On a mathematical model in ice sheet dynamics

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Abstract: In this talk we consider a three-dimensional isothermal model for ice sheet dynamics in Glaciology. The model is derived from the Continuum Mechanics principles and well-known experimental results carried out in Glaciology. The final formulation of the model gives rise to a degenerate quasi-linear elliptic-parabolic equation for the ice-thickness function. Under appropriate initial and Dirichlet boundary conditions, we discuss the existence and uniqueness of weak solutions for this mathematical model. Then, we prove the localization properties of finite speed of propagations and waiting time for the ice-thickness function. To establish these properties we use here a suitable energy method.

Key–Words: ice sheet dynamics, existence, uniqueness, finite speed of propagations, waiting time.

1 Introduction

Ice sheets are vast and slow-moving edifices of solid ice, which are mainly concentrated in Antarctic and much smaller in Greenland. They flow under their own weight by solid state creep processes such as the creep of dislocation in the crystalline lattice structure of the ice. In this resemble rivers, expect they move more slowly and are consequently much thicker. Ice sheets have thickness of several kilometers and move at velocities of 10-100 meters per year. Despite their slow movement and apparent changelessness, ice sheets exhibits various interesting dynamic phenomena. In polar climate regions the snow accumulates on the uplands, is compressed into ice and flows out to cover the region under the action of gravity. Ice flows as highly viscous solids from the central parts, where the thickness is great, towards the margins. If the margins are near the coast, it can be formed floating ice shelves. The ice sheet equilibrium can be maintained through a balance between accumulation in the center and ablation at the margins. Accumulation occurs mainly through solid precipitation and ablation can occur either through evaporation or melting of the ice in the warmer climate at the margin, or through calving of icebergs.

The common Fluid Mechanics model adopted for cold ice is a non-Newtonian, viscous, heat-conducting, incompressible fluid. It should be pointed out that, strictly speaking, it is not possible to assume ice to be incompressible and yet still presume density variations under phase changes. It is, however, justified to ignore density variations since associated changes in bulk density are very small. On the other hand, it is worth to know that ice sheets are assumed to be isotropic materials, but they can develop an induced anisotropy when stressed over sufficiently long time scales. The model adopted for ice sheet flows result from the basic principles of Fluid Mechanics:

- the conservation of mass

$$\mathrm{div}\mathbf{u} = 0; \tag{1}$$

- the conservation of momentum

$$\mathbf{0} = \rho\,\mathbf{g} + \mathrm{div}\mathbf{T}. \tag{2}$$

Note that in (2) we have neglected the inertial terms because we are in the presence of very slow flows. Moreover, we have not written the equation for temperature, which results from the conservation of energy, because in the sequel we shall consider isothermal motions only. This brings some controversy to the model, because isothermal models are not quantitatively very realistic. However they are mathematically nice and it is not our aim to produce the most realistic model incorporating as much realism as possible. The notation used in (1)-(2) is well known: $\mathbf{u}$ is the velocity field, p is the pressure, ρ is the constant density, $\mathbf{g}$ is the gravitational force and $\mathbf{T}$ is the Cauchy stress tensor:

$$\mathbf{T} = -p\mathbf{I} + \mathbf{S}\,; \tag{3}$$

$\mathbf{I}$ is the unit tensor and $\mathbf{S}$ is the deviatoric part of $\mathbf{T}$. Notice that from (1), $\mathrm{tr}(\mathbf{S}) = 0$. Extra stress tensor

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$\mathbf{S}$ and strain rate tensor $\mathbf{D}$ are related by a rheological flow law. According to the common usage in Glaciology to write stretching as a function of stress, this law states that the strain rate $\mathbf{D}$, at a given strain, is proportional to the stress $\mathbf{S}$ raised to the power n:

$$\mathbf{D} = A(\theta)\mathrm{sgn}(\mathbf{S})|\mathbf{S}|^n, \quad \mathrm{sgn}(\mathbf{S}) = \frac{\mathbf{S}}{|\mathbf{S}|}. \tag{4}$$

This law was suggested by J.W. Glen and, for this reason, is called Glen's law in Glaciology. The basic postulate is that ice is an incompressible nonlinear viscous fluid. Here n is a positive constant and the function A may depend on the temperature and usually is postulated an Arrhenius-type relationship

$$A(\theta) = A_0 \exp\left(-\frac{Q}{k\theta}\right), \tag{5}$$

where Q is the so-called activation energy, k the Boltzman constant, θ the absolute temperature and A_0 a constant. The temperature-depending rate factor in (5) causes A to vary $\pm 3^\circ$ over a temperature range of 50° K. Concerning the exponent n, experimental results showed that it varies from about $1,9$ to $4,8$ in secondary creep (the strain rate is approximately constant) and reaches values as high as 10 in tertiary creep (the strain rate accelerates). In good approximation we can assume that in deforming ice masses like ice sheets, secondary creep prevails for low temperatures (below -10°C), whereas tertiary creep prevails for higher temperatures. Therefore there is general agreement now to use $n = 3$, although Glen concludes that $n = 3,5$ would be more appropriate. See Hutter [13] and Paterson [18] for a better understanding of these issues concerning theoretical glaciology.

2 Dynamics of ice sheets

A thorough analysis of ice sheets dynamics is made in many monographs, for instance, Hutter [13] and Paterson [18]. However, many authors deal only with 2D mathematical models, see *e.g.* Fowler [10]. Present-day 3D mathematical models including full thermomechanical coupling are those developed by Huybrechts [14], Greve [12] and Patyn [19], to name a few. The mathematical model approach is based on the continuum dynamics equations (1)-(4). We consider a Cartesian coordinate system (x, y, z) with the z-axis vertically pointing upward and being $z = 0$ at the mean sea level.

Field Equations. Denoting the velocity components in the correspondingly directions as (u, v, w), (1) can be rewritten as

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0. \tag{6}$$

Once that the gravitational force is only important in the vertical direction, *i.e.* considering $\mathbf{g} = (0, 0, -g)$, (2) becomes

$$\frac{\partial T_{xx}}{\partial x} + \frac{\partial T_{xy}}{\partial y} + \frac{\partial T_{xz}}{\partial z} = 0, \tag{7}$$

$$\frac{\partial T_{yx}}{\partial x} + \frac{\partial T_{yy}}{\partial y} + \frac{\partial T_{yz}}{\partial z} = 0, \tag{8}$$

$$\frac{\partial T_{zx}}{\partial x} + \frac{\partial T_{zy}}{\partial y} + \frac{\partial T_{zz}}{\partial z} = \rho\, g, \tag{9}$$

where T_{ij} means stress in the i-plane ($i = constant$) along j-direction.

Dynamic Boundary Condition. At the free surface, say $z = h(x, y, t)$, the model assumes that there is no applied traction, *i.e.*

$$\mathbf{T} \cdot \mathbf{n} = \mathbf{0} \quad \text{on} \quad z = h(x, y, t), \tag{10}$$

where $\mathbf{n}$ is the exterior unit normal to the ice sheet top surface $z = h(x, y, t)$. Since (10) is related with stresses, it is usually called the dynamic boundary condition of the free surface. If we write the free surface in the implicit form $z - h(x, y, t) = 0$, then $\mathbf{n} = |\nabla s|^{-1}\nabla s$, where $s(x, y, t) = z - h(x, y, t)$. It is a matter of practical evidence that everywhere in an ice sheet the slopes of the free surface $z = h(x, y, t)$ are small, except in a small neighborhood of ice domes and ice margins. Thus the normal unit vector of the free surface $z = h(x, y, t)$ is approximately vertical and (10) reduces to

$$T_{xz} = 0\,,\ T_{yz} = 0\,,\ T_{zz} = 0 \ \text{on}\ z = h(x, y, t)\,, \tag{11}$$

Hydrostatic Approximation. Applying the hydrostatic approximation in the vertical direction, *i.e.* $p_z = -\rho\, g$, then (9) reduces to

$$\frac{\partial T_{zz}}{\partial z} = \rho\, g\,. \tag{12}$$

This means that, in all parts of an ice sheet, the shear stresses T_{xz} and T_{yz} are small compared to the vertical normal stress T_{zz}. Therefore the variational stress in the z-plane can be neglected. On the other hand, if we neglect atmospheric pressure, an integration of (12) from the surface $h(x, y, t)$ to a height z in the ice body together with the usage of (11), gives us an expression for the vertical normal stress

$$T_{zz} = \rho\, g(z - h)\,. \tag{13}$$

From (3) and (13), the pressure p reads

$$p = \rho\, g\,(h - z) - S_{xx} - S_{yy} \tag{14}$$

and the horizontal normal stresses can be expressed as

$$T_{xx} = 2S_{xx} + S_{yy} - \rho\, g\, (h - z)\,, \tag{15}$$

$$T_{yy} = S_{xx} + 2S_{yy} - \rho\, g\, (h - z)\,. \tag{16}$$

Inserting (15) and (16) in the horizontal components (x, y) of (7) and (8), we achieve to

$$\frac{\partial}{\partial x}\,(2S_{xx} + S_{yy}) + \frac{\partial T_{xy}}{\partial y} + \frac{\partial T_{xz}}{\partial z} = \rho\, g \frac{\partial h}{\partial x}\,, \tag{17}$$

$$\frac{\partial}{\partial y}\,(S_{xx} + 2S_{yy}) + \frac{\partial T_{xy}}{\partial x} + \frac{\partial T_{yz}}{\partial z} = \rho\, g \frac{\partial h}{\partial y}\,. \tag{18}$$

Shallow-Ice Approximation. The major simplification of the model ensues by considering the shallow-ice approximation. This is justified, since we assume a physical process in which important length scales in the longitudinal directions are much larger, compared to those in the transverse directions. For instance, present-day Antarctic ice sheets has a thickness of 1 Km whilst its lateral extent is typically on the order of 1000 Km. Consistently, $x\,,\ y \gg z$, and also $u\,,\ v \gg w$, and thus the dominant stresses are the shear stresses in the horizontal plane, S_{xz} and S_{yz}, which are supported by the basal drag. Moreover, normal stresses S_{xx}, S_{yy}, S_{zz} are negligible , as well the shear stress S_{xy} in the vertical planes. In consequence,

$$T_{xx} = T_{yy} = T_{zz} = -p\,. \tag{19}$$

and, from (10)-(13), the pressure is close to hydrostatic

$$p = \rho\, g(h - z)\,. \tag{20}$$

Then the horizontal components of (17)-(18) simplify to

$$\frac{\partial T_{xz}}{\partial z} = \rho\, g \frac{\partial h}{\partial x}, \tag{21}$$

$$\frac{\partial T_{yz}}{\partial z} = \rho\, g \frac{\partial h}{\partial y}\,. \tag{22}$$

On the free surface $z = h(x, y, t)$ we obtain, after using (11) and (19),

$$T_{xz} = 0\,,\ \ T_{yz} = 0\,,\ \ p = 0 \ \text{ on } \ z = h(x, y, t)\,, \tag{23}$$

Then a vertical integration of (21)-(22) from $h(x, y, t)$ to a height z in the ice body, and the usage of (23), leads us to

$$T_{xz} = -\rho\, g(h - z)\frac{\partial h}{\partial x}, \tag{24}$$

$$T_{yz} = -\rho\, g(h - z)\frac{\partial h}{\partial y}\,. \tag{25}$$

From (4), strain rates are related with deviatoric stresses by

$$\mathbf{D} = A(\theta)\tau^{n-1}\mathbf{S}\,, \quad \tau = \sqrt{II_{\mathbf{S}}}\,, \tag{26}$$

where $II_{\mathbf{S}}$ denotes the second invariant of $\mathbf{S}$. Notice that (1) implies $\tau = \sqrt{\frac{1}{2}\mathrm{tr}(\mathbf{S}^2)}$ and from the simplifications of the shallow ice approximation, especially (24)-(25),

$$\tau = \sqrt{T_{xz}^2 + T_{yz}^2} = \rho\, g(h - z)|\nabla h|\,. \tag{27}$$

A common assumption in ice sheet modeling, and which is valid for most of the ice sheet domain, is that horizontal gradients of the vertical velocity are small compared to the vertical gradient of the horizontal velocity, *i.e.* $w_x \ll u_z$ and $w_y \ll v_z$. Using this assumption, (24)-(25) and (27), we obtain from (26)

$$\frac{\partial u}{\partial z} = -2A(\theta)\,[\rho\, g(h - z)]^n\, |\nabla h|^{n-1} \frac{\partial h}{\partial x}\,, \tag{28}$$

$$\frac{\partial v}{\partial z} = -2A(\theta)\,[\rho\, g(h - z)]^n\, |\nabla h|^{n-1} \frac{\partial h}{\partial y}\,. \tag{29}$$

Integrating (28) and (29) from the ice base, say $z = b(x, y, t)$, to an arbitrary point z in the ice sheet, we obtain

$$u = u_b - 2(\rho\, g)^n |\nabla h|^{n-1} \frac{\partial h}{\partial x} \int_b^z A(\theta)(h - s)^n ds\,, \tag{30}$$

$$v = v_b - 2(\rho\, g)^n |\nabla h|^{n-1} \frac{\partial h}{\partial y} \int_b^z A(\theta)(h - s)^n ds\,, \tag{31}$$

where $\mathbf{u}_b = (u_b, v_b)$ is the ice velocity at the ice base. This term is usually called the basal sliding velocity and results from assuming the ice sheet slides, with velocity $\mathbf{u}_b$, over its base. This happens when basal ice reaches the melting point and consequently basal melt water is produced. This water can lubricate the bed sufficiently that the ice slides over the bed. But, once the base reaches the melting point, we assume the ice above remains cold.

Kinematic Boundary Conditions. Now, we shall derive boundary conditions at the free surface $z = h(x, y, t)$ and at the ice base $z = b(x, y, t)$. The possible presence of attached ice shelves will be ignored. If we write the free surface in the implicit form $s(x, y, t) = 0$, with $s(x, y, t) = z - h(x, y, t)$, then its exterior unit normal is given by $\mathbf{n} = |\nabla s|^{-1}\nabla s$. Let $\mathbf{u}$ and $\mathbf{w}$ denote, respectively, the ice surface velocity and the velocity at which the free surface points move. Then $\mathbf{w} \cdot \mathbf{n}$ represents the normal speed of propagation of the free surface and

$$a_h = (\mathbf{w} - \mathbf{u}) \cdot \mathbf{n} \tag{32}$$

is the ice volume flux through the free surface, also known as the accumulation/ablation function. The sign is chosen such that a supply (accumulation) is counted as positive and a loss (ablation) as negative. Then the time derivative of $s(x, y, t)$ following the motion of the free surface with velocity $\mathbf{w}$ must vanish and, by using (32), we obtain

$$\frac{\partial h}{\partial t} + u\frac{\partial h}{\partial x} + v\frac{\partial h}{\partial y} - w = a_h N_h\,, \tag{33}$$

where $N_h = \sqrt{h_x^2 + h_y^2 + 1}$. A similar boundary condition can be derived for the ice base. We proceed as above, considering the implicit form of the ice base $r(x,y,t) = 0$ ($r(x,y,t) = b(x,y,t) - z$), its exterior unit normal given by $\mathbf{n} = |\nabla r|^{-1}\nabla r$ and the ice volume flux through the ice base is given by

$$a_b = (\mathbf{w} - \mathbf{u}) \cdot \mathbf{n}\,. \tag{34}$$

Now $\mathbf{w}$ is the velocity at which the ice base points move and $\mathbf{w}\cdot\mathbf{n}$ represents the normal speed of propagation of the ice base. Arguing as before, we obtain

$$\frac{\partial b}{\partial t} + u\frac{\partial b}{\partial x} + v\frac{\partial b}{\partial y} - w = a_b N_b\,, \tag{35}$$

where $N_b = \sqrt{b_x^2 + b_y^2 + 1}$. In both cases, free surface and ice base, their interior sides are identified with the ice and therefore the exterior sides are identified with the atmosphere and the lithosphere, respectively. Since (33) and (35) have been derived by geometrical considerations only, they are called kinematic boundary conditions. Provided that accumulation/ablation functions (32) and (34) are given, equations (33) and (35) govern the evolution of the free surface and ice base, respectively.

Ice-Thickness Equation. Using kinematic boundary conditions (33) and (35) and the conservation of mass equation (6), we can derive an evolution equation which expresses the change of ice thickness, say $H(x,y,t) = h(x,y,t) - b(x,y,t)$. We integrate (6) along the vertical from the ice base $z = b(x,y,t)$ to the free surface $z = h(x,y,t)$ and we use (33) and (35) to obtain

$$\frac{\partial}{\partial x}\int_b^h u\,dz + \frac{\partial}{\partial y}\int_b^h v\,dz + \frac{\partial h}{\partial t} - N_h a_h - \frac{\partial b}{\partial t} + N_b a_b = 0\,. \tag{36}$$

Replacing, in (36), u and v by its expressions (30) and (31), we obtain, after an integration by parts, the following evolution equation for the ice sheet thickness

$$\frac{\partial H}{\partial t} + \mathbf{u}_b \cdot \nabla H = \operatorname{div}\left(\int_b^h \mathcal{A}(\theta)(h-z)^{n+1}dz|\nabla h|^{n-1}\nabla h\right) + a\,, \tag{37}$$

where $\mathbf{u}_b = (u_b, v_b)$ is the sliding velocity, $\mathcal{A}(\theta) = 2(\rho g)^n A(\theta)$ and $a = a_h - a_b$ is the accumulation/ablation rate. We already have seen that everywhere in an ice sheet the slopes of the free surface $z = h(x,y,t)$ are small. The same happens with the slopes of the ice base $z = b(x,y,t)$. Then the exterior normal vectors to $z = h(x,y,t)$ and to $z = b(x,y,t)$ are approximately vertical and this justifies why we have taken $N_h = N_b = 1$ in (37).

3 Statement of the problem

In this section we introduce the mathematical problem we shall work with and define the notion of solutions we are interested. As we already have mentioned in Section 1, we shall consider the isothermal case which causes in (37) that $\mathcal{A}$ does not depend on θ anymore. This can be a consequence of approximately zero changes of temperature in the ice sheet, or more generally if in the Arrhenius relationship (5) $|-Q/(k\theta)| \ll 1$ and $A_0 = 1$. Another simplification of the model, results from an usual assumption in ice sheet modeling, the base $b(x,y,t)$ is a horizontal flat surface, *i.e.* $b = constant$. Under these assumptions, and after an integration procedure (see [10]), (37) comes

$$\frac{\partial H}{\partial t} + \mathbf{u}_b \cdot \nabla H = \operatorname{div}\left(\frac{H^{n+2}}{n+2}|\nabla H|^{n-1}\nabla H\right) + a\,. \tag{38}$$

A different mathematical model was considered by the authors in [3, 4]. There, it was used the arguing of Fowler [11] to justify the replacement of the sliding velocity $\mathbf{u}_b$ by $-\nabla H$. Mathematically, (38) is a nonlinear diffusion equation for the ice-thickness, with the additional convective term $\mathbf{u}_b \cdot \nabla H$, and which degenerates for $n > 1$ at points where $\nabla H = \mathbf{0}$ (see Díaz [8]). On the other hand, it should be pointed out that, from the considerations we have made in the previous section, the ice-thickness must be non-negative.

Strong formulation. When formulating mathematical models for the study of ice sheets, usually it is necessary to take into account that the flow domain is not prescribed and is itself part of the solution (see Calvo *et al.* [9] and Rodrigues and Santos [20]). However, once in this work we are mainly interested with local properties of the ice sheet thickness, we may assume that the ice sheet based domain is known. We assume the ice sheet occupies a sufficiently large area where there can possibly occur the vanishing of the ice-thickness in some relatively small subareas. In the boundary of this large area we assume the ice-thickness vanishes. Let us then consider the cylinder

$$Q_T := \Omega \times (0,T) \subset \mathbb{R}^2 \times \mathbb{R}^+$$

whose boundary is defined by $\Gamma_T := \partial\Omega \times (0,T)$ and where Ω is assumed to be a large enough open bounded domain with a sufficiently smooth boundary $\partial\Omega$. Then the strong formulation of the problem can be stated in the following terms. Given an accumulatoin/ablation rate function $a = a(x, y, t)$ and a sliding velocity $\mathbf{u}_b = \mathbf{u}_b(x, y)$ defined in Q_T, and an initial ice-thickness $H_0 = H_0(x, y) \geq 0$, bounded and compactly supported in Ω, to find a sufficiently smooth function $H = H(x, y, t)$ defined on Q_T such that (38) is fulfilled in Q_T,

$$H = H_0 \quad \text{in} \quad \Omega \quad \text{for} \quad t = 0\,, \tag{39}$$

$$H = 0 \quad \text{on} \quad \Gamma_T\,. \tag{40}$$

The mathematical (strong) solutions of (38)-(40) must be physically admissible, *i.e.* they have to be non-negative compactly supported solutions.

General formulation. In order to obtain a more general framework than (38)-(40), let us introduce the new functions $\nu = \nu(x, y, t)$ and $b = b(s)$ defined by

$$\nu := H^m = \psi(H) \Longrightarrow \psi^{-1}(\nu) = \nu^{\frac{1}{m}} := b(\nu)\,, \tag{41}$$

where $m = 2(n + 1)/n$. Notice that the new variable $\nu := H^m$ is motivated by the relation

$$\frac{H^{n+2}}{n+2} |\nabla H|^{n-1} \nabla H = \frac{m^{1-p}}{n+2} |\nabla H^m|^{p-2} \nabla H^m\,,$$

with $p = n + 1$. Let us assume that:

$$a \in \mathrm{L}^\infty(\Omega)\,; \tag{42}$$

$$\operatorname{div} \mathbf{u}_b = 0 \quad \text{in} \quad Q_T\,; \qquad \mathbf{u}_b \in \mathbf{L}^\infty(Q_T)\,; \tag{43}$$

$$\nu_0 \in \mathrm{L}^\infty(\Omega). \tag{44}$$

Notice that, according to (41), condition (44) is equivalent to assume that $H_0 \in \mathrm{L}^\infty(\Omega)$. Then the general formulation of (38)-(40) can be stated in terms of ν and b as follows. Given Ω, a constant $k = m^{1-p}/(n + 2)$ and a, $\mathbf{u}_b$ and H_0 satisfying (42)-(44), to find a function ν defined by (41) and solution of

$$\frac{\partial\, b(\nu)}{\partial\, t} = \operatorname{div}\left(k|\nabla\nu|^{p-2}\nabla\nu - \mathbf{u}_b b(\nu)\right) + a\,, \tag{45}$$

$$b(\nu) = b(\nu_0) \quad \text{in} \quad \Omega \quad \text{for} \quad t = 0\,, \tag{46}$$

$$\nu = 0 \quad \text{on} \quad \Gamma_T\,. \tag{47}$$

It is worth to notice that, according to (41), ν and H have the same support and have the same value on the boundary Γ_T. Moreover, if H is a solution of (38)-(40) then ν is a solution of (45)-(47) and reciprocally. The general formulation (45)-(47) is the one used to establish existence and uniqueness of solutions (see Calvo *et al.* [9]) and goes back to mathematical works on quasi-linear elliptic-parabolic differential equations (see Alt and Luckhaus [1], Otto [17], Benilan and Wittbold [5], Carrillo and Wittbold [7], Ivanov and Rodrigues [15]), being our problem a particular case.

4 Weak formulation

We start this section by introducing the notion of solutions to the problem (45)-(47) we shall work with in the sequel. We multiply (45) by a test function ζ and integrate by parts over Q_T to obtain

$$\begin{aligned} \int_{Q_T} \left(b(\nu)\frac{\partial\zeta}{\partial t} + a\zeta \right) d\mathbf{z} \; + \int_\Omega b(\nu_0)\zeta_0\, d\mathbf{x} = \\ \int_{Q_T} (k|\nabla\nu|^{p-2}\nabla\nu \; - b(\nu)\mathbf{u}_b)\cdot\nabla\zeta\, d\mathbf{z}\,, \end{aligned} \tag{48}$$

where $\zeta_0 = \zeta(\cdot, 0)$ and where we have set $\mathbf{x} = (x, y)$ and $\mathbf{z} = (x, y, t)$. Then the definition of weak solution follows as usual (see Alt and Luckhaus [1]).

Definition 1 . *Let (42)-(44) be fulfilled. A function* ν *is a weak solution of the problem (45)-(47), if:*
1. $\nu \geq 0$ *a.e. in* Q_T *and* $\nu \in \mathrm{L}^p(0, T; \mathrm{W}_0^{1,p}(\Omega))$*;*
2. $b(\nu) \in \mathrm{L}^\infty(0, T; \mathrm{L}^1(\Omega))$ *and* $b(\nu)_t \in \mathrm{L}^{p'}(0, T;\mathrm{W}^{1,-p}(\Omega))$*;*
3. The relation (48) holds for every $\zeta \in \mathrm{L}^p(0, T; \mathrm{W}_0^{1,p}(\Omega)) \cap \mathrm{W}^{1,1}(0, T; \mathrm{L}^\infty(\Omega))$*, such that* $\zeta(\cdot, T) = 0$.

There are now many existence and uniqueness results which can be applied directly to the problem (45)-(47) (Alt and Luckhaus [1], Otto [17], Benilan and Wittbold [5], Ivanov and Rdrigues [15], Carrillo and Wittbold [7], to name a few). One of the first references to appear was the paper by Alt and Luckhaus [1], where is proved (Theorem 1.7) the existence of a weak solution to a general problem which includes the case of Definition 1. The existence result there is proved for any

$$u_0 = b(\nu_0) \quad \text{with} \quad B(\nu_0) \in \mathrm{L}^1(\Omega)$$

(see (52) bellow for the definition of B) and

$$a \in \mathrm{L}^{p'}(0, T; \mathrm{W}^{-1,p'}(\Omega)).$$

In order to apply Alt and Luckhaus [1, Theorem 1.7] to the problem (45)-(47), let us define the following functions

$$b : \mathbb{R} \to \mathbb{R}\,, \quad b(u) = u^{\frac{1}{m}}\,, \tag{49}$$

$$a : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}^2\,,\; a(\mathbf{v}, u) = k|\mathbf{v}|^{p-2}\mathbf{v} - u\mathbf{u}_b\,, \tag{50}$$

where m, k and $p = n + 1$ are constants and $\mathbf{u}_b$ is a given vector - the sliding velocity at the ice base. One can easily see that (49) is a nondecreasing continuous function in $\mathbb{R}$ such that $b(0) = 0$ and (50) is a vector-valued continuous function in $\mathbb{R}^2 \times \mathbb{R}$ such that the growth condition

$$|a(\nabla\nu, b(\nu))|^{p'} \leq C_1 \left(1 + |\nabla\nu|^p + B(b(\nu))\right)\,, \tag{51}$$

hold. In (51), $C_1 = const. \geq 0$ and $B(b(\nu))$ is the Legendre transform of the primitive of $b(\nu)$

$$B(b(\nu)) := \int_0^\nu s\, d\, b(s)\,. \tag{52}$$

It should be noticed that B is super-linear in the sense that for any $\delta > 0$, there exists a $C(\delta) < \infty$, such that for all $u \in \mathbb{R}$, $|u| \leq \delta\, B(u) + C(\delta)$. From this property of B it is a easy task to prove (51). The proof of the strict monotonicity condition

$$(a(\mathbf{v}, u) - a(\mathbf{w}, u)) \cdot (\mathbf{v}, \mathbf{w}) \geq C_2\, |\mathbf{v} - \mathbf{w}|^p\ , \tag{53}$$

$C_2 = const. > 0$, is more involved. In fact, after some algebraic manipulations, we can prove successively

$$k^{-1} a(\mathbf{v}, u) - a(\mathbf{w}, u)) \cdot (\mathbf{v}, \mathbf{w}) =$$
$$|\mathbf{v}|^p + |\mathbf{w}|^p - \left(|\mathbf{v}|^{p-2} - |\mathbf{w}|^{p-2}\right) \mathbf{v} \cdot \mathbf{w} =$$
$$\left(|\mathbf{v}|^{p-2} + |\mathbf{w}|^{p-2}\right) |\mathbf{v} - \mathbf{w}|^2 + |\mathbf{v}|^{p-1} |\mathbf{w}| + |\mathbf{v}||\mathbf{w}|^{p-1} \geq$$
$$C|\mathbf{v} - \mathbf{w}|^p\,, \quad C = C(p)\,, \quad p \geq 2\,.$$

For our purposes, it is enough to consider $p \geq 2$, because $p = n + 1$ and, in Glaciology, it is usual to take $n = 3$. An extension of the result presented in Alt and Luckhaus [1] to the case $1 < p < 2$ is given by Ivanov and Rodrigues [15]. Moreover, Alt and Luckhaus [1] have shown that the natural energy associated to a weak solution ν of the problem (45)-(47) is given by finite sum

$$\sup_{t \in (0,T)} \int_\Omega B(b(\nu(\cdot, t)))\, d\mathbf{x} + \int_{Q_T} |\nabla \nu|^p d\mathbf{z} < \infty\,, \tag{54}$$

where $B(b(\nu(\cdot, t)))$ is defined in (52). Benilan and Wittbold [5] under rather general assumptions than Alt and Luckhaus [1], and using the nonlinear semigroup theory, have proved the existence of mild solutions, which under certain conditions were shown to be weak solutions. Uniqueness of weak solutions of (45)-(47) is a much more difficult task because of the nonlinear term $b(\nu)$. The usual approach consists in to prove the L^1-contraction principle

$$\int_\Omega |b(\nu_1(\cdot, t) - b(\nu_2(\cdot, t)|\, d\mathbf{x} \leq e^{Lt} \int_\Omega |b_0(\nu_1) - b_0(\nu_2)|\, d\mathbf{x} \tag{55}$$

for any two weak solutions ν_1 and ν_2 satisfying (54) - L is the Lipschitz constant of $a(\cdot, \cdot)$. Under the additional continuity property

$$|a(\mathbf{v}, u) - a(\mathbf{v}, z)|^{p'} \leq C(1 + B(u) + B(z) + |\mathbf{v}|^p)|u - z|\,, \tag{56}$$

Alt and Luckhaus [1, Theorem 2.3] also have proved the uniqueness of a weak solution ν provided

$$\frac{\partial \nu}{\partial t} \in \mathrm{L}^1(Q_T)\,. \tag{57}$$

It is a easy task to prove that (50) satisfies (56). Latter, Otto [17], by using Kruzhkov method of doubling variables both in space and time, have proved (55), and consequently the uniqueness result, for ν_i, $i = 1,\ 2$, satisfying (54) without assuming (57). Carrillo and Wittbold [7] have generalized the uniqueness result of Otto [17] and have proved a comparison result by using also Kruzhkov method.

5 Localization properties

In this section we shall establish the localization properties of finite speed of propagations and waiting time for the solutions H to the problem (38)-(40). Existence and uniqueness of a weak solution $\nu = H^m$ to the equivalent problem (45)-(47) have been established in the previous section. According to (41), that results allow us to state the existence and uniqueness of a weak solution H for (38)-(40) and such that, for every $\zeta \in \mathrm{L}^{n+1}(0, T; \mathrm{W}_0^{1,n+1}(\Omega)) \cap \mathrm{W}^{1,1}(0, T; \mathrm{L}^\infty(\Omega))$, with $\zeta(\cdot, T) = 0$, the equivalent of (48) holds:

$$\int_{Q_T} \left(H \frac{\partial \zeta}{\partial t} + a\zeta\right) d\mathbf{z} + \int_\Omega H_0 \zeta_0\, d\mathbf{x} =$$
$$\int_{Q_T} \left(\frac{H^{n+2}}{n+2} |\nabla H|^{n-1} \nabla H - H\mathbf{u}_b\right) \cdot \nabla \zeta\, d\mathbf{z}\,.$$

We define the energy associated with the problem (38)-(40) by

$$E(Q_T) := \sup_{t \in [0,T]} \int_\Omega |H(\cdot, t)|^2 d\mathbf{x} + \int_{Q_T} |H|^{n+2} |\nabla H|^{n+1} d\mathbf{z}\,, \tag{58}$$

which, by the same reasoning used to obtain (54), can be proved to be finite. In order to define the notions of the properties we want to establish, let us fix $\mathbf{x}_0$ in Ω and assume that

$$H_0(\mathbf{x}) = 0 \quad \text{for } \mathbf{x} \in B_{\rho_0}(\mathbf{x}_0) = \{\mathbf{x} \in \Omega : |\mathbf{x} - \mathbf{x}_0| < \rho_0\} \subset \Omega\,, \tag{59}$$

where $\rho_0 \in (0, \operatorname{dist}(\mathbf{x}_0, \partial\Omega))$.

Definition 2 *The weak solutions of problem (38)-(40) possess the property of:*
1. finite speed of propagation, if for some $\mathbf{x}_0 \in \Omega$ and $t^* \in (0, T)$

$$H(\mathbf{x}, t) = 0 \quad \textit{a.e. in } B_{\rho(t)}(\mathbf{x}_0) \quad \forall\, t \in [0, t^*]\,;$$

2. waiting time, if for some $\mathbf{x}_0 \in \Omega$ and $t^* \in (0, T)$

$$H(\mathbf{x}, t) = 0 \quad \textit{a.e. in } B_{\rho_0}(\mathbf{x}_0) \quad \forall\, t \in [0, t^*]\,.$$

In this section we shall assume that

$$\operatorname{div} \mathbf{u}_b = 0 \text{ in } Q_T\,, \quad \mathbf{u}_b \in \mathrm{C}^1(0,T;\mathbf{C}^\alpha(\Omega))\,, \tag{60}$$

with $0 < \alpha < 1$. To proceed our study, let us consider the Lagrange variables $\mathbf{X}$ defined as usual in Continuum Mechanics (see, *e.g.*, Meirmanov *et al.* [16]):

$$\frac{d\mathbf{X}(\mathbf{x},t)}{dt} = \mathbf{u}_b(\mathbf{X},t), \quad t \in (0,T)\,; \tag{61}$$

$$\mathbf{X}(\mathbf{x},0) = \mathbf{x}, \quad \mathbf{x} \in \Omega. \tag{62}$$

Under conditions expressed in (60), there exists a unique solution $\mathbf{X}(\mathbf{x},t)$ of the problem (61)-(62), which is a homeomorphism between Ω and

$$\Omega^t = \{\mathbf{y} : \mathbf{z} = \mathbf{X}(\mathbf{x},t), \quad \mathbf{x} \in \Omega\}$$

for any $t \in [0,T]$. This solution transforms the ball $B_\rho(\mathbf{x}_0)$ into

$$B^t_\rho(\mathbf{x}_0) = \{\mathbf{z} : \mathbf{y} = \mathbf{X}(\mathbf{x},t), \text{for some } \mathbf{x} \in B_\rho(\mathbf{x}_0)\}.$$

Moreover, the following formula hold

$$\frac{d}{dt}\int_{B^t_\rho(\mathbf{x}_0)} \Phi\, d\mathbf{z} = \int_{B^t_\rho(\mathbf{x}_0)} \left(\frac{\partial \Phi}{\partial t} + \mathbf{u}_b \nabla \Phi\right) d\mathbf{z}\,, \tag{63}$$

$$\begin{aligned} \frac{dJ}{dt} &= J \operatorname{div} \mathbf{u}_b, \quad J = \det\left(\frac{\partial \mathbf{X}}{\partial \mathbf{x}}\right), \\ J(\mathbf{x},0) &= \det\left(\frac{\partial \mathbf{X}(\mathbf{x},0)}{\partial \mathbf{x}}\right) = 1. \end{aligned} \tag{64}$$

In the considered case, we have that $J(\mathbf{x},0) = J(\mathbf{x},t) = 1$. We introduce the energy functions

$$\begin{aligned} E(\rho,t) &:= \int_0^t \int_{B^t_\rho(\mathbf{x}_0)} \frac{|H|^{n+2}}{n+2} |\nabla H|^{n+1} d\mathbf{z} dt, \\ B(\rho,t) &= \int_{B^t_\rho(\mathbf{x}_0)} |H|^2 d\mathbf{z}\,. \end{aligned} \tag{65}$$

Notice that

$$\frac{\partial E(\rho,t)}{\partial \rho} = \int_0^t \int_{S^t_\rho(\mathbf{x}_0)} \frac{H^{n+3}}{n+2} |\nabla H|^{n-1} \nabla H \cdot \mathbf{n} dS\,, \tag{66}$$

where $S^t_\rho(\mathbf{x}_0)$ is the boundary of $B^t_\rho(\mathbf{x}_0)$, *i.e.* $S^t_\rho(\mathbf{x}_0) = \partial B^t_\rho(\mathbf{x}_0)$ and $\mathbf{n}$ is the unit exterior normal to $S^t_\rho(\mathbf{x}_0)$ Then, applying the results of Antontsev *et al.* [2, Chapter 3], we can prove the following theorem.

Theorem 3 *Let H be a non-negative weak solution to the problem (38)-(40). Assume $\mathbf{u}_b$ satisfies (60) and (58) is finite.*

1. *If (59) is verified, then there exists t^*, $0 < t^* < T$, such that*

   $$H(\mathbf{x},t) = 0 \quad \textit{a.e. in } B_{\rho(t)}(\mathbf{x}_0), \quad \forall\, t \in [0,t^*],$$

   with $\rho(t)$ given by

   $$\rho^\nu(t) = \rho_0^\nu - \frac{\nu}{\gamma C} t^\lambda E^\gamma(\rho_0,0),$$

   for some positive constants ν, λ and γ.

2. *If additionally to (59), the following condition holds*

   $$\int_{B_\rho(\mathbf{x}_0)} |H_0|^2 d\mathbf{x} \le D(\rho - \rho_0)^\mu\,,$$

   for some $\rho > \rho_0$, $\mu = \mu(n) > 0$, $D > 0$. then, there exist t^*, $0 < t^* < T$, and $D^* > 0$, $0 < D \le D^*$, such that

   $$H(\mathbf{x},t) = 0 \quad \textit{a.e. in } B_{\rho_0}(\mathbf{x}_0), \quad \forall\, t \in [0,t^*]\,.$$

PROOF. We formally multiply (38) by H, a weak solution of (38)-(40) and integrate by parts over $B^t_\rho(\mathbf{x}_0) \times (0,s)$, with $s \le t \le T$. To be precise, we should multiply (38) by a regularized H function, with compact support in Ω, and then pass to the limit in the obtained integral equation. Using $(60)_1$ and the notations introduced in (65)-(66), we obtain the following energy relation

$$\frac{1}{2}B(\rho,t) + E(\rho,t) = \frac{1}{2}B(\rho,0) + \frac{\partial E(\rho,t)}{\partial \rho}, \tag{67}$$

We notice that $B(\rho,0) = 0$ if $\rho \le \rho_0$, which corresponds to the first assertion. In this case, taking the supreme, for $s \in [0,t]$, of (67), the results of Antontsev *et al.* [2, §3.2] lead us to the ordinary differential inequality

$$\left(\sup_{0 \le s \le t} B(\rho,s) + E(\rho,t)\right)^\gamma \le C t^\lambda \rho^{1-\nu} \left(\frac{\partial E(\rho,t)}{\partial \rho}\right),$$

where $0 < \gamma < 1$, and λ, $\nu > 0$. Integrating the last inequality, we come to the estimate

$$E^\gamma(\rho,t) \le E^\gamma(\rho_0,0) - \frac{\gamma}{\nu} C t^{-\lambda} \left(\rho_0^{1-\nu} - \rho^{1-\nu}\right),$$

which lead us to

$$E(\rho,t) = 0, \quad \text{if} \quad \rho^{1-\nu} \le \rho_0^{1-\nu} - \frac{\nu}{\gamma C} t^\lambda E^\gamma(\rho_0,0).$$

First assertion of the theorem is thus proved. In the second case, we come to the nonhomogeneous inequality with $\rho \ge \rho_0$

$$\left(\sup_{0 \le s \le t} B(\rho,s) + E(\rho,t)\right)^\gamma \le C\left[t^\lambda \left(\frac{\partial\, E(\rho,t)}{\partial \rho}\right) + D^\gamma (\rho - \rho_0)^{\gamma\mu}\right],$$

where $\mu \geq 1/(1-\gamma)$. According to Antontsev *et al.* [2, §3.3], all solutions of the last inequality permit the majority

$$E(\rho, t) \leq C^{\gamma}(\rho - \rho_0), \quad \rho \geq \rho_0$$

if $D > 0$ and $t > 0$ are sufficiently small. Second assertion of the theorem is proved. □

The results of Theorem 3 are still valid for a global non-zero accumulation/ablation rate. Indeed, finite speed of propagations property holds, provided we assume $a = 0$ in $B_{\rho_0}(\mathbf{x}_0) \times [0, t^*]$. As for the waiting time property, it holds if we assume $a = 0$ in $B_{\rho_0}(\mathbf{x}_0) \times [0, T]$.

A detailed paper including the results established in this text will be published elsewhere as soon as possible.

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