Approximate Solutions of Heat Conduction Problems in Multi-dimensional Cylinder Type Domain by Conservative Averaging Method, Part 2

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Abstract: - In this second part of paper the description of conservative averaging method for partial differential (or integro-differential) equation with discontinuous coefficients in cylinder type domain is given. The conservative averaging is carried out in two orthogonal directions. Different types of boundary conditions are examined.

Key-Words: - partial differential equations, discontinuous coefficient, conservative averaging, various boundary conditions.

1 Introduction

In praxis very often important mathematical models consist of partial differential equations (PDE) with discontinuous coefficients [1]-[3]. They describe various physical processes in piecewise homogeneous media, e.g. in layered structure. Conservative averaging as special analytical (or analytically-numerical) method was developed for partial differential equations with discontinuous coefficients in layered media. In part 1 of this paper we extended the method of conservative averaging for partial differential (or integro-differential) equations with continuous coefficients in cylindrical domain. Here we generalize our investigation for the situation when base of cylinder consists of two sub-domains, i.e. the main equation has discontinuity its coefficients on the cylindrical domain. The conservative averaging here is realized in two orthogonal directions. Thus this paper generalizes the results of paper [4] in two senses. Firstly, we realize averaging in two directions. Secondly, we consider different types of boundary conditions (BC) for generalized main PDE.

2 Conservative Averaging Method for Two-layer Cylinder Type Domain

We will start with the statement of problem for finite cylinder type domain.

2.1 Original Problem

We will consider the cylinder type domain $\tilde{D}$, where $\tilde{D} = \{(x,\tilde{y}) : x \in [0,H] \times \tilde{G}\} \subset R^{n+1}$. Here the basis $\tilde{G}$ of the cylinder $\tilde{D}$ is bounded (or unbounded) domain $\tilde{y} = (z, y_2, ..., y_n) \in \tilde{G} \subset R^n$. The closure of the domain (base) is represented as union of two closed sub-domains $\bar{\tilde{G}} = \bar{G} \cup \bar{G}_0$. The sub-domain G_0 is the cylindrical domain of finite height δ:

$$G_0 = \{z \in (-\delta, 0)\} \times \{y = (y_2, ..., y_n) \in G_0^{n-1}\},$$

where $G_0^{n-1} \subseteq R^{n-1}$ is bounded or unbounded domain. The definition of the sub-domain is following: $G = \{z > 0\} \times G^{n-1}$. Here with notation $z > 0$ we understand that the domain G is located on the right from domain G_0 relatively the coordinate z. Accordingly are defined the sub-domain $D_0 = \{x \in (0,H)\} \times G_0$ and the second sub-domain $D = \{x \in (0,H)\} \times G$. Shared border-hyper plane between domains D_0 and D we denote as H:

$$H := \bar{D}_0 \cap \bar{D} = \left\{(x,z,y) \in \tilde{D} : z = 0\right\}.$$

Right border of the domain G_0 we denote as H_0:

$$H_0 = \left\{(x,z,y) \in \bar{D}_0 : z = -\delta\right\}.$$

Sometimes we will use short notation $z = 0$ and $z = -\delta$ for hyper planes H and H_0.

As in first part of this paper one of components $y_2, ...y_n$ again could be time variable t.

The main equation in the sub-domain D_0 for the solution (function $U_0(x,z,y)$) in general form looks as follows:

$$\frac{\partial}{\partial x}\left(k_0\frac{\partial U_0}{\partial x}\right)+\frac{\partial}{\partial z}\left(k_0\frac{\partial U_0}{\partial z}\right)+ L^0\left(U_0\right)=-F_0(x,z,y). \quad (1)$$

Here the *linear* differential (integral or integro-differential) operator L^0 is operator concerning to vector argument y with coefficients related to the same argument (and concerning argument x for the first averaging procedure, acting in the $z-$direction; see sub-section 2.2).

Accordingly the main equation in the sub-domain D for the solution $U(x,z,y)$ is:

$$\frac{\partial}{\partial x}\left(k\frac{\partial U}{\partial x}\right)+\frac{\partial}{\partial z}\left(k\frac{\partial U}{\partial z}\right)+ L\left(U\right)=-F(x,z,y). \quad (2)$$

On the hyper-plane H between both sub-domains the conjugation conditions are given:

$$U_0\big|_{z=-0}=U\big|_{z=+0}, \quad (3)$$

$$k_0\frac{\partial U_0}{\partial z}\bigg|_{z=-0}=k\frac{\partial U}{\partial z}\bigg|_{z=+0}. \quad (4)$$

On border H_0 the boundary condition in general form is written:

$$\left[-k_0\nu_0\frac{\partial U_0}{\partial z}+\lambda_0 U_0\right]\bigg|_{z=-\delta}=\varphi^0(x,y). \quad (5)$$

At this moment it is not necessary to concretize the boundary conditions on the rest of the borders:

$$\tilde{l}(\tilde{U})=\tilde{\Psi}(x,z,y),\ (x,z,y)\in\partial\breve{D}=\partial\tilde{D}\setminus H_0. \quad (6)$$

Here we have introduced the function $\tilde{U}(x,z,y)$ which is equal the function $U_0(x,z,y)$ on sub-domain $\bar{D}_0$ and the function $U(x,z,y)$ on the sub-domain $\bar{D}$.

2.2 Transformation of the Original Problem by Conservative Averaging According to the Coordinate z

We will transform the original problem (1) – (6). As in the part 1, to make difference between these two problems clearer, we denote the new solution of the equation (2) as $u(x,z,y)$ instead of the original solution $U(x,z,y)$.Then the main equation (2) on the sub-domain $\bar{D}$ looks as follow:

$$\frac{\partial}{\partial x}\left(k\frac{\partial u}{\partial x}\right)+\frac{\partial}{\partial z}\left(k\frac{\partial u}{\partial z}\right)+L\left(u\right)= -F(x,z,y),\ (x,z,y)\in D. \quad (7)$$

We introduce integral averaged function in direction z:

$$u_0(x,y)=\frac{1}{\delta}\int_{-\delta}^{0}U_0(x,z,y)dz\,. \quad (8)$$

As the first step we integrate the main equation (1). This gives exact equality:

$$\delta\frac{\partial}{\partial x}\left(k_0\frac{\partial u_0}{\partial x}\right)+\delta L^0\left(u_0\right)+ k_0\frac{\partial U_0}{\partial z}\bigg|_{z=-\delta}^{z=-0}=-\delta f_0(x,y), \quad (9)$$

$$f_0(x,y)=\delta^{-1}\int_{-\delta}^{0}F_0(x,z,y)dz.$$

We shall call this equality *principal relation.* Again (as in part 1) principal relation is underdetermined equation because of presence of two different functions: $u_0(x,y)$ and $U_0(x,z,y)$ in one equation (9). It means that connection between these functions must be established. Next steps in our approach (method) depend on two factors:

1) Assumption about the behavior of the function $U_0(x,z,y)$ in $z-$direction at fixed (x,y);
2) The concrete type of the BC on the hyper-plane H_0.

The simplest assumption regarding the behavior of the function $U_0(x,z,y)$ is: the function is weakly depending on variable z. Then we can assume following sequence of equalities:

$$U_0(x,z,y)\cong u_0(x,y)\cong U(x,0,y)\equiv u(x,0,y). \quad (10)$$

Let it be given the second type of BC on H_0:

$$-k_0\frac{\partial U_0}{\partial z}\bigg|_{z=-\delta}=\varphi^0(x,y). \quad (11)$$

The principal relation (9) by means of second conjugations condition (4) immediately gives the following equation:

$$k\frac{\partial u}{\partial z}\bigg|_{z=0}+\delta\frac{\partial}{\partial x}\left(k_0\frac{\partial u}{\partial x}\right)+\delta L^0\left(u\right)= -\left[\varphi^0(x,y)+\delta f_0(x,y)\right]. \quad (12)$$

This equation is independent from argument z and contains the term $k\frac{\partial u}{\partial z}$ at hyper-plain $z=0$. That means we can consider equation (12) as non-classical BC (given on H) for main equation (7).

The approximation of the solution $U_0(x,z,y)$ in $z-$direction by linear function

$$U_0(x,z,y)=u(x,0,y)-x\frac{\varphi^0(x,y)}{k_0(y)}$$

leads to similar to equation (12) transformed BC (the difference between both formulae is in the right hand side terms; approximation by linear function instead of constant gives last two additional terms):

$$k\frac{\partial u}{\partial z}\bigg|_{z=0}+\delta\frac{\partial}{\partial x}\left(k_0\frac{\partial u}{\partial x}\right)+\delta L^0(u)=$$
$$-\left\{\varphi^0(x,y)+\delta f_0(x,y)+\frac{\delta^2}{2}\left[\frac{\partial^2\varphi^0}{\partial x^2}+L^0\left(\frac{\varphi^0}{k_0}\right)\right]\right\}.$$

We can use the second order polynomial for the more accurate approximation of the function $U_0(x,z,y)$:

$$U_0(x,z,y)=u(x,0,y)+\frac{z}{\delta}u_1(y)+\left(\frac{z}{\delta}\right)^2u_2(y).$$

Such approximation finally gives the system of two BC on H for equation (7) (see [4] for details):

$$\begin{cases}\frac{3k_0}{\delta}(u-u_0)+\delta\frac{\partial}{\partial x}\left(k_0\frac{\partial u_0}{\partial x}\right)+\delta L^0(u_0)= \\ -\left(\frac{3}{2}\varphi^0+\delta f_0\right), \\ k\frac{\partial u}{\partial z}\bigg|_{z=+0}=\frac{3k_0}{\delta}\left(u\big|_{z=+0}-u_0\right)+\frac{\varphi^0}{2},\end{cases}$$

or, in other form:

$$\begin{cases}k\frac{\partial u}{\partial z}+\delta\frac{\partial}{\partial x}\left(k_0\frac{\partial u_0}{\partial x}\right)+\delta L^0(u_0)=-\left(\varphi^0+\delta f_0\right), \\ k\frac{\partial u}{\partial z}\bigg|_{z=+0}=\frac{3k_0}{\delta}\left(u\big|_{z=+0}-u_0\right)+\frac{\varphi^0}{2}.\end{cases}\qquad(13)$$

The system of BC (13) can be reduced to one equation by excluding the averaged function $u_0(x,y)$:

$$k\frac{\partial u}{\partial z}+\delta\frac{\partial}{\partial x}\left[k_0\frac{\partial}{\partial x}\left(u-\frac{k\delta}{3k_0}\frac{\partial u}{\partial z}\right)\right]+\delta L^0(u_0)$$
$$=-\left\{\varphi^0+\delta f_0+\frac{\delta^2}{6}\frac{\partial}{\partial x}\left[k_0\frac{\partial}{\partial x}\left(\frac{\varphi^0}{k_0}\right)\right]\right\}.\qquad(13')$$

After the solving of the new transformed problem we can approximately reconstruct the solution (the function $U_0(x,z,y)$) on sub-domain $\bar{D}_0$ by formula:

$$\bar{U}_0(x,z,y)=u\big|_{z=0}-z\frac{k}{k_0}\frac{\partial u}{\partial z}\bigg|_{z=0}-$$
$$\frac{z^2}{2k_0}\left[L^0(u)+\frac{\partial}{\partial x}\left(k_0\frac{\partial u}{\partial x}\right)+F_0(x,z,y)\right]\bigg|_{z=0}.$$

The estimation of the error ΔU_0 between the solutions of the original and the transformed problems at the end point $z=-\delta$ is similar to with that given in paper [4] (only one additional term appears). E.g., in case of approximation by constant we obtain following expression:

$$\Delta U_0\le\frac{\delta}{k_0}\left[k\left|\frac{\partial u}{\partial z}\right|+\frac{\delta}{2}\left|L^0(u)+\frac{\partial}{\partial x}\left(k_0\frac{\partial u}{\partial x}\right)+F_0\right|\right]\bigg|_{z=0}$$

The process of obtaining the new non-classical BC in case of third type BC

$$-k_0\frac{\partial U_0}{\partial z}\bigg|_{z=-\delta}+h_0u=\varphi^0(x,y)$$

in the initial statement of problem is similar to the case of the given second type BC. For the simplest approximation instead of BC (12) we obtain following new BC:

$$k\frac{\partial u}{\partial z}\bigg|_{z=0}+\delta\frac{\partial}{\partial x}\left(k_0\frac{\partial u}{\partial x}\right)-$$
$$h_0u+\delta L^0(u)=-\left[\varphi^0+\delta f_0\right].$$

The first type of BC on H_0

$$U_0\big|_{z=-\delta}=\varphi^0(x,y)\qquad(14)$$

requires different consideration. The direct use of the simplest assumption (10) gives:

$$U_0(x,z,y)\cong u_0(x,y)=$$
$$u(x,0,y)=\varphi^0(x,y),\qquad(15)$$

i.e.

$$u\big|_{z=0}=\varphi^0(x,y).$$

That means that from the new formulation of the problem all the physical and geometrical properties of the sub-domain D_0 have disappeared. That is inadmissible decision. The right way is to employment the principal relation with the usage of the equality (15) for first term and operator L^0, the continuity condition (4) for first flux term and neglecting second flux term. Then, instead of first type BC (14) we obtain following generalization of second type BC:

$$k\frac{\partial u}{\partial z}+\delta\frac{\partial}{\partial x}\left(k_0\frac{\partial u}{\partial x}\right)+\delta L^0(u)= -\delta f_0(x,y),(x,y)\in H. \quad (16)$$

We can modify the transformed BC by using obvious equality:

$$u_0(x,y)=\frac{u(x,0,y)+\varphi^0(x,y)}{2}. \quad (17)$$

Then instead of BC (16) we obtain following non-classical BC:

$$k\frac{\partial u}{\partial z}+\frac{\delta}{2}\left[\frac{\partial}{\partial x}\left(k_0\frac{\partial u}{\partial x}\right)+L^0(u)\right]= -\delta\left\{f_0(x,y)+\frac{1}{2}\left[\frac{\partial}{\partial x}\left(k_0\frac{\partial\varphi^0}{\partial x}\right)+L^0\left(\varphi^0\right)\right]\right\}. \quad (18)$$

The next step is the approximation of the solution $U_0(x,z,y)$ by linear function. The equality (17) together with evident equality

$$k_0\frac{\partial U_0}{\partial z}\bigg|_{z=-\delta}=\frac{k_0}{\delta}\left[u(x,0,y)-\varphi^0(x,y)\right]$$

gives such generalization of third type BC on the hyper-plane H :

$$u+\frac{\delta}{k_0}\left\{k\frac{\partial u}{\partial z}+\frac{\delta}{2}\left[\frac{\partial}{\partial x}\left(k_0\frac{\partial u}{\partial x}\right)+L^0(u)\right]\right\}= \varphi^0-\frac{\delta^2}{k_0}\left\{f_0+\frac{1}{2}\left[\frac{\partial}{\partial x}\left(k_0\frac{\partial\varphi^0}{\partial x}\right)+L^0\left(\varphi^0\right)\right]\right\}. \quad (19)$$

The approximation the solution $U_0(x,z,y)$ by linear function means the constant flux at any point $z\in[-\delta,0]$. This concept brings us to other form of non-classical BC:

$$\left[\frac{\partial}{\partial x}\left(k_0\frac{\partial u}{\partial x}\right)+L^0(u)\right]=-2f_0(x,y)+ \left[\frac{\partial}{\partial x}\left(k_0\frac{\partial\varphi^0}{\partial x}\right)+L^0\left(\varphi^0\right)\right].$$

The following step to increase accuracy of new non-classical BC on H consists of the usage of second order polynomial

$$U_0(x,z,y)=u(x,0,y)+\frac{z}{\delta}u_1(y)+\left(\frac{z}{\delta}\right)^2u_2(y). \quad (20)$$

The BC (14) together with conjugation condition (4) and definition (8) gives the first of two new BC on the hyper-plane H :

$$\frac{\delta k}{2k_0}\frac{\partial u}{\partial z}=2u+\varphi^0-3u_0. \quad (21)$$

The representation (20) allows obtaining following expression for the difference of fluxes:

$$k_0\frac{\partial U_0}{\partial z}\bigg|_{z=-\delta}^{z=-0}=\frac{6k_0}{\delta^2}\left(u+\varphi^0-2u_0\right).$$

Then the principal relation (9) easy gives the second new BC:

$$u-\frac{\delta k}{k_0}\frac{\partial u}{\partial z}-\frac{\delta^2}{2k_0}\left[\frac{\partial}{\partial x}\left(k_0\frac{\partial u_0}{\partial x}\right)+L^0(u_0)\right] =\varphi^0+\frac{\delta^2}{2k_0}f_0(x,y). \quad (22)$$

The system of two BC (21), (22) can be reduced to one equation by excluding from it the averaged function $u_0(x,y)$.

2.2 Conservative Averaging According the Coordinate x

Now the original problem consists of the main equation (7). The solution of this equation (together with appropriate BC) we denote again as $u(x,z,y)$:

$$\frac{\partial}{\partial x}\left(k\frac{\partial u}{\partial x}\right)+\frac{\partial}{\partial z}\left(k\frac{\partial u}{\partial z}\right)+L(u) =-F(x,z,y),\ (x,z,y)\in D. \quad (23)$$

Now we introduce second integral averaged function, in $x-$direction:

$$v_0(y)=\frac{1}{H}\int_0^H u_0(x,y)dx. \quad (24)$$

Then we integrate the principal relation (9). This gives exact equality again:

$$L^0(v_0)+\frac{k_0}{\delta}\frac{\partial v_0}{\partial z}\bigg|_{z=-\delta}^{z=-0}+\frac{k_0}{H}\frac{\partial u_0}{\partial x}\bigg|_{x=0}^{x=H} =-g_0(y),\ g_0(y)=\frac{1}{H}\int_0^H f_0(x,y)dx. \quad (25)$$

We shall call this equality *second principal relation.* Second principal relation (25) is also an underdetermined equation. It contains two different functions: $v_0(y)$ and $u_0(x,y)$ in one equation.

Conservative averaging in $z-$directions with assumptions which led to non-classical BC (12) immediately gives following relation:

$$\frac{k}{\delta}\frac{\partial u}{\partial z}+L^0(u)+\frac{k_0}{H}\frac{\partial u}{\partial x}\bigg|_{x=0}^{x=H}= -\left[\frac{\phi^0(y)}{\delta}+g_0(y)\right]. \quad (26)$$

Here

$$\phi_0(y) = \frac{1}{H}\int_0^H \varphi_0(x,y)dx.$$

For non-classical system of two BC (13) we will have system of two relations:

$$\begin{cases} k\frac{\partial u}{\partial z} + \frac{k_0\delta}{H}\frac{\partial v_0}{\partial x}\Big|_{x=0}^{x=H} + \delta L^0\left(u_0\right) = \\ -\left[\phi^0 + \delta g_0 + \delta^2 L^0\left(\frac{\phi^0}{6k_0}\right)\right], \\ k\frac{\partial u}{\partial z} = \frac{3k_0}{\delta}\left(u - \frac{v_0\big|_{x=0}^{x=H}}{H}\right) + \frac{\phi^0}{2}. \end{cases} \tag{27}$$

It remains to repeat the conservative averaging in $x-$direction as in part 1 of our paper.

3 Some examples of Transformed Problems

We will start with the first type BC for heat transfer problem. To simplify the explanation we assume absence of other space arguments, i.e. $y' \equiv y = t$. Further, let the operator L^0 is the time derivative in the heat equation:

$$L^0(u) := -c_0\rho_0\frac{\partial u}{\partial t}.$$

Then the equation (16) gives following non-classical BC on hyper-plane $z = 0$ for the main PDE (7):

$$c_0\rho_0\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(k_0\frac{\partial u}{\partial x}\right) + \frac{k}{\delta}\frac{\partial u}{\partial z} + f_0(x,t). \tag{28}$$

This BC by absent second space coordinate x reduces to so-called "concentrate heat capacity" condition with one flux term (see [1], [5]).

By the way, we can easy demonstrate how we can obtain from principal relation (9) the well known convective heat (mass) exchange BC. Let it be given the first type BC (14). We assume the absence of operator L^0, of source term and of conduction term in equation (9). Then this equation for function $U_0(z,y)$ reduces to following simple equality:

$$k_0\frac{\partial U_0}{\partial z}\Big|_{z=-\delta}^{z=-0} = 0.$$

Assuming the linearity of solution in $z-$direction for the lower flux term and using second conjugations condition (4) for the upper flux term we obtain:

$$k\frac{\partial u}{\partial z} = h(u - \varphi^0), h = \frac{k_0}{\delta}. \tag{29}$$

We will finish with mathematical model for diffusion-convection problem in orthotropic media, which generalize simplest BC (29). We take the PDE (1), (2) in form:

$$\begin{aligned} &\frac{\partial}{\partial x}\left(k_1\frac{\partial U}{\partial x}\right) + \frac{\partial}{\partial z}\left(k_2\frac{\partial U}{\partial z}\right) - v\frac{\partial U}{\partial x} \\ &-c\frac{\partial U}{\partial t} = -F(x,z,y),\ (x,z,t)\in D, \end{aligned} \tag{1_1}$$

$$\begin{aligned} &\frac{\partial}{\partial x}\left(k_{0,1}\frac{\partial U_0}{\partial x}\right) + \frac{\partial}{\partial z}\left(k_0\frac{\partial U_0}{\partial z}\right) - v_0\frac{\partial U_0}{\partial t} \\ &-c_0\frac{\partial U_0}{\partial t} = -F_0(x,z,t),\ (x,z,t)\in D_0 \end{aligned} \tag{2_1}$$

together with conjugation conditions (3), (4) and BC (14). We also need other initial and BC, but they are not important for averaging process. The averaging procedure by means of principal relation (9) leads to following formulation. We obtain the main PDE on sub-domain $(x,z,t)\in D$:

$$\begin{aligned} &\frac{\partial}{\partial x}\left(k_1\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial z}\left(k_2\frac{\partial u}{\partial z}\right) - \\ &v\frac{\partial u}{\partial x} - c\frac{\partial u}{\partial t} = -F(x,z,y) \end{aligned} \tag{30}$$

with following non-classical BC on hyper-plane $H\left(z=0\right)$:

$$\begin{aligned} &\frac{\partial}{\partial x}\left(k_{0,1}\frac{\partial u}{\partial x}\right) + \frac{k}{\delta}\frac{\partial u}{\partial z} - \frac{k_0}{\delta}u - v_0\frac{\partial u}{\partial t} \\ &-c_0\frac{\partial u}{\partial t} = -\left(f_0(x,t) + \frac{k_0}{\delta}\varphi^0\right). \end{aligned} \tag{31}$$

The non-classical BC of type (31) for main PDE (30) allows us to describe different physical processes in thin border layer, e.g. [7], [8].

4 Conclusions

Conservative averaging method can be applied to steady-state and non-stationary problems, to problems with continuous or discontinuous coefficients. It can be applied to different types of boundary conditions.

Acknowledgements:
Research was supported by Council of Sciences of Latvia (grant 05.1525) and University of Latvia (project Y2- ZP09-100).

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