Approximate Solutions of Heat Conduction Problems in Multi-dimensional Cylinder Type Domain by Conservative Averaging Method, Part 1

MARGARITA BUIKE, ANDRIS BUIKIS
Institute of Mathematics and Computer Science
University of Latvia
Raina bulv. 29., Riga, LV1459
LATVIA
http://www.lza.lv/scientists/buikis.htm

Abstract: - In this paper the description of the simplest variant of conservative averaging method for partial differential (or integro-differential) equations in cylinder type domain is given. Different types of boundary conditions (both linear and non-linear) are considered. As application of the method the process of intensive steel quenching as the time inverse ill-posed problem for the hyperbolic heat conduction for is studied. The conservative averaging method leads to the inverse well-posed problem. This problem is solved in closed form.

Key-Words: - partial differential equations, conservative averaging, various boundary conditions, hyperbolic heat conduction, intensive steel quenching, inverse problem, well-posed.

1 Introduction

Mathematical models of various physical processes very often contain partial differential equations (PDE) with piecewise constant coefficients. If some of coefficients are relatively large and/or some of sub-domains are relatively thin in literature (e.g. [1], [2]) various types of averaging procedure based of some particular properties of the concrete problem are applied. Conservative averaging method was developed by A. Buikis in his doctoral thesis [3] (in Russian language) and in several papers [4] - [7] etc. as unified analytical (or analytically-numerical) approach for PDE with discontinuous coefficients independent of peculiarities of concrete problem. The conservative averaging method is substantial extension and generalization of such non-classical conditions as the concentrated heat capacity [1], [7] and generalized non-ideal contact condition [1]. Geometrically-physically the class of problems for which the conservative averaging method was adjusted could be described as mathematical models of various physical processes in layered media. In this paper we extend the method of conservative averaging for PDE (or integro-differential equations) with continuous coefficients in cylindrical domain.

2 Conservative Averaging Method in Cartesian coordinates for Cylindrical Domain in R^{n+1}

We will start with the statement of problem for differential (or integro-differential) equation with continuous coefficients. We consider solutions in *classical* sense, i.e. all the highest derivatives of the considered differential equation are continuous.

2.1 Original Problem

Let us look at domain D (see fig.1), where $D = \left\{(x, y) : x \in [0, H] \times G \subset R^{n+1}\right\}$. Here the basis G of the cylinder D is bounded (or unbounded) domain $y = (y_1, ..., y_n) \in G \subset R^n$.

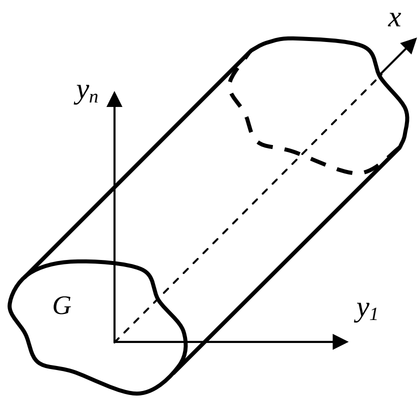

Fig.1

It is important to underline immediately that one of vector argument y components could be time variable t . It means the method of conservative averaging is applicable side by side to steady-state and transient processes.

The main equation in general form looks as follows:

$$\frac{\partial}{\partial x}\left(k\frac{\partial U}{\partial x}\right)+L(U)=-F(x,y),\ (x,y)\in D. \quad (1)$$

Here the operator $L(\bullet)$ is *linear* differential (either integral or integro-differential) operator related to vector-argument y with coefficients which could depend on variable y, but not on argument x.

On the basis of cylindrical domain (on G) we propose the boundary condition (BC) in general form:

$$l_0(U)=\Phi_0(y)+L_0(U).$$

Explicitly:

$$-\nu_0 k\frac{\partial U}{\partial x}+\lambda_0 U=\Phi_0(y)+L_0(U). \quad (2)$$

The operator $L_0(U)$ in BC (2) has properties identical with operator $L(U)$; such operator can arise by applying the conservative averaging method to similar with (1) PDE on thin cylinder $G_0=G\times\{x\in[-\delta,0]\}$. After the conservative averaging procedure, the thin cylinder G_0 disappears from original statement of the problem and we obtain the problem for PDE (1) with non-typical BC (2), see, e.g. [6].

Further we consider mainly the BC without operator L_0. The coefficients $\lambda_0=1,\ \nu_0=0$ give the first type BC:

$$U(0,y)=\Phi_0(y). \quad (2_1)$$

The second type BC we obtain by $\lambda_0=0,\nu_0=1$:

$$-k\frac{\partial U}{\partial x}=\Phi_0(y). \quad (2_2)$$

Finally, the coefficients $\lambda_0=h,\nu_0=1$ give the third type BC, where the function $\Theta_0(y)$ represents physically the environment temperature:

$$-k\frac{\partial U}{\partial x}+h_0U=\Phi_0(y),\Phi_0(y)=h_0\Theta_0(y). \quad (2_3)$$

The conservative averaging method is also applicable to non-linear BC:

$$-k\frac{\partial U}{\partial x}+\beta\left[U-\Theta(t)\right]^m=0.$$

(2_4)

If here the power m is equal to 10/3 it is the case of the so called nucleate boiling process on the surface of material.

Another interesting and famous case of BC is the heat transfer by radiation, where ε is the emissivity of the surface, $\sigma-$ the Stefan-Boltzmann constant:

$$-k\frac{\partial U}{\partial x}+\varepsilon\sigma\left[U^4-\Theta^4(y)\right]=0.$$

(2_5)

On the second basis of cylindrical domain (at $x=H$) we propose the same type of BC:

$$\nu_1 k\frac{\partial U}{\partial x}+\lambda_1 U=\Phi_1(y)+L_1(U). \quad (3)$$

The type of boundary condition (or conditions) on the side surface (on all the side surface or only on part of it) $\Gamma\subseteq\{\partial G=\bar{G}\setminus G\}\times\{0\le x\le H\}$ of the cylindrical domain D depends on the type of the operator $L(\bullet)$. We will specify the concrete form of the BC later and now we restrict ourselves to writing this *linear* BC in following general form:

$$l(U)=\Psi(x,y),\ (x,y)\in\Gamma. \quad (4)$$

Nevertheless it is important to make immediately two following additional remarks about BC (4). Firstly, as it was already written, Γ can be a part of the entire side surface. For example, if the basis of cylindrical domain is rectangle and one of vector argument y components is the time, then the BC (4) could be given for $t=0$, but not for the $t=T$. Secondly, on some parts of the side surface could be given two (or even more) BC, e.g. for biharmonic or hyperbolic operator L.

2.2 Transformation of the original problem

We will transform problem (1) – (4), it means that we will have a different problem instead of the original problem. To make difference between these two problems clearer, we denote the new solution of the equation (1) as $u(x,y)$ instead of the function $U(x,y)$ and in addition we introduce integral averaged function in direction x:

$$u_0(y)=H^{-1}\int_0^H U(x,y)dx. \quad (5)$$

Now we integrate the main equation (1). This gives exact equality (the operator $L(u)$ is linear operator!):

$$L(u_0)+\frac{k}{H}\frac{\partial U}{\partial x}\bigg|_{x=0}^{x=H}=-f(y),$$

$$f(y)=H^{-1}\int_0^H F(x,y)dx.$$

We shall call this equality *principal relation.* As one can see, principal relation is underdetermined equation because of presence of two different functions: $u_0(y)$ and $U(x,y)$, respectively $u(x,y)$ in one equation. It means that connection between these functions must be determined. Next steps in our approach (method) depend on two factors:

1) Assumption about the behavior of the function $U(x,y)$ respectively $u(x,y)$ in $x-$direction at fixed y;
2) The concrete type of the BC on both bases of the cylindrical domain.

The simplest assumption; the behavior of the function $u(x,y)$ in $x-$direction: it is weakly depending on variable x. Then we can assume following sequence of equalities:

$$U(x,y)=u(x,y)\approx u(y)=u_0(y). \qquad (6)$$

The second type of BC on both bases immediately gives the following new main equation (from the principal relation):

$$L(u_0)=-\left[\frac{\Phi_0(y)+\Phi_1(y)}{H}+f(y)\right]. \qquad (7_2)$$

In case of the third type of BC (2_3) we express the "flux" term at lower base (similar action must be done for the upper base):

$$-k\frac{\partial U}{\partial x}=h_0\Theta_0(y)-h_0u(0,y).$$

Finally, instead of (7_2) we obtain following main equation:

$$L(u_0)-\frac{h_0+h_1}{H}u_0=\left\{\frac{h_0\Theta_0(y)+h_1\Theta_1(y)}{H}+f(y)\right\}. \qquad (7_3)$$

Analogously as in case of the Stefan-Boltzmann type of BC we obtain such main equation:

$$L(u_0)-\frac{2\varepsilon\sigma}{H}u_0^4=\left\{\varepsilon\sigma\frac{[\Theta_0(y)]^4+[\Theta_1(y)]^4}{H}+f(y)\right\}.$$

For the BC (2_4) we obtain following main equation:

$$L(u_0)-\frac{\beta_0}{H}[u_0-\Theta_0(y)]^m-\frac{\beta_1}{H}[u_0-\Theta_1(y)]^m=-f(y). \qquad (7_4)$$

More attention must be paid to case of first type BC (2_1). Formally from principal relation and equalities (6) follow that both "flux" terms vanish (become zero) and as main equation we have:

$$L(u_0)=-f(y).$$

The weakness of such averaged main equation is evident: it is independent from right hand side functions $\Phi_i(y), i=0,1$ of BC. Assumption that integral averaged value $u_0(y)$ is placed near the middle point $x=H/2$ gives (by replacing both derivatives with finite differences) such main equation:

$$L(u_0)-\frac{4k}{H^2}u_0=\left\{\frac{2k}{H^2}[\Phi_0(y)+\Phi_1(y)]+f(y)\right\}. \qquad (7_1)$$

The general case (the operators $L_0(L_1)$ are present in the first type BC) instead of equation (7_1) gives following averaged equation:

$$L(u_0)+\frac{2k}{H^2}[L_0(u_0)+L_1(u_0)-2u_0]=-\left\{f+\frac{2k}{H^2}[\Phi_0(u_0)+\Phi_1(u_0)]\right\}.$$

In all cases, to the main equation (7_i) must be added BC on $\gamma\subseteq\{\partial G=\bar{G}\setminus G\}$. The linearity of BC (4) immediately gives from (5) following new BC:

$$l(u_0)=\psi(y), y\in\gamma,$$
$$\psi(y)=H^{-1}\int_0^H\Psi(x,y)dx. \qquad (8)$$

3 Application of Conservative Averaging Method for Time Inverse Hyperbolic Heat Conduction Problem

In paper [8] was proposed mixed initial-boundary problem for the hyperbolic heat equation as new mathematical model for intensive steel quenching process. This inverse problem was reduced to the first kind Fredholm integral equation.

3.1 Original Problem

We start with the formulation of the one-dimensional mathematical model for intensive steel quenching as in paper [8]:

$$\tau_r\frac{\partial^2U}{\partial t^2}+\frac{\partial U}{\partial t}=a^2\frac{\partial^2U}{\partial x^2}+f(x,t), \qquad (9)$$
$$x\in(0,H), t\in(0,T), H<\infty,$$

$$\left(-k\frac{\partial U}{\partial x}+hU\right)=h\Theta(t), x=0, t\in[0,T], \quad (10)$$

$$\frac{\partial U}{\partial x}=0, x=H, \quad (11)$$

$$U=U^0(x), t=0, x\in[0,H]. \quad (12)$$

Here k is heat conductivity coefficient, c - specific heat, h - heat exchange coefficient, $a^2=k/(c\rho)$ and ρ is the steel density.

From the practical point of view the second initial condition is unrealistic. The initial heat flux

$$\frac{\partial U}{\partial t}=V_0(x), x\in[0,H], \ t=0 \quad (13)$$

can't be measured experimentally and must be calculated. As additional condition we assume experimentally realizable condition –the temperature distribution at the end of process is given:

$$U(x,T)=U_T(x), x\in[0,H]. \quad (14)$$

In paper [8] it was shown that the solution of this problem can be written in very well known form:

$$U(x,t)=\int_0^H U^0(x)G(x,\xi,t)d\xi+\frac{h}{c\rho}\int_0^t\Theta(\tau)G(x,0,t-\tau)d\tau$$
$$+\int_0^t d\tau\int_0^H\left[f(\xi,\tau)-\tau_r\frac{\partial^2 U(\xi,\tau)}{\partial\tau^2}\right]G(x,\xi,t-\tau)d\xi.$$

Here the function $G(x,\xi,t)$ is the Green function for the classical heat conduction equation

$$\frac{\partial U}{\partial t}=a^2\frac{\partial^2 U}{\partial x^2}, x\in(0,H), t\in(0,T)$$

with BC (10), (11) and initial condition (12).

The representation given above can be written in shorter form:

$$U(x,t)=G_2(x,\xi,t)-$$
$$\tau_r\int_0^t d\tau\int_0^H\frac{\partial^2 U(\xi,\tau)}{\partial\tau^2}G(x,\xi,t-\tau)d\xi. \quad (15)$$

Here $G_2(x,\xi,t)$ is the known part of the solution's representation:

$$G_2(x,\xi,t)=\int_0^H U^0(x)G(x,\xi,t)d\xi$$
$$+\frac{h}{c\rho}\int_0^t\Theta(\tau)G(x,0,t-\tau)d\tau+$$
$$\int_0^t d\tau\int_0^H f(\xi,\tau)G(x,\xi,t-\tau)d\xi.$$

Now it is possible to use the given temperature distribution at the end of process (14) and representation (15) immediately gives the Fredholm type integral equation of the first kind for the second time derivative of the temperature:

$$\int_0^T d\tau\int_0^H\frac{\partial^2 U}{\partial\tau^2}G(x,\xi,T-\tau)d\xi$$
$$=\frac{G_2(x,T)-U_T(x)}{\tau_r}. \quad (16)$$

This problem is ill-posed, but this integral equation can be solved numerically, e.g., by Tikhonov regularization method. Then, see [8]:

$$V_0(x)=\frac{\partial G_2(x,+0)}{\partial t}-\tau_r\frac{\partial^2\tilde{U}(x,+0)}{\partial t^2}. \quad (17)$$

Here $\frac{\partial^2\tilde{U}(x,t)}{\partial t^2}$ is approximate (regularized) solution of the integral equation (16).

3.2 The Approximate Solution by Conservative Averaging Method

By applying conservative averaging method to the problem (9)-(14) we obtain relatively the integral average temperature $u_0(t)$ following boundary problem for ordinary differential equation:

$$\tau_r\frac{du_0^2}{dt^2}+\frac{du_0}{dt}+\frac{h}{c_H}u_0=$$
$$\frac{h}{c_H}\Theta(t)+f(t), c_H=c\rho H, \quad (18)$$

$$u_0(0)=u_0^0, \quad (19)$$

$$u_0(T)=u_T. \quad (20)$$

We are interested to determine

$$v_0=\frac{du_0(0)}{dt}. \quad (21)$$

To solve this problem we split it in two sub-problems:

$$u_0(t)=\bar{u}(t)+\bar{w}(t). \quad (22)$$

First of them is homogeneous main equation:

$$\tau_r\frac{d\bar{u}}{dt^2}+\frac{d\bar{u}}{dt}+\frac{h}{c_H}\bar{u}=0 \quad (23)$$

together with non-homogeneous initial conditions:

$$\bar{u}(0)=u_0^0, \ \frac{d\bar{u}(0)}{dt}=v_0. \quad (24)$$

This problem can be solved in traditional way and its solution is:

$$\bar{u}(t)=\left[u_0^0\cosh(\beta t)+v_0\beta^{-1}\sinh(\beta t)\right]e^{-\frac{t}{2\tau_r}}. \quad (25)$$

Here $\beta=\frac{1}{2\tau_r}\sqrt{1-\frac{4h\tau_r}{c_H}}$. (26)

The condition $4h\tau_r < c_H$ is mathematically caused: it shows the boundedness of possibility to approximate the solution $U(x,t)$ by constant regarding x.

The second problem has non-homogeneous main equation and homogeneous initial conditions:

$$\tau_r \frac{d\overline{w}^2}{dt^2} + \frac{d\overline{w}}{dt} + \frac{h}{c_H}\overline{w} = \frac{h}{c_H}\Theta(t) + f(t) \quad (27)$$

$$\overline{w}(0) = \frac{d\overline{w}(0)}{dt} = 0.$$

The solution of this problem has following form:

$$\overline{w}(t) = e^{-\frac{t}{2\tau_r}} \int_0^t q(t-\tau)\Phi(\tau)d\tau,$$

$$\Phi(t) = \tau_r^{-1}\left[\gamma\Theta(t) + f(t)\right]e^{\frac{t}{2\tau_r}}, \gamma = \frac{h}{c_H}.$$

Here $q(t)$ is solution of the differential equation (23) with special initial conditions:

$$q(0) = 0, \ \frac{dq(0)}{dt} = 1,$$

i.e. $q(t) = \beta^{-1}\sinh(\beta t)$.

Hence:

$$\overline{w}(t) = \beta^{-1}e^{-\frac{t}{2\tau_r}} \int_0^t \sinh[\beta(t-\tau)]\Phi(\tau)d\tau. \quad (28)$$

Consequently, we have finally obtained the solution of the problem (18),(19),(21) as:

$$u_0(t) = \left[u_0^0\cosh(\beta t) + \frac{v_0}{\beta}\sinh(\beta t)\right]e^{-\frac{t}{2\tau_r}} + \frac{e^{-\frac{t}{2\tau_r}}}{\beta\tau_r}\int_0^t \sinh[\beta(t-\tau)]\left[\gamma\Theta(\tau) + f(\tau)\right]e^{\frac{\tau}{2\tau_r}}d\tau. \quad (29)$$

As the last step we use the additional information: the condition (20), i.e. known value at the end of the process. This information allows us to express unknown second initial condition in closed and simple form:

$$v_0 = \frac{\beta u_T e^{\frac{T}{2\tau_r}}}{\sinh(\beta T)} - \beta u_0^0\coth(\beta T) - \tau_r^{-1}\int_0^T \frac{\sinh(\beta(T-\tau))}{\sinh(\beta T)}\left[f(\tau) + \gamma\Theta(\tau)\right]e^{\frac{\tau}{2\tau_r}}d\tau. \quad (30)$$

We can increase the order of the approximation for the solution of the original problem (9)-(14) by the representation with polynomial of second degree (the same polynomial form was used for integral parabolic spline in our papers [9], [10]):

$$U(x,t) = u_0(t) + m(t)\left(x - \frac{H}{2}\right) + \frac{e(t)}{kH}\left[\left(x - \frac{H}{2}\right)^2 - \frac{H^2}{12}\right]. \quad (31)$$

The unknown coefficients $m(t), e(t)$ we determined from BC (10), (11) and it finally gives:

$$U(x,t) = u_0(t) + \frac{\left[\Theta(t) - u_0(t)\right]}{H\left(H + \frac{3k}{h}\right)}\left(\frac{3}{2}x^2 - 3Hx + H^2\right).$$

The integration over interval $x \in [0, H]$ of the main equation (9) practically gives the same ordinary differential equation (18). The only difference is in the same coefficient at two terms:

$$\tau_r \frac{du_0^2}{dt^2} + \frac{du_0}{dt} + \frac{3kh}{c_H\left(3k + hH\right)}u_0 = \frac{3kh}{c_H\left(3k + hH\right)}\Theta(t) + f(t). \quad (32)$$

The additional conditions (19)-(21) remains the same. It means that we can use obtained above formulae (25), (28)-(30) replacing the parameters β, γ by following expressions:

$$\beta = \frac{1}{2\tau_r}\sqrt{1 - \frac{4h\tau_r}{c_H\left(1 + \frac{hH}{3k}\right)}}, \ \gamma = \frac{h}{c_H\left(1 + \frac{hH}{3k}\right)}. \quad (33)$$

As it was mentioned earlier, the restriction $\frac{4h\tau_r}{c_H} < 1$ shows the boundedness of possibility to approximate the solution $U(x,t)$ by constant. For the approximation of the function $U(x,t)$ by the representation (31) from (33) we obtain weaker restriction:

$$\frac{4h\tau_r}{c_H} < 1 + \frac{hH}{3k}.$$

This, on the one hand, is the weakness of the approximate solutions of inverse problem. On the other hand, we have obtained solutions of well posed problem in closed form. This solution (29) can be used as initial approximation for integrated over $x \in [0, H]$ equation (17).

3.3 The Approximate Solution by Other Boundary Conditions

Conservative averaging method is applicable in cases of given non-linear boundary conditions as well. We will explain the employment of our method for the intensive steel quenching problem from sub-section 3.1 with following BC: on the left end of the slab we have nucleate boiling type BC, but on the right endpoint the Stefan-Boltzmann radiation takes place:

$$\tau_r \frac{\partial^2 U}{\partial t^2} + \frac{\partial U}{\partial t} = a^2 \frac{\partial^2 U}{\partial x^2} + f(x,t), \quad x \in (0,H), t \in (0,T), H < \infty, \tag{34}$$

$$-k\frac{\partial U}{\partial x} + \beta_0 \left[U - \Theta_0(t)\right]^m = 0, \quad x = 0, t \in [0,T], \tag{35}$$

$$k\frac{\partial U}{\partial x} + \varepsilon_1 \sigma \left[U^4 - \Theta_1^4(t)\right] = 0, \quad x = H, t \in [0,T], \tag{36}$$

$$U\big|_{t=0} = U^0(x), x \in [0,H]. \tag{37}$$

We will use the simplest approximation for the solution of the problem (34)-(37) by the constant (see formula (6)). Then we finally obtain following main ordinary equation:

$$\tau_r \frac{du_0^2}{dt^2} + \frac{du_0}{dt} - \frac{\varepsilon_1\sigma}{c_H} u_0^4 - \beta_0 \left[U - \Theta_0(t)\right]^m = -\frac{\varepsilon_1\sigma}{c_H}\Theta_1^4(t) + f(t). \tag{38}$$

This differential equation must be solved together with given temperatures at initial and final moments:

$$u_0(0) = u_0^0, \tag{39}$$

$$u_0(T) = u_T. \tag{40}$$

This boundary problem can be solved numerically. After solving the problem (38)-(40) we determine the initial heat flux

$$v_0 = \frac{du_0(0)}{dt}.$$

4 Conclusions

Conservative averaging method can be applied to steady-state and non-stationary problems. It can be applied to different boundary conditions, including non-linear and non-classical.

Even approximation by constant might be sufficient in practical heat transfer problems. Sometimes the conservative averaging method can transform an inverse ill-posed problem to corresponding well-posed problem.

Acknowledgements:

Research was supported by Council of Sciences of Latvia (grant 05.1525) and University of Latvia (project Y2-ZP09-100).

References:

[1] Samarskii, A.A., Vabishchevich, P.N. *Computational Heat Transfer. Vol.1, Mathematical Modelling.* John Wiley&Sons Ltd., 1995.

[2] Bear, J. *Hydraulics of Groundwater.* Prentice-Hall, Inc., 1987.

[3] Buikis, A. The modelling of filtration processes in layered media by conservative averaging method. *Doctoral thesis.* Kasan, 1987. 374p. (In Russian)

[4] Buikis, A. Aufgabenstellung und Loesung einer Klasse von Problemen der mathematischen Physik mit nichtklassischen Zusatzbedingungen. *Rostock. Math. Kolloq.*, 1984, 25, p. 53-62. (In German)

[5] Buikis, A., The change of formulation of problems of mathematical physics with dis-continuous coefficients in compound domains. *The Electronic Modelling*, 1986, Vol.8, No.6, p. 81-83. (In Russian)

[6] Vilums, R., Buikis, A. Conservative averaging method for partial differential equations with discontinuous coefficients. *WSEAS Transactions on Heat and Mass Transfer.* Vol. 1, Issue 4, 2006, p. 383-390.

[7] Buikis, A. Conservative averaging as an approximate method for solution of some direct and inverse heat transfer problems. *Advanced Computational Methods in Heat Transfer, IX.* WIT Press, 2006. p. 311-320.

[8] Buikis, A., Guseinov, Sh. Solution of Reverse Hyperbolic Heat equation for intensive carburized steel quenching. *Proceedings of ICCES'05 (Advances in Computational and Experimental Engineering and Sciences).* December 1-6, 2005, IIT Madras. p. 741-746.

[9] Buike, M., Buikis, A. Analytical approximate method for modelling of three-dimensional transport processes in layered media. *Proceedings of the 4th IASME/WSEAS Int. Conf. on Heat Transfer, Thermal Engineering and Environment.* Elounda, Agios Nikolaos, Crete Island, Greece, August 21-23, 2006, p.232-237.

[10] Buike, M., Buikis, A. Modelling of Three-Dimensional Transport Processes in Anisotropic Layered Stratum by Conservative Averaging Method. *WSEAS Transactions of Heat and Mass Transfer*, 2006, Issue 4, Vol. 1, p. 430-437.