Couette-Poiseuille Flow of Non-Newtonian Fluids in Concentric Annuli

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Abstract: - Two types of laminar flows of non-Newtonian fluids in annuli with stationary outer cylinder are analysed from the viewpoint of existence of an exact analytical or almost exact analytical (quasisimilarity) solution relating volumetric flow rate and axial pressure gradient. The exact analytical solution is derived for the Vocadlo (Robertson-Stiff) fluids when flow is caused simultaneously by the inner cylinder moving along its axis and by the pressure gradient imposed in the axial direction. Both cases - either pressure gradient assists to the moving cylinder or opposes - are considered. The quasisimilarity solution is derived for the power-law fluids when the inner cylinder rotates under a constant torque and pressure force is imposed in the axial direction as in the preceding case.

Key-Words: - Concentric annuli, Power-law model, Vocadlo (Robertson-Stiff) model

1 Introduction

The flow of non-Newtonian fluids through an annulus is often encountered in various industrial processes such as transportation of drilling fluids in petroleum industry and extrusion of polymers (in a mandrel region).

In the annular flow one of the most difficult complications consists in the inhomogeneous distribution of shear stresses in the annular region. The analysis of annular flow originated by a combination (Couette-Poiseuille flow) of the drag (Couette flow) and pressure (Poiseuille flow) forces is further complicated by the fact that no superposition principle takes place; in other words, this flow field is not possible to obtain as a mere superposition of corresponding Couette and Poiseuille flow fields. This is a direct consequence of the dependence of fluid viscosity on velocity field invariants.

At present there exist two basic classes of constitutive equations describing rheological quantities of the materials: classical empirical models (relating stress, rate of deformation, viscosity) and more sophisticated ones (differential, integral or integro-differential equations based on physical grounds). The latter ones provide more complex characterisation of materials including simultaneous description of the individual rheological quantities, their number of adjustable parameters is limited, however their accuracy is not always acceptable in all aspects. On the other hand usefulness of the classical empirical models is examined through a number of decades, from the practical point of view only the usage of these models gives a chance for derivation of analytical solutions.

Roughly speaking there are two approaches how to cope with the description of the Couette-Poiseuille flow situations. The numerical approach aims at a calculation of the quantities (e.g. velocity components, flow rate) describing the concrete problem, and with an arbitrary change of the entry parameters (geometry, kinematics, rheological characteristics) it is necessary to repeat the whole procedure from the beginning.

The other approach lays emphasis on the functional participation of the individual entry parameters in the whole solution. This method enables to decide which parameters should be altered (and in which way) to obtain the more favourable results e.g. from the viewpoint of production rate. In this case the optimum approach is represented by an explicit solution or ‘almost explicit’ one (as e.g. so-called quasisimilarity solution) deviating negligibly from the exact one.

In the present contribution this second approach is presented for two types of steady laminar isothermal Couette-Poiseuille flows of incompressible fluids in concentric annuli. For both types it is supposed that an outer cylinder is stationary and pressure is exerted in an axial direction. The difference is in kinematics of an inner cylinder - either moving along (application of the Vocadlo rheological model) or rotating round (application of the power-law model) its axis.

2 Non-helical and helical flows of power-law fluids in concentric annuli

In this section there will be presented the possibilities how to determine analytically or almost analytically the relation volumetric flow rate vs. axial pressure gradient for Poiseuille (both cylinders are stationary) and Couette-Poiseuille (outer cylinder is stationary, inner cylinder is rotating round its axis) flows under the exertion of pressure in an axial direction, see Fig.1.

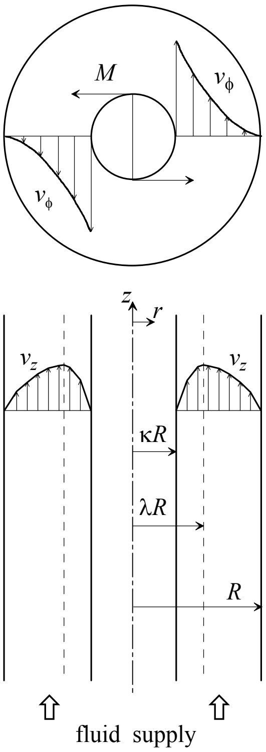

Fig.1 Definition sketch.

The power law model relates the rate of strain tensor D

$$D=\begin{bmatrix} 0 & r\dfrac{dW}{dr} & \dfrac{dU}{dr} \\ r\dfrac{dW}{dr} & 0 & 0 \\ \dfrac{dU}{dr} & 0 & 0 \end{bmatrix} \tag{1}$$

and shear stress tensor T

$$T=\begin{bmatrix} 0 & T_{r\phi} & T_{rz} \\ T_{r\phi} & 0 & 0 \\ T_{rz} & 0 & 0 \end{bmatrix} \tag{2}$$

by the following expression (η denotes shear viscosity)

$$D=\frac{1}{\eta}\cdot T\,,\quad \eta=k^{\frac{1}{n}}\cdot(II_T)^{\frac{n-1}{2n}}\,,\quad II_T=T_{r\phi}^2+T_{rz}^2 \quad . \tag{3}$$

Denoting velocity components by

$$v_r=0,\quad v_\phi=r.W(r),\quad v_z=U(r) \tag{4}$$

we can formulate balance equations in the form

$$\frac{dT_{rz}}{dr}+\frac{1}{r}\cdot T_{rz}=P,\quad \frac{dT_{r\phi}}{dr}+\frac{2}{r}\cdot T_{r\phi}=0, \qquad \frac{\partial p}{\partial r}=-\rho r W^2(r) \tag{5}$$

where

$$P\equiv\frac{\partial p(r,z)}{\partial z}=\frac{p(r,L)-p(r,0)}{L}<0 \quad . \tag{6}$$

The boundary conditions are

$$U(\kappa R)=0,\,U(R)=0,\,W(\kappa R)=\Omega,\,W(R)=0\,. \tag{7}$$

For the case of non-helical case the problem formulation is simplified due to the fact that $M\equiv 0$ (and consequently $W\equiv 0$, $v_\phi\equiv 0$).

2.1 Non-helical flow

The first theoretical contribution dealing with the flow of a power-law fluid through a concentric annulus with steady cylinders is a paper by Fredrickson and Bird [11]. They derived a relation between flow rate and axial pressure gradient by means of infinite series for the cases when a reciprocal value of the flow behaviour index n is a natural number. In their derivation a parameter λ (λR is a location of maximum velocity) plays a crucial role. For its determination it is necessary to solve an integral equation

$$\int_\kappa^\lambda\left(\frac{\lambda^2}{\xi}-\xi\right)^{\frac{1}{n}}d\xi=\int_\lambda^1\left(\xi-\frac{\lambda^2}{\xi}\right)^{\frac{1}{n}}d\xi \quad . \tag{8}$$

Due to this fact and to the form of the derived solution (infinite series) the role of the individual entry parameters is not sufficiently elucidated.

Substantial improvement was presented by Hanks and Larsen [14] who derived an analytical relation between flow rate and pressure gradient for an arbitrary value of flow behaviour index n. Nevertheless, their relation still requires knowledge of a parameter λ - thus demanding the numerical solution of the integral equation introduced in Fredrickson and Bird [11]. The same result was derived by Prasanth and Shenoy [18].

Non-helical flow was also analysed by Bird et al. [1] using a variational method. They minimised the corresponding functional using the supposed one-parametrical distribution of velocity. They finally obtained a relation that eliminated the parameter λ but in fact they derived a relation identical to that in McKelvey [17] for flow of a power-law fluid between two parallel plates (in other words, they implicitly supposed $\lambda=(1+\kappa)/2$). The inaccuracy of their result to the numerical solution is in full correspondence with Fig.8 in Worth [26]. Worth discussed the difference between the solution for a concentric annulus with that for parallel plates. This difference does not exceed 2% for $\kappa\geq 0.5$ and $n\geq 0.25$. Another possible approach is given in David and Filip [4].

The given problem is also possible to treat from the viewpoint of the similarity behaviour. It was shown (David and Filip [3]) that a solution exhibits various features of similarity behaviour – not in an exact form but only approximately (it implies the term 'quasisimilarity'). Nevertheless, even this 'weak' similarity enables one to derive a 'universal' solution which is possible to rewrite to a concrete form for given entry parameters by means of certain derived transformations. This fully eliminates the role of the parameter λ; however, quasisimilarity is not valid in the whole range of entry parameters κ, n. Nevertheless, based on this quasisimilarity behaviour, the approximate relations (David and Filip [5]) were derived for the whole range of entry parameters. The inaccuracy of these approximate relations to the precise ones is negligible (in comparison with experimental setting of parameters k,n of a power-law model), these approximate relations do not include the parameter λ and with respect to their accuracy provide functional dependence of entry parameters in the resulting expression between flow rate and pressure gradient.

2.2 Helical flow

Rivlin [21], and Coleman and Noll [2] ranged among first who derived balance equations for the case of the rotating inner cylinder. A series of authors published papers dealing with various constitutive equations and experimental set-ups – e.g. Tanner [22, 23], Dierckes and Showalter [6], Rea and Showalter [19], Rigbi and Galili [20], Winter [25], Garcia-Ramirez and Isayev [12], Dostal et al. [7]. For a more complete list of references (dealing not only with the power-law fluids and including also a non-helical case) see also Escudier et al. [8].

In the following the quasisimilarity analysis will be used to determine the relation volumetric flow rate-pressure drop-torque.

The quasisimilarity solution is possible to derive in three consequent steps: derivation of a dimensional solution, its transformation to a non-dimensional form, and finally application of suitable scales to the resulting relations (obtaining of the 'universal' profiles).

A dimensional solution and suitable non-dimensional transformations are introduced in Filip and David [10]. The non-dimensional form of the solution is as follows

$$t_{\xi\zeta} = \frac{\lambda^2}{\xi} - \xi, \quad t_{\xi\phi} = -\frac{\beta}{\xi^2} \tag{9}$$

$$\eta^* = \xi^{2\left(\frac{1}{n}-1\right)} \cdot \left(\beta^2 + \xi^2\left(\lambda^2 - \xi^2\right)^2\right)^{\frac{n-1}{2n}} \tag{10}$$

$$u(\xi) = f_3 - \lambda^2 \cdot f_2\,, \; w(\xi) = \beta \cdot f_1(\xi)\,, \; \omega = \beta \cdot j_1 \tag{11}$$

where

$$f_i(\xi) = \int_{\xi}^{1} \frac{\overline{\xi}^{2i-5}}{\eta^*(\overline{\xi})} d\overline{\xi}\,, \; j_i = f_i(\kappa R) \quad (i = 1,2,3,4) \tag{12}$$

$$\beta = \frac{\omega}{j_1}, \quad \lambda^2 = \frac{j_3}{j_2} \quad . \tag{13}$$

The relations for dimensionless axial flow rate and torque are of the form

$$q = \pi \cdot (j_4 - \lambda^2 j_3)\,, \quad m = 2\pi l \beta \quad . \tag{14}$$

The answer to a question how to obtain suitable scales transforming the dimensionless solutions for various entry parameters to a unique one, proceeds in the following way:

a) behaviour of the quantities q and ω will be determined for the limiting cases $\beta \to 0$ and $\beta \to \infty$;
b) the intersection of these limiting cases will be taken as a reference point for derivation of the resulting transformations;
c) the asymptotic planes for flow rate will be derived using these new transformations;
d) existence of a quasisimilarity solution will be shown.

ad a1) case $\beta \to 0$ (vanishing influence of rotation)
In this case we obtain

$$q \to q_0 = \frac{\pi n}{1+3n} \cdot \left(\left(1-\lambda^2\right)^{1+\frac{1}{n}} - \kappa^{1-\frac{1}{n}} \cdot \left(\lambda^2 - \kappa^2\right)^{1+\frac{1}{n}}\right), \omega \to 0 \tag{15}$$

where

$$j_i \to \int_{\kappa}^{1} \overline{\xi}^{-2(i-2)-\frac{1}{n}} \cdot \left(\left(\lambda^2 - \overline{\xi}^2\right)^2\right)^{\frac{1-n}{2n}} d\overline{\xi} \quad (i = 1,2,3,4)\,. \tag{16}$$

The relation for q_0 was derived by Hanks and Larsen [14].

ad a2) case $\beta \to \infty$ (prevailing influence of rotation)
From the relations obtained in the second step it follows that

$$q \to q_\infty = \pi \cdot \left(a_4 - \frac{a_3^2}{a_2}\right) \cdot \beta^{\frac{1-n}{n}}, \quad \omega \to \omega_\infty = a_1 \cdot \beta^{\frac{1}{n}} \tag{17}$$

where

$$j_i \to \beta^{\frac{1-n}{n}} \cdot a_i \quad (i = 1,2,3,4) \quad , \tag{18}$$

$$a_i = \int_{\kappa}^{1} \overline{\xi}^{2\left(i-\frac{1}{n}\right)-3} d\overline{\xi} \quad (i = 1,2,3,4) \quad . \tag{19}$$

ad b) The relations for q_0, q_∞ determine in the plot $\log(q)$ against $\log(\beta)$ two asymptotic straight lines intersecting at the point (q_0, β_0) where

$$q_0 = q_\infty \quad \Rightarrow \quad \beta_0 = \left(q_0 \cdot \left(\pi\left(a_4 - \frac{a_3^2}{a_2}\right)\right)^{-1}\right)^{\frac{n}{1-n}} \tag{20}$$

$$\omega_0 = a_1 \cdot \left(q_0 \cdot \left(\pi\left(a_4 - \frac{a_3^2}{a_2}\right)\right)^{-1}\right)^{\frac{1}{1-n}} \quad . \tag{21}$$

These three quantities q_0, ω_0, β_0 represent the scales that enable one to introduce the resulting transformations

$$q_r = \frac{q}{q_0}, \quad \omega_r = \frac{\omega}{\omega_0}, \quad \beta_r = \frac{\beta}{\beta_0} \quad . \tag{22}$$

ad c) From above it follows that in the coordinate system [$\log(q_r)$, $\log(\beta_r)$, $\log(\omega_r)$] there exist two asymptotic planes given by the relations

$$\lim_{\beta_r \to \infty} \left(\frac{q_r \cdot \beta_r}{\omega_r} \right) = 1 \Rightarrow \log(q_r) + \log(\beta_r) - \log(\omega_r) = 0 \tag{23}$$

$$\lim_{\beta_r \to 0} (\log(q_r)) = 0 \Rightarrow \log(q_r) = 0 \quad . \tag{24}$$

It is evident that the straight line given by the intersection of these asymptotic planes represents all Newtonian cases. Moreover, and this is important for the whole quasisimilarity analysis, neither plane depends on the flow behaviour index n and aspect ratio κ or any other entry parameter.

ad d) This implies that it is possible to determine approximate resulting values of the whole problem from the following Figs.2-6 taken from Filip and David [10] (Figs.5,6 are projections onto the corresponding planes in the coordinate system [$\log(q_r)$, $\log(\beta_r)$, $\log(\omega_r)$]) where the value κ=0.99 was taken as a basis for a 'universal' profile. This value was chosen due to the following reasons:

- for given n and β a deviation between the quasimilarity and exact solutions dramatically decreases with the increasing value of aspect ratio κ,
- with increasing value of κ this deviation does not exceed the acceptable limit for still broader range of the parameters n and β.

From the entry parameters we obtain the values q_0, ω_0, β_0, consequently the values q_r, ω_r, and from the dimensionless transformations the resulting final values (as e.g. flow rate Q). If we restrict the quasisimilarity region by the conditions

$$\kappa \geq 0.4 \quad , \quad n \geq 0.2 \quad , \quad \beta \geq \frac{17}{2} \cdot \left(1 - \kappa^2\right) \cdot \left(1 + \kappa\right) \tag{25}$$

then the inaccuracy of the approximate solution from the precise one does not exceed 2%. This value is attained in the very limited region along the part of the boundary given by rel.(25); this deviation decreases with the increasing aspect ratio κ as well as with the increasing flow behaviour index n.

For n=1 (Newtonian fluid) the relative flow rate q_r is independent on β_r and ω_r ($q = q_0$); in the coordinate system [$\log(q_r)$, $\log(\beta_r)$, $\log(\omega_r)$] flow of Newtonian fluids is represented by the straight line $\log(q_r)$=0, $\log(\beta_r)$=$\log(\omega_r)$.

For dilatant fluids (n>1) the deviation between the quasisimilarity solution and the exact one is completely negligible, i.e. for every set of entry parameters (and not only for those restricted by rel.(24)) the true curve sticks very closely to the asymptotic planes (23, 24). In this case the derived functional behaviour fully describes the problem studied. The increase of flow behaviour index n and/or torque M results in the decrease of flow rate.

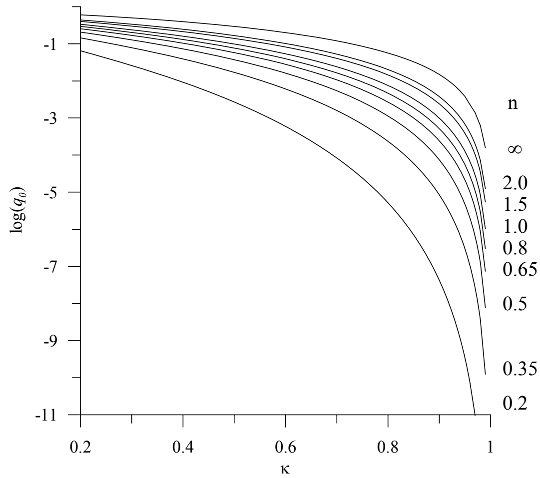

Fig.2 Dependence of q_0 on entry parameters κ, n.

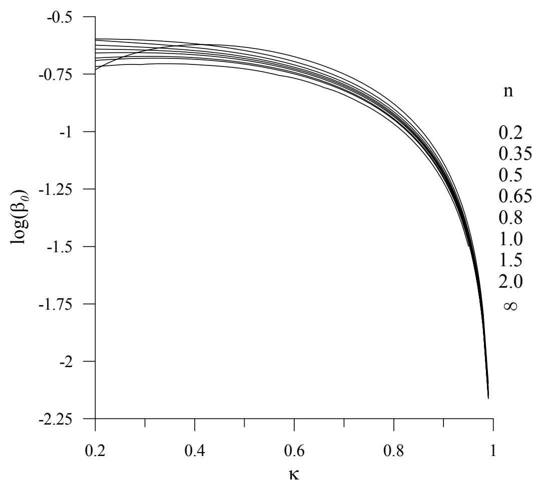

Fig.3 Dependence of β_0 on entry parameters κ, n.

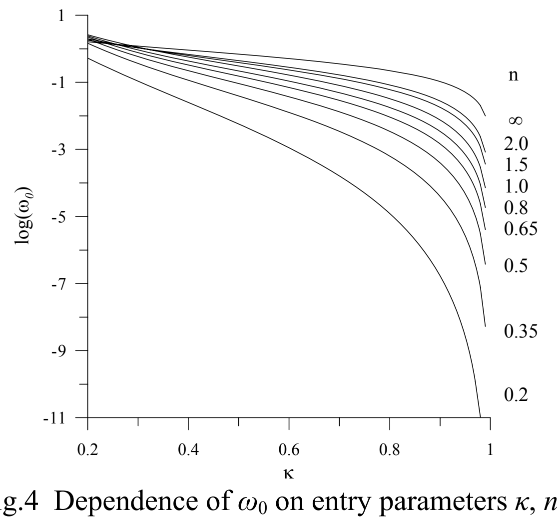

Fig.4 Dependence of ω_0 on entry parameters κ, n.

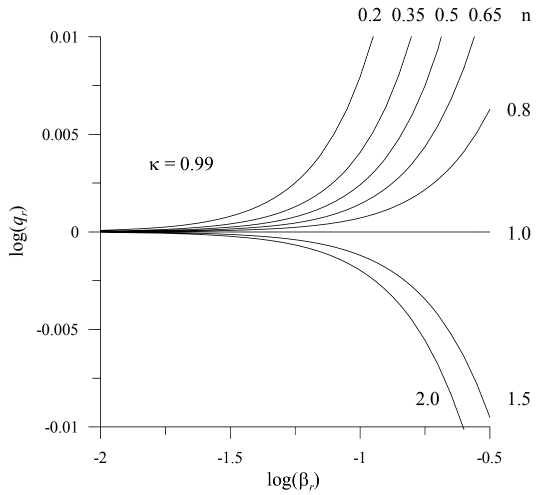

Fig.5 Dependence of q_r on β_r, n.

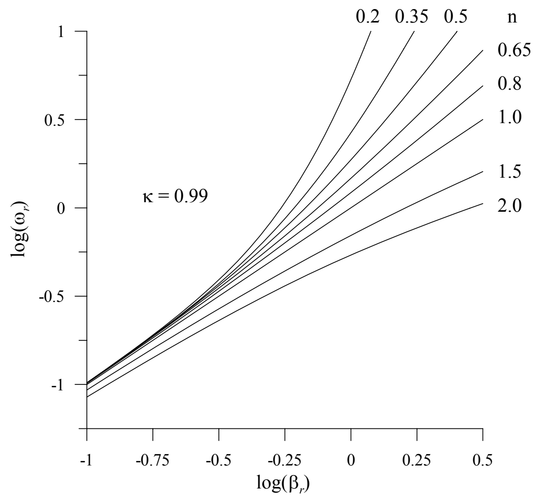

Fig.6 Dependence of ω_r on β_r, n.

The derived quasisimilarity solution subjects to a choice of the resulting transformations. The advantage of the relations (22) over the other possibilities consists in a 'proper' quasisimilarity behaviour of the flow rate q_r in dependence on β_r and ω_r; in other words, the course of the flow rate is very close (deviation less than 2% under the condition (25)) to the asymptotic planes representing classical similarity behaviour.

3 Longitudinal flow of Vocadlo fluids in concentric annuli

The literature on the axially moving inner cylinder (under simultaneous action of pressure gradient) is scarcer. For the case of the inner cylinder moving along its axis Wadwha [24] obtained the integral form for the axial velocity profile in the case of Ellis fluid. Lin and Hsu [15] studied power-law fluids and obtained the integral form for the flow rate. For the same problem Malik and Shenoy [16] succeeded to derive the semi-analytical form, furthermore they considered both directions of the exerted pressure gradient and provided criteria for diversification among the individual possible cases.

This contribution deals with axial flow through the concentric annulus. The inner cylinder is moving at a constant velocity along its axis and simultaneously axial pressure gradient is exerted to Vocadlo fluid. The aim is to provide a unique classification of all possible cases (including possible appearance of plug flow regions) and for each case to derive volumetric flow rate in a semi-analytical form.

As already mentioned above, geometrical and kinematical conditions correspond to those in Malik and Shenoy [16], geometrical and rheological to those in Gücüyener and Mehmetoğlu [13]. Unfortunately the present case is not possible to obtain (including power-law model) as a simple superposition due to nonlinearity of the whole problem.

The problem is formulated as follows

$$\frac{1}{r}\frac{d(r\tau_{rz})}{dr} = P\ ,\quad v_z(\kappa R) = V > 0,\quad v_z(R) = 0 \tag{26}$$

with the Vocadlo (Robertson-Stiff) model

$$\tau_{rz} = \left[K^{\frac{1}{n}} \left| \frac{dv_z}{dr} \right|^{\frac{n-1}{n}} + \tau_0^{\frac{1}{n}} \left| \frac{dv_z}{dr} \right|^{-\frac{1}{n}} \right]^n \left(-\frac{dv_z}{dr} \right) \text{ for } |\tau_{rz}| \geq \tau_0\ ,$$

$$\frac{dv_z}{dr} = 0 \quad \text{for } |\tau_{rz}| \leq \tau_0 \tag{27}$$

where K and n are consistency and flow behaviour indices, respectively; τ_0 represents yield stress.

For the sake of simplicity the following transformations converting the problem to the dimensionless form are used

$$\xi = \frac{r}{R}\ ,\quad \varphi = \frac{v_z}{V}\ ,\quad T = \frac{2\tau_{rz}}{|P|R}\ ,\quad T_0 = \frac{2\tau_0}{|P|R}\ ,$$
$$\Lambda = \frac{|P|R}{2K}\left(\frac{R}{V}\right)^n\ ,\quad Q = \frac{q}{2\pi R^2 V} \tag{28}$$

In the derivation of fully analytical solution it is necessary to take into account whether pressure gradient assists or opposes the drag on the fluid caused by the moving inner cylinder. Each of these two cases is possible to classify into three situations diversified with respect to the location of plug flow region. Moreover, each of these six situations is possible to determine a priori by the classification criteria derived in Filip and David [9].

As an example the following relation illustrates the relation between volumetric flow rate and pressure gradient for the case when pressure gradient opposes the drag on the fluid caused by the moving inner cylinder and plug flow region is formed within the annulus

($\kappa<\lambda_i<\lambda_o<1$; inner and outer λ_i, λ_o denote the dimensionless boundary values of the plug flow region)

$$Q = \frac{1}{2}\left(\frac{1-s}{3+s}\lambda^2-\kappa^2\right)+\left[\frac{\kappa^3-\lambda_i^3}{6}-\frac{1-s}{2(3+s)}\lambda^2(\kappa-\lambda_i)\right]\Lambda^s T_o^s - -\frac{\Lambda^s}{2(3+s)}\left[\lambda_i^{1-s}\left(\lambda^2-\lambda_i^2\right)^{1+s}-\kappa^{1-s}\left(\lambda^2-\kappa^2\right)^{1+s}\right] \quad (29)$$

where $s=1/n$. This relation is applicable for a description of back-extrusion technique

4 Conclusion

Poiseuille-Couette flows in concentric annuli are analysed for fluids obeying frequently used power-law and Vocadlo empirical models. It is shown that for the types of flows studied there exist analytical or at least quasisimilarity solutions enabling to determine the participation of the entry parameters (rheological, geometrical and kinematical) in the relation volumetric flow rate vs. pressure drop (vs. torque).

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