Theoretical Advances on Generalized Fractals with Applications to Turbulence

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Abstract: - Theoretical advances toward a purely meshless framework for analysis of the generalized fractal dimension, with applications to turbulence, are considered. The key basic theoretical idea is the formulation of the probability density function of the minimum distance to the nearest part of the flow feature of interest, e.g. a turbulent interface, from any location randomly chosen within a reference flow region that contains the feature of interest. The probability density function of the minimum-distance scales provides a means to define and evaluate the generalized fractal dimension as a function of scale. This approach produces the generalized fractal dimension in a purely meshless manner, in contrast to box-counting or other box-based approaches that require meshes. This enables the choice of a physical reference region whose shape can be based on physical considerations, for example the region of fluid enclosed by the turbulent interface, in contrast to box-like boundaries necessitated by box-counting approaches. The purely meshless method is demonstrated on spiral interfaces as well as high-resolution experimental turbulent jet interfaces. Examination of the generalized fractal dimension as a function of scale indicates strong scale dependence, at the large energy-containing scales, that can be described theoretically using exponential Poisson analytical relations.

Keywords: - Fractals, Dimensions, Distributions of Scales, Turbulence, Self-Similarity.

1. Introduction

Turbulent flows and other multiscale flows exhibit highly complex geometrical behaviour as is directly evident in quantitative and qualitative visualizations in nature as well as in laboratories [e.g. 1, 2]. Because of this geometrical complexity, which results directly from highly irregular multiscale dynamics, advancing the knowledge of geometrical properties remains a challenging task [e.g. 1–3]. In turbulence, geometrical aspects are useful both for fundamental studies and for applications [e.g. 1–3], in a wide range of problems with phenomena ranging from molecular diffusion to electromagnetic wave propagation. Fundamentally as well as in applications, it is desirable to develop new ideas that can facilitate the study of geometrical multiscale properties of a variety of flow aspects such as fluid interfaces, vortices, or dissipation regions.

In previous studies of geometrical aspects of turbulence and multiscale flows, a key concept that has emerged is the notion of fractional, i.e. non-integral, dimensions or fractal dimensions. The earliest suggestions of the use of the fractal concept in fluid mechanics can be traced to Richardson [4], who cited Weierstrass regarding what are now known as fractal functions, i.e. self-similar functions. Extensive studies of fractals in turbulence started with the work of Mandelbrot [5] . The words fractal and self-similar, both

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coined by Mandelbrot, reflect ideas concerning multiscale behaviour of direct interest to turbulence studies, with the former etymologically meaning fragmentation and the latter denoting repetition of structure at multiple scales, i.e. scale independence or scale invariance. However, other studies starting with the work of Takayasu [6] indicate scale dependence as an additional, or alternative, concept for the multiscale geometrical nature of turbulence [e.g. 2]. In fact, level-crossing measurements in turbulence [e.g. 7] in conjunction with general theoretical results on fractal dimensions [8] indicate scale dependence rather than scale independence.

As an idea as well as a tool, the fractal dimension is invaluable in light of the need to quantify the multiscale geometry of turbulence. Reviews of some of the key developments on fractals in turbulence are available in a number of articles [e.g. 2, 3]. Findings of a constant or nearly-constant fractal dimension in a range of scales imply self-similarity [e.g. 3]. Observations of scale dependence of the fractal dimension [e.g. 2, 8] can imply either intrinsic variation of structure with scale, or intermittently self-similar behaviour, or finite-size effects of upper and lower cutoff scales. In practice, the conventional approach to fractal aspects of turbulence is in terms of box-counting methods that require partitioning of box-shaped regions that are chosen to contain the turbulent features of interest [e.g. 3].

In the present work, theoretical advances toward a purely meshless approach to the analysis of generalized fractal dimensions are investigated. The probability density function of the shortest distance from random point locations to the flow feature of interest is considered. The method is demonstrated and investigated using spiral interfaces as well as fully-developed turbulent jet interfaces. Results are presented concerning the generalized fractal behaviour and demonstrating the utility of the meshless method for the study of the multiscale geometry of turbulent flows.

2. The Minimum Distance Method and Purely Meshless Analytical Framework

Throughout most previous studies of fractal aspects of turbulence [e.g. 1, 3, 5], statistical self-similarity has been assumed in a range of scales in which a turbulent feature of interest has a fractal, i.e. fractional, dimension D_d that is scale-independent and quantifies the multiscale geometrical behaviour:

$$0 \leq D_d \leq d, \tag{1}$$

within a reference region of Euclidean dimension d. In other studies, however, it has been suggested to generalize the original notion of a fractal dimension to allow for possible dependence on scale, i.e. a dimension $D_d(\lambda)$ that can be a function of scale λ [e.g. 2, 6, 8]. This has been denoted as a differential or scale-dependent fractal dimension. We denote it here as a generalized fractal dimension.

Motivated by previous theoretical work [8], we consider as a basic idea toward a purely meshless framework, as shown in figure 1, the identification of any random location from which the minimum distance to the nearest part of the turbulent feature is identified. This can be performed for several random locations, as shown for example in figure 1, that are selected from within a reference boundary.

Theoretically, the shortest distances from all possible such locations, i.e. an infinity of such locations, determine the probability density of these distances. In practice, a finite number of such locations depending on the scale resolution desired for the probability density function can be expected to be sufficient.

The method, therefore, provides an intrinsically Monte Carlo approach. The procedure is repeated for as many locations as needed depending on the scale resolution desired for the resulting generalized fractal dimension function. As will be shown below, the shortest-distance scales can be used directly to compute the generalized fractal dimension function.

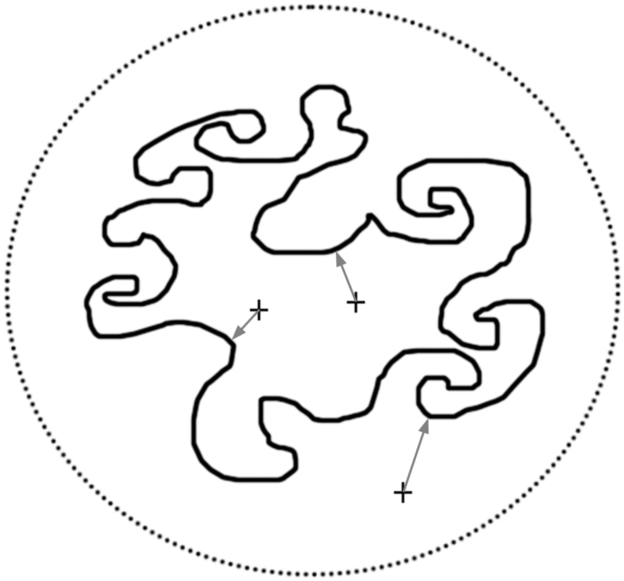

Figure 1: In the purely meshless approach, a boundary region of any shape can be chosen. In this schematic, a circular boundary is shown (dotted). Multiple random locations (crosses) selected within this boundary are utilized to evaluate the minimum distance from each location to the nearest flow feature.

The shortest-distance probability density function, $g_d(\lambda)$, in general d-dimensional space, is defined as follows:

Minimum-Distance PDF $g_d(\lambda)$:

> We define the quantity $g_d(\lambda)$ as the probability density function (pdf) of the minimum distance λ from a randomly located point, within the reference boundary, to the nearest part of the flow feature of interest.

By minimum distance, in d dimensions, we mean the shortest distance in a general sense, i.e. in space, in time, or in space–time, as appropriate, from each point randomly chosen but contained within a $(d-1)$-dimensional reference boundary, to the flow feature of interest. Along with the shortest-distance probability density function, i.e.

$$g_d(\lambda) = \mathcal{P}\{\lambda \text{ is min. distance to flow feature}\}, \tag{2}$$

we therefore have the associated shortest-distance probability $g_d(\lambda)\,\mathrm{d}\lambda$. The normalization integral, over the region contained within the reference boundary, is:

$$\int_0^{\lambda_C} g_d(\lambda)\mathrm{d}\lambda \;=\; 1\,, \tag{3}$$

where λ_C denotes a scale, with $\lambda \leq L$, and L denotes the largest characteristic scale of the flow. In general, i.e. for multiscale flow features, we can expect that there will be a scale above which $g_d(\lambda) = 0$ yet this scale need not be as large as the largest characteristic scale L of the flow. This is because the convolutions of the flow feature directly limit the maximum possible value of the shortest distance. We can define λ_C as the maximum shortest distance scale, i.e. such that $g_d(\lambda > \lambda_C) = 0$ or $g_d(\lambda \leq \lambda_C) > 0$:

$$\lambda_C \;=\; \max_\lambda \{\lambda \,:\, g_d(\lambda) \,>\, 0\}\,. \tag{4}$$

We note that this scale can also be effectively identified using $\min_\lambda \{\lambda \;:\; g_d(\lambda) \;=\; 0\}$. The cumulative minimum-distance distribution function $G_d(\lambda)$ associated with the shortest-distance probability density function $g_d(\lambda)$ is, i.e.

$$G_d(\lambda) = \int_0^\lambda g_d(\lambda')\,\mathrm{d}\lambda'\,,\ \text{or}\,,\ g_d(\lambda) = \frac{\mathrm{d}G_d(\lambda)}{\mathrm{d}\lambda}\,, \tag{5}$$

with limiting values $G_d(\lambda \to 0) \;\to\; 0$ and $G_d(\lambda \;\to\; \lambda_C) \;\to\; 1$ at the smallest and largest scales, respectively. The minimum-distance probability density function $g_d(\lambda)$ and the minimum-distance cumulative distribution function $G_d(\lambda)$ can now be utilized directly, in analogy with previous theory [8], to obtain the generalized fractal dimension $D_d(\lambda)$ and scaling exponent $\alpha_d(\lambda)$:

$$D_d(\lambda) \;\equiv\; d \;-\; \frac{\mathrm{d}\log G_d(\lambda)}{\mathrm{d}\log\lambda}\,, \tag{6}$$

and

$$\alpha_d(\lambda) \;\equiv\; -\,\frac{\mathrm{d}\log g_d(\lambda)}{\mathrm{d}\log\lambda}\,. \tag{7}$$

Substituting for $G_d(\lambda)$, using equation 5 and the normalization equation 3, we have:

$$D_d(\lambda) \;=\; d \;-\; \frac{\lambda\, g_d(\lambda)}{\displaystyle\int_0^{\lambda} g_d(\lambda')\,\mathrm{d}\lambda'}\,, \qquad (8)$$

or,

$$D_d(\lambda) \;=\; =\; d \;-\; \frac{\lambda\, g_d(\lambda)}{1 \;-\; \displaystyle\int_{\lambda}^{\lambda_C} g_d(\lambda')\,\mathrm{d}\lambda'}\,, \quad (9)$$

and,

$$g_d(\lambda) = \frac{E_d(\lambda)}{\lambda}\,\exp\left\{ -\int_{\lambda}^{\lambda_C} \left[E_d(\lambda') \right] \frac{\mathrm{d}\lambda'}{\lambda'} \right\}\,, \qquad (10)$$

where $E_d(\lambda) \equiv d - D_d(\lambda)$ and the improvement with respect to the earlier framework [8] is that this approach is purely meshless, i.e. it is free of any box-like constraints. Thus, the method enables the evaluation of the generalized fractal dimension $D_d(\lambda)$ from the minimum-distance probability density function $g_d(\lambda)$.

3. Applications of the Minimum Distance Framework to Spiral and Turbulent Interfaces

Application to spiral interfaces, in figure 2, shows that the purely meshless method is more accurate than box-based methods because it captures very well the transition in the dimension at the scale of the spacing between spiral turns. As is evident in figure 1, the concept of the minimum distance is applicable to all randomly-chosen points within the reference boundary. For each such point, the shortest distance $\lambda \geq 0$. If the random location is a part of the flow feature, then $\lambda = 0$. If the random location is not part of the flow feature, then $\lambda > 0$. Because the randomly chosen locations must be within the reference boundary, the probability density function $g_d(\lambda)$ is therefore a conditional probability density function of shortest-distance scales.

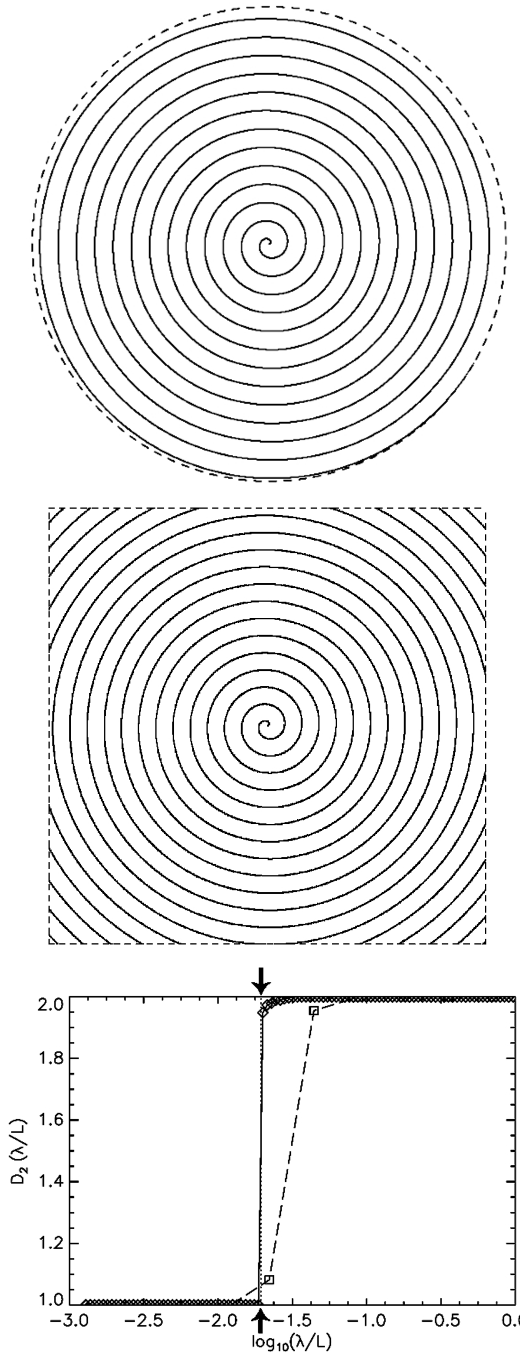

Figure 2: Spiral of Archimedes, i.e. $r \sim \theta$, and the generalized fractal dimension as a function of scale for a circular boundary using the purely meshless approach (dashed) and for a box boundary using box counting (solid). The purely meshless approach captures accurately the transition in the generalized fractal dimension, which occurs at the scale given by the spacing between spiral turns, as shown by the arrows.

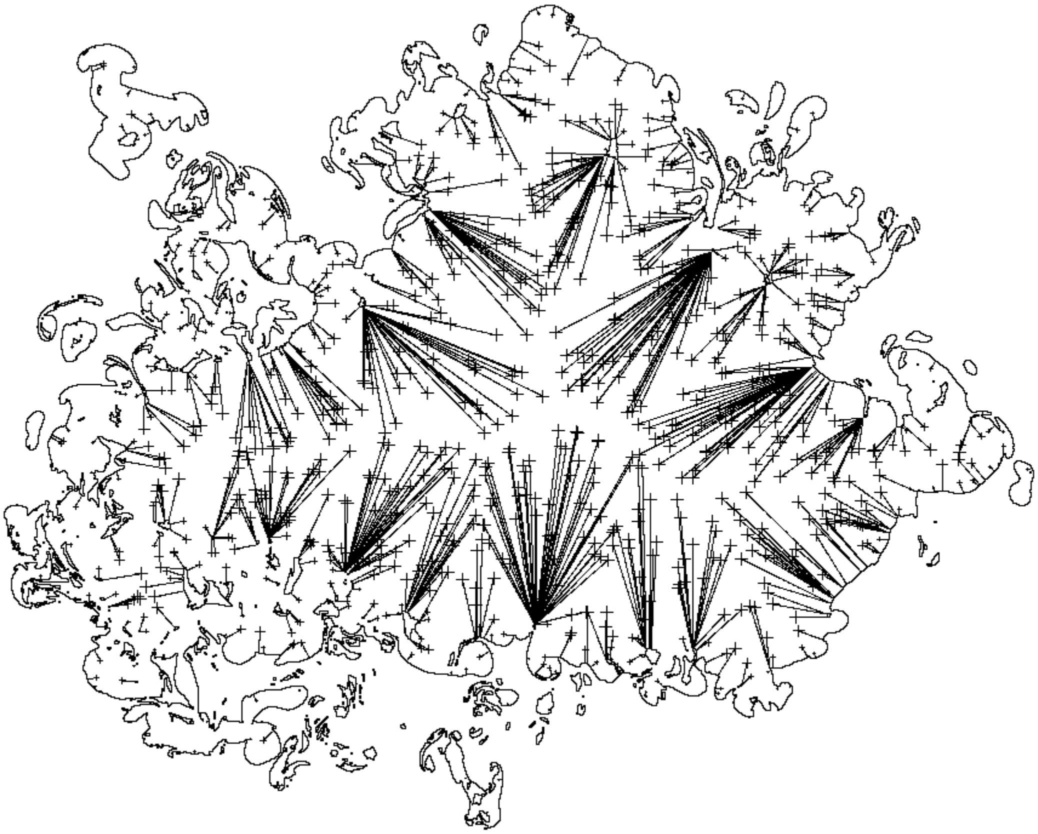

Figure 3: Example of conditional randomly chosen locations and minimum-distance scales within an outer turbulent jet scalar interface, with the interface as the reference boundary. The Reynolds number is $Re \sim 20,000$ and the Schmidt number is $Sc \sim 2,000$.

Application to experimental fully-developed turbulent jet interfaces is shown in figure 3. The minimum-distance scales from conditionally random locations chosen within the outer turbulent jet scalar regions are indicated in figure 3. In order to focus on the behavior of the generalized fractal dimension at the energy-containing large scales, coarse graining was performed to retain large scales and filter small scales at the Taylor scale threshold.

The corresponding results on the generalized fractal dimension as a function of scale are shown in figure 4. The results clearly indicate a strong scale dependence of the generalized fractal dimension as a function of scale at large scales, which in particular appears to correspond to an exponential distribution of scales, i.e., Poisson statistics, as is evident by comparing the behavior in figure 4 to previous theoretical studies of exponential probability density functions of scales [8].

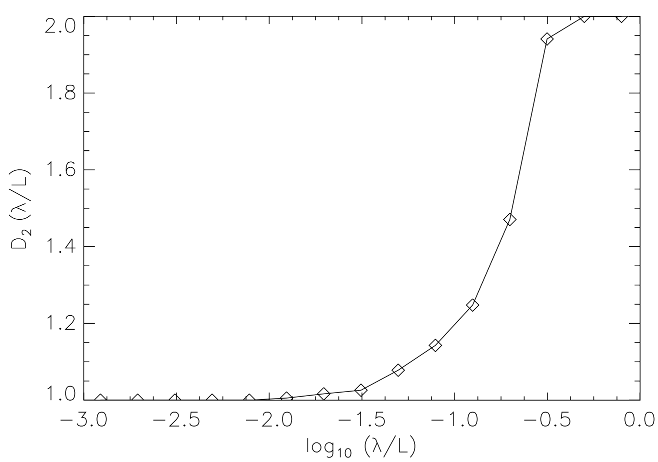

Figure 4: Generalized fractal dimension as a function of logarithmic scale, evaluated using the shortest-distance scales from conditionally random locations in the coarse grained scalar turbulent jet regions, created by spatial averaging combined with bilinear interpolation.

The experimental data shown in figure 4, which are for coarse-grained turbulent jet scalar interfaces, are in very close agreement with exponential statistics [8]. Specifically, the theoretical formula for the generalized fractal dimension as a function of scale for exponential statistics is as follows:

$$D_2(\lambda) \;=\; 2 \;-\; \frac{\lambda/l_{\rm m}}{e^{\lambda/l_{\rm m}} \;-\; 1}\,. \qquad (11)$$

Thus, this result shows that the large-scale statistical geometrical behavior in turbulent jets can be quantified in terms of exponential Poisson statistics for the generalized fractal dimension as a function of scale. This is specifically the case in the range of energy-containing scales which, for the present data, is $7.8 \times 10^{-3} \sim \lambda/L \sim 1$. We note that this scale is consistent with an estimate based on large-scale dynamics, i.e.

$$\frac{\lambda_{\rm L}}{L} \;\sim\; Re^{-1/2} \;\sim\; (2 \times 10^4)^{-1/2} \;\sim\; 7.1 \times 10^{-3}\,, \qquad (12)$$

for the present conditions of Reynolds number $Re \sim 20{,}000$.

4. Conclusions

The present findings show that the purely meshless minimum-distance theoretical framework provides a practical method for determining the generalized fractal dimension as a function of scale. We have demonstrated the use of the method for spiral interfaces and for fully-developed turbulent interfaces. For example, the present observation of Poisson behavior of the generalized fractal dimension as a function of scale, at large scales, quantifies the scale dependence in turbulent jets. This is crucial in several applications, e.g. for modeling the mixture fraction, because recent work has shown that the large-scale geometry of the outer scalar interfaces provides the dominant contribution to the mixture fraction [2]. We note that the large-scale geometrical aspects of turbulent scalar interfaces are of direct practical interest in various applications such as the mixing efficiency of fluids engineering devices, laser propagation through turbulence, and flow optimization.

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