(i)}\}$ and $\tilde{u}_2^{(i)} = \{u_2^{(i)}\ v_2^{(i)}\ \theta_2^{(i)}\}$

are the matrices of the displacements of the node N1 and the node N2, respectively, for the element (*i*), we have

$$\begin{bmatrix} \tilde{u}_1^{(1)} \\ \tilde{u}_2^{(1)} \\ \tilde{u}_1^{(2)} \\ \tilde{u}_2^{(2)} \\ \tilde{u}_1^{(3)} \\ \tilde{u}_2^{(3)} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} O \\ \tilde{U}_1 \\ \tilde{U}_2 \\ \tilde{U}_3 \end{bmatrix} \tag{4}$$

or

$$u = AU \tag{5}$$

The external loading, which correspond to the displacements *U* is denoted by the matrix *F*, such that

$$F = \{ F_R\ F_1\ F_2\ F_3 \} \tag{6}$$

where the components of the block F_k represent the external forces in the direction of the components of the displacements block $\tilde{U}_k$ and F_R the reaction forces from the fix point **O**. To relate the external forces *F* to the corresponding displacements *U*, we use the principle of virtual work, [3], [8], which states that an elastic structure is in equilibrium under a given system of loads, if for any virtual displacements δu from a compatible state of deformation *u*, the virtual work is equal to the virtual strain energy. Finally, we obtain the matrix equation

$$F = KU \tag{7}$$

where $$K = A^t\, k\, A \tag{8}$$

Using (4) and (5) we obtain the stiffness matrix *K* for the complete structure (regarded as a free body) of the form

$$K = \begin{bmatrix} k_{11} & k_{12} & 0 & 0 \\ k_{13} & k_{14} + k_{21} & k_{22} & 0 \\ 0 & k_{23} & k_{24} + k_{31} & k_{32} \\ 0 & 0 & k_{33} & k_{34} \end{bmatrix} \tag{9}$$

In the matrix (6), F_R is the column block that contains the reaction forces from the **O** point. Therefore, *K* is a singular matrix. In order to calculate the displacements

$$U_A = \{\tilde{U}_1, \tilde{U}_2, \tilde{U}_3\} \tag{10}$$

in the active forces directions

$$F_A = \{ F_1, F_2, F_3 \} \tag{11}$$

we partition the matrix *K* in the form

$$\begin{bmatrix} K_{RR} & K_{AR}^t \\ K_{AR} & K_{AA} \end{bmatrix} \cdot \begin{bmatrix} O \\ U_A \end{bmatrix} = \begin{bmatrix} F_R \\ F_A \end{bmatrix} \tag{12}$$

where

$$K_{RR} = k_{11} \qquad K_{AR} = \begin{bmatrix} k_{12} \\ 0 \\ 0 \end{bmatrix}$$

$$K_{AA} = \begin{bmatrix} k_{14} + k_{21} & k_{22} & 0 \\ k_{23} & k_{24} + k_{31} & k_{32} \\ 0 & k_{33} & k_{34} \end{bmatrix} \tag{13}$$

From the system (12), the matrices U_A and F_R can be obtained with the following relations

$$U_A = K_{AA}^{-1} \cdot F_A \tag{14}$$

$$F_R = K_{AR}^t \cdot U_A \tag{15}$$

Movable structure

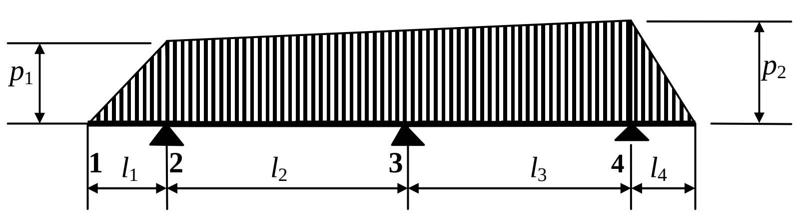

Fig. 3

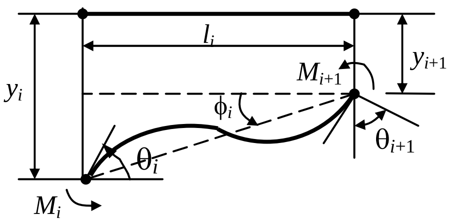

Fig. 4

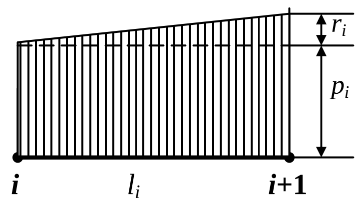

Fig.5

In the figure 3 is presented a rudder beam attached to a deflecting fin structure at the 3 points, where support **2** corresponds to the maximum displacement *U*. Using the definitions of the elements of the matrix *U* from (3), we find the vertical displacements y_i of the supports **1, 2** and **3** (Fig. 3): $y_2 = V_3,\ \ y_3 = V_2,\ \ y_4 = V_1$.

Let us now determine the bending moment at the supports under the combined action of transverse loading and of the support settling. For a member bounded by two end joints, the end moments can be expressed in terms of the end rotations. Furthermore for static equilibrium, the sum of the end moments on the members meeting at a joint must be equal to zero. With the help of these equations of static equilibrium, the unknown joints rotations and the end moments can be computed. Finally, the reactions in the supports are found. At the beginning, the following values will be evaluated (Fig. 3 and Fig.4).

a) The swing of the member $i \div i+1$:

$$\phi_i^{(1)} = \frac{y_i^{(1)} - y_{i+1}^{(1)}}{l_i} = -\frac{\Delta_i}{l_i} \tag{16}$$

b) The moments due to any applied loads on the beam, when considered as fixed ends

$$m_{i,i+1} = -\frac{(5p_i + 2r_i).l_i^2}{60};\quad m_{i+1,i} = \frac{(5p_i + 3r_i).l_i^2}{60} \tag{17}$$

For sign convention: the rotation of a joint or a member is positive, if it turns in a clockwise direction; the end moment is considered positive if it tends to rotate the end of the member clockwise or the joint counter clockwise.

c) The moments in the *i* – point due to distortion of the supports, [2] are

$$M_{i,i+1}^{(1)} = 2K_i \cdot (2\theta_i + \theta_{i+1} - 3\phi_i) + m_{i,i+1} \tag{18}$$

$$M_{i,i-1}^{(1)} = 2K_{i-1} \cdot (2\theta_i + \theta_{i-1} - 3\phi_{i-1}) + m_{i,i-1} \tag{19}$$

where $K_i = \dfrac{E \cdot IX_i}{l_i}$, $E -$ modulus of elasticity.

For static equilibrium of the joint $i : M_{i,i+1}^{(1)} + M_{i,i-1}^{(1)} = 0$ and using (25), the following system is obtained

$$\overline{K} \cdot \overline{\theta} = \overline{F} \tag{20}$$

where

$$\overline{\theta} = \{\theta_2, \theta_3, \theta_4\}$$

$$\overline{F} = \overline{K_1} \cdot \overline{\phi} - \overline{M}^*$$

and the stiffness matrix is of the form

$$\overline{K} = \begin{bmatrix} 4K_2 & 2K_2 & 0 \\ 2K_2 & 4(K_2 + K_3) & 2K_3 \\ 0 & 2K_3 & 4K_3 \end{bmatrix}$$

$$\overline{K_1} \cdot \overline{\phi} = 6\begin{bmatrix} K_2 & 0 & 0 \\ K_2 & K_3 & 0 \\ 0 & 0 & K_3 \end{bmatrix} . \begin{bmatrix} \phi_2 \\ \phi_3 \\ \phi_4 \end{bmatrix}, \tag{21}$$

$$\overline{M}^* = \begin{bmatrix} M_{2,1} + m_{2,3} \\ m_{3,2} + m_{3,4} \\ m_{4,3} + M_{4,5} \end{bmatrix}$$

Solving the (20) for $\overline{\theta}$ we obtain the bending moments at the supports by (18) and (19). With the help of the material strength computing techniques, we get the following matrix relation for find the corresponding support reactions $R_k^{(1)}$, $k = 2, 3, 4$:

$$\begin{bmatrix} R_2^{(1)} \\ R_3^{(1)} \\ R_4^{(1)} \end{bmatrix} = \begin{bmatrix} \frac{3l_1+2l_2}{6} & \frac{l_2}{6} & 0 \\ \frac{l_2}{6} & \frac{l_2+l_3}{3} & \frac{l_3}{6} \\ 0 & \frac{l_3}{6} & \frac{3l_4+2l_3}{6}\end{bmatrix} \cdot \begin{bmatrix} p_2\\ p_3\\ p_4\end{bmatrix} + \begin{bmatrix} -\frac{1}{l_2} & \frac{1}{l_2} & 0 \\ \frac{1}{l_2} & -\frac{1}{l_2}-\frac{1}{l_3} & \frac{1}{l_3} \\ 0 & \frac{1}{l_3} & -\frac{1}{l_3}\end{bmatrix}\cdot \begin{bmatrix} -\frac{p_1 l_1^2}{6} \\ M_3 \\ -\frac{p_4 l_4^2}{6}\end{bmatrix} \quad (22)$$

where $M_3 = M_{3,4}$ and by $R_k^{(1)}$ we denote the reaction from the support k at the step 1. The loading p_k (daN/cm) is the value of this that corresponds to the support point k.

Step 2

We consider now the deflecting supporting structure loaded with the air loads as well as with the reactions determined by the step1. Using the superposition principle that sum up the individual effects of every load, which acts upon a structure, we can obtain the displacement functions for the supports. The displacements corresponding air loads were computed in the step 1 and now we have to concentrate on the displacement caused by the three reactions, $R_k^{(1)}$ (fig.6). We denote:

$$F_1 = \begin{bmatrix} 0 \\ R_4^{(1)} \\ 0 \end{bmatrix}, \quad F_2 = \begin{bmatrix} 0 \\ R_3^{(1)} \\ 0 \end{bmatrix}, \quad F_3 = \begin{bmatrix} 0 \\ R_2^{(1)} \\ 0 \end{bmatrix} \qquad (23)$$

Using the relation (14), where F_A is defined for the loading defined by (23), we find the new displacements $V_k', k = 1, 2, 3$ of the supports. Finally, we compute the displacements that corresponding to the step 2 with the formula

$$\bar{v}_k = V_k + V_k', \; k = 1, 2, 3$$

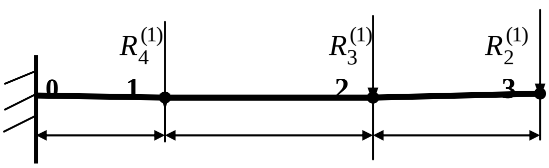

Fig. 6

The vertical displacements of the supports accordingly with (14) are

$$y_2 = \bar{v}_3, \quad y_3 = \bar{v}_2, \quad y_4 = \bar{v}_1$$

The new reactions $R_2^{(2)}, R_3^{(2)}, R_4^{(2)}$ are evaluated with the matrix relation (22).

The iterations cease if

$$\max_i \left\{ \left| R_2^{(i)} - R_2^{(i-1)} \right|, \left| R_3^{(i)} - R_3^{(i-1)} \right|, \left| R_4^{(i)} - R_4^{(i-1)} \right| \right\} \le \varepsilon . \qquad (24)$$

where ε is a given small number.

3 Numerical example

Supporting structure

Let us consider the moment of inertia of the fin, $I_z = 3325$ cm^4 and the modulus of elasticity of the material, $E = 735000$ daN/cm^2. The air load on the fin is shown in the Fig.7. Using the finite element method we obtain:

$$V_1 = 0.161\,\text{cm}; V_2 = 0.983\,\text{cm}; V_3 = 2.122\,\text{cm}$$

from the relation (14).

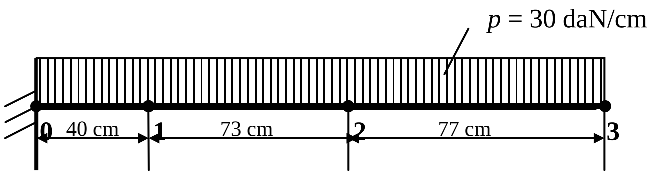

Fig. 7

Movable structure

Let us now consider a rudder (Fig.3) with: l_1=14 cm; l_2 = 77cm; l_3 = 73 cm, l_4 = 20 cm, p_1 =3 daN/cm, p_2 =11.2 daN/cm and using the values V_i, we get y_2 = 2.12 cm, y_3 = 0.983 cm, y_4 = 0.161 cm. From (17) we have

$$m_{3,2} = \frac{77^2(5 \cdot 3 + 3 \cdot 4)}{60} = 2668 \; daNcm$$

$$m_{2,3} = -\frac{77^2(5 \cdot 3 + 2 \cdot 4)}{60} = -2273 \; daNcm$$

$$m_{3,4} = -\frac{73^2(5 \cdot 7 + 2 \cdot 4.2)}{60} = -3855 \; daNcm$$

Te beam has a constant section, hence we get $K_2 = EI/l_2$ = 315000 daNcm; $K_3 = EI/l_3$ = 453082 daNcm.

<u>*Span* 2 – 3</u>

The settlement of support **2** with respect to support **3**, Δ_2 = 2.12 – 0.98 = 1.14 cm and

$$\phi_2 = -\frac{\Delta_2}{l_2} = -\frac{1.14}{77} = -0.015 \text{ rad.} \qquad (25)$$

Since the joint **2** turns in counterclockwise with respect to **3,** the sign of ϕ is negative.

<u>*Span*</u> <u>3 – 4</u>

$\Delta_3 = 0.98 - 0.16 = 0.82$ cm and

$$\phi_3 = -\frac{\Delta_3}{l_3} = -\frac{0.82}{73} = -0.011 \text{ rad.}$$

The moment $M_{2,1} = 21 \cdot 14/3 = 98$ daNcm and from the static equilibrium of joint **2**, $M_{2,3} = -M_{2,1}$. Substituting in equations (18) and (19) we get

$$M_{2,3} = 2K_2(2\theta_2 + \theta_3 - 3\phi_2) + m_{2,3}$$
$$-98 = 1260000\,\theta_2 + 630000\,\theta_3 + 29484 - 2273$$

For static equilibrium of joint **3**, $M_{3,2} + M_{3,4} = 0$, where

$$M_{3,2} = 2 \cdot 315000(2 \cdot \theta_3 + \theta_2 + 3 \cdot \phi_2) + 2668$$
$$M_{3,4} = 2 \cdot 453082(2 \cdot \theta_3 + \theta_4 + 3 \cdot \phi_3) - 3855 \qquad (26)$$

Finally, for static equilibrium of joint **4**

$M_{4,3} + M_{4,5} = 0$ where

$$M_{4,5} = -112 \cdot 20/3 = -747 \text{ daNcm} \text{ and}$$
$$M_{4,3} = 2 \cdot 453082(2\theta_4 + \theta_3 + 3\phi_3) + 4228$$

The equation (20) becomes

$$10^4\begin{bmatrix}126 & 63 & 0\\ 63 & 307.2 & 90.6\\ 0 & 90.6 & 181.2\end{bmatrix}\begin{bmatrix}\theta_1\\ \theta_2\\ \theta_3\end{bmatrix} = 6\begin{bmatrix}315000 & 0\\ 315000 & 453082\\ 0 & 453082\end{bmatrix}\begin{bmatrix}-0.015\\ -0.011\end{bmatrix} - \\ -\begin{bmatrix}-273+98\\ 2668-3855\\ 4228-747\end{bmatrix}.$$

We find the following values: θ_2 = -0.014 rad; θ_3 = -0.012 rad; θ_4 = -0.012 rad and from (26), M_3 = - 8585 daNcm, M_2 = -98 daNcm and M_4 = - 747 daNcm. Finally, the reactions from the support points have the values: $R_2^{(1)} = 107.31$ daN; $R_3^{(1)} = 695.3$ daN; $R_4^{(1)} = 394.2$ daN.

The displacement V_k of a support k due to the air load on the fin must be added now to the displacement V_k' due to the forces $R_2^{(1)}$, $R_3^{(1)}$, $R_4^{(1)}$ determined in step 1. We calculate the displacements V_k'', $k = 1,2,3$ of the support points by (14) for the reactions $R_k^{(1)}$, $k = 1,2,3$ and get the following values: $V_1 = 0.046$ cm; $V_2 = 0.297$ cm; $V_3 = 0.652$ cm. Using the superposition principle, the supporting points of the movable control surface are displaced with the values: y_2 = 2.12 + 0.65 = 2.77 cm, y_3 = 0.983 + 0.297 = 1.28 cm and y_4 = 0.161 + 0.046 = 0.207 cm. With these new, y_i, we repeat the calculation of the reactions as to the step 1 for the movable structure.

4 Conclusions

The proposed analysis allows us to find with a great accuracy the reactions in the attachment points of the movable surfaces to the support beam. Thus, theirs fittings will be calculated correctly. This algorithm is applicable for structures of variable bending rigidity and loaded by various types of external forces and may be easily generalized for *N*- supports points. Also, it reduces the number of equations to be solved simultaneously and it presents equations that easily and rapidly formulated.

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