What Slip Boundary Conditions Induce a Well–Posed Problem for the Navier–Stokes Equation?

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Abstract: - We discuss an initial–boundary value problem for the Navier–Stokes equation with several types of slip boundary conditions. We mainly pay attention to boundary conditions based on vorticity.

Key-Words: - The Navier–Stokes equation, boundary conditions

1 Introduction

We deal with the Navier–Stokes system

$$\partial_t \boldsymbol{u} - \nu\Delta\boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} + \nabla p = \boldsymbol{f} \qquad (1)$$

$$\operatorname{div}\boldsymbol{u} = 0 \qquad (2)$$

in $\Omega\times(0,T)$, where Ω is a bounded simply connected domain in $\mathbb{R}^3$ with the boundary of the class $C^{2,1}$ and $T>0$. We consider the initial condition

$$\boldsymbol{u}(\boldsymbol{x},0) = \boldsymbol{u}_0(\boldsymbol{x}) \qquad \text{in } \Omega. \qquad (3)$$

The system (1), (2) describes the motion of a viscous incompressible fluid with a constant density (we assume that it equals one). We denote by $\boldsymbol{u}$ the velocity, by p the pressure, by $\boldsymbol{f}$ the specific body force and by ν the coefficient of viscosity. The equation (1) expresses the balance of momentum and the equation (2) represents the condition of incompressibility.

By a well–posed problem we mean a problem which possesses the existence of a weak solution and under an additional assumption on smoothness of the solution, also its uniqueness. In order to obtain such a problem, we must add an appropriate boundary condition. The system (1), (2) is usually considered with the homogeneous *Dirichlet boundary condition*

$$\boldsymbol{u}\,|_{\partial\Omega} = \boldsymbol{0} \qquad (4)$$

in the case when $\partial\Omega$ is a fixed wall. This condition was suggested by G. G. Stokes in 1845 and it expresses the requirement that

$$\boldsymbol{u}\cdot\boldsymbol{n}|_{\partial\Omega} = 0, \qquad (5)$$

$$\boldsymbol{u}_\tau|_{\partial\Omega} = \boldsymbol{0}, \qquad (6)$$

where $\boldsymbol{u}_\tau = \boldsymbol{n}\times\boldsymbol{u}\times\boldsymbol{n}$ denotes the projection of $\boldsymbol{u}$ onto the plane tangential to $\partial\Omega$. While the first condition (5) naturally follows from the impermeability of the wall, the second one (6) is often called the "no–slip condition" because it is believed that the fluid cannot slip on the boundary due to its viscosity. The mathematical theory of the Navier–Stokes equation with the Dirichlet boundary condition is relatively well elaborated, and this assertion also includes the case of an inhomogeneous condition of the type (4).

On the other hand, if we confine ourselves only to the condition (5) and assume that the law of conservation of momentum holds up to the boundary then we naturally obtain, from physical considerations, a complementary boundary condition to (5) in the form

$$(\mathbb{T}\cdot\boldsymbol{n})_\tau + k\boldsymbol{u} = \boldsymbol{0} \qquad (7)$$

where $\mathbb{T}$ is the dynamic stress tensor, $(\mathbb{T}\cdot\boldsymbol{n})_\tau$ denotes the tangential component of the surface force acting on the boundary and k is a coefficient of proportionality. Let us explain in greater detail what we mean by this: Let V be an arbitrary control domain in Ω such that its boundary ∂V consists of two surfaces $\Gamma_0\subset\partial\Omega$ and $\Gamma_1\subset\Omega$. The difference of momenta in V between the times t_2 and t_1 is

$$\int_{t_1}^{t_2}\int_V (\partial_t\boldsymbol{u})\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t.$$

The flux of the momentum through the boundary of V in the time interval $[t_1,t_2]$ is

$$-\int_{t_1}^{t_2}\int_{\partial V}\boldsymbol{u}\,(\boldsymbol{u}\cdot\boldsymbol{n})\,\mathrm{d}S\,\mathrm{d}t$$

$$= \; -\int_{t_1}^{t_2}\int_V (\boldsymbol{u}\cdot\nabla)\boldsymbol{u}\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t.$$

The surface force consists of $\mathbb{T}\cdot\boldsymbol{n} = (\mathbb{T}\cdot\boldsymbol{n})_\tau + \tau_n\,\boldsymbol{n}$ on Γ_1 and $\tau_n\,\boldsymbol{n} - k\,\boldsymbol{u}$ on Γ_0. Here we denote by τ_n the normal component of the surface force coming from the stress tensor, acting on the boundary of V (i.e. $\tau_n = \boldsymbol{n}\cdot\mathbb{T}\cdot\boldsymbol{n}$) and k (> 0) is the coefficient of friction between the fluid and the wall. Naturally, since $(\mathbb{T}\cdot\boldsymbol{n})_\tau$ expresses the tangential component of the contact force on the boundary of the type "fluid–fluid" (i.e. on Γ_1), it must be replaced by $-k\,\boldsymbol{u}$ on the boundary of the type "fluid–wall" (which is in our case Γ_0). Thus, the impulse of the surface force is

$$\int_{t_1}^{t_2}\int_{\Gamma_1}[(\mathbb{T}\cdot\boldsymbol{n})_\tau + \tau_n\,\boldsymbol{n}]\,\mathrm{d}S\,\mathrm{d}t \\ + \int_{t_1}^{t_2}\int_{\Gamma_0}[\tau_n\,\boldsymbol{n} - k\boldsymbol{u}]\,\mathrm{d}S\,\mathrm{d}t.$$

The impulse of the body force is

$$\int_{t_1}^{t_2}\int_V \boldsymbol{f}\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t.$$

Due to the conservation of momentum, we have

$$\int_{t_1}^{t_2}\int_V[\partial_t\boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} - \boldsymbol{f}]\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t \\ = \int_{t_1}^{t_2}\int_{\Gamma_0}[\tau_n\boldsymbol{n} - k\,\boldsymbol{u}]\,\mathrm{d}S\,\mathrm{d}t \\ + \int_{t_1}^{t_2}\int_{\Gamma_1}[(\mathbb{T}\cdot\boldsymbol{n})_\tau + \tau_n\boldsymbol{n}]\,\mathrm{d}S\,\mathrm{d}t.$$

If we consider $\Gamma_1 \to \Gamma_0$ then the volume of V tends to zero (and consequently the volume integral vanishes) and $\boldsymbol{n}\,|_{\Gamma_1} \to -\boldsymbol{n}\,|_{\Gamma_0}$. Hence we obtain

$$\int_{t_1}^{t_2}\int_{\Gamma_0}[-(\mathbb{T}\cdot\boldsymbol{n})_\tau - k\,\boldsymbol{u}]\,\mathrm{d}S\,\mathrm{d}t \;=\; 0.$$

This implies, due to the possibility of the arbitrary choice of Γ_0, the condition (7). The condition (5) is called *Navier's boundary condition*, sometimes however this name also automatically involves (7). It is necessary to add that the coefficient k depends on the normal stress τ_n, similarly as the friction between a body towed on a desk depends not only on the area of the contact surface, but also on the force the body acts onto the surface with.

The condition (7) naturally follows from the weak formulation of the problem (1), (2), (3), (5) which sounds: Given $\boldsymbol{u}_0 \in \boldsymbol{L}^2_\sigma(\Omega)$ (the space of divergence–free in the sense of distributions vector functions in Ω whose normal component in the sense of traces equals zero on the boundary) and $\boldsymbol{f}$ in $L^2(0,T;\,\boldsymbol{W}^{-1,2}_\sigma(\Omega))$. ($\boldsymbol{W}^{-1,2}_\sigma(\Omega)$ is the dual to $\boldsymbol{W}^{1,2}_\sigma(\Omega) = \boldsymbol{W}^{1,2}(\Omega)\cap\boldsymbol{L}^2_\sigma(\Omega)$.) We search for $\boldsymbol{u} \in L^2(0,T;\,\boldsymbol{W}^{1,2}_\sigma(\Omega))\cap L^\infty(0,T;\,\boldsymbol{L}^2_\sigma(\Omega))$ such that

$$\int_0^T\int_\Omega[-\boldsymbol{u}\cdot\partial_t\boldsymbol{\phi} + \mathbb{T}\cdot\nabla\boldsymbol{\phi} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u}\cdot\boldsymbol{\phi}]\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t \\ + \int_{\partial\Omega}k\boldsymbol{u}\cdot\boldsymbol{\phi}\,\mathrm{d}S\,\mathrm{d}t \\ = \int_\Omega \boldsymbol{u}_0\cdot\boldsymbol{\phi}(.\,,0)\,\mathrm{d}\boldsymbol{x} + \int_0^T\langle\boldsymbol{f},\boldsymbol{\phi}\rangle_\Omega\,\mathrm{d}t \qquad (8)$$

for all $\boldsymbol{\phi} \in C^\infty(0,T;\,\boldsymbol{W}^{1,2}_\sigma(\Omega))$ such that $\boldsymbol{\phi}(.\,,T) = \boldsymbol{0}$. Here $\langle\,.\,,\,.\,\rangle_\Omega$ is the duality between $\boldsymbol{W}^{-1,2}_\sigma(\Omega)$ and $\boldsymbol{W}^{1,2}_\sigma(\Omega)$. Indeed, if $\boldsymbol{u}$ is a "smooth" solution of this problem then, considering at first the test functions $\boldsymbol{\phi}$ with a compact support in $\Omega\times[0,T)$, we find out that there exists an appropriate pressure p such that $\boldsymbol{u}$, p satisfy the equation (1) a.e. in $\Omega\times(0,T)$. Then, considering all admissible test functions and integrating by parts in (8), we arrive at the identity

$$\int_0^T\int_{\partial\Omega}[\mathbb{T}\cdot\boldsymbol{n} + k\boldsymbol{u}]\cdot\boldsymbol{\phi}\,\mathrm{d}S\,\mathrm{d}t \;=\; 0,$$

which implies (7).

Let us note that the coefficient k is often considered to be zero. The correctness of this step is, of course, a matter of discussion and depends on the real smoothness of the boundary of Ω. Although the number of works on the Navier–Stokes equation with Navier's boundary condition is not as high as with Dirichlet's boundary condition, it is possible to state that the theory which considers Navier's condition is also relatively well developed.

2 Generalized impermeability boundary conditions

There is a wide range of other possibilities between (4) and (5). In this section, we wish to discuss the case when, in addition to the condition of impermeability (5), we assume that the 2D flow on the boundary of Ω is non–rotational, which means that it satisfies

$$\mathbf{curl}\,\boldsymbol{u}\cdot\boldsymbol{n}\,|_{\partial\Omega} \;=\; 0. \qquad (9)$$

The natural weak formulation of the initial–boundary value problem (1)–(3), (5), (9) comes from the Navier –Stokes equation written in the form

$$\partial_t\boldsymbol{u} + \nu\,\mathbf{curl}^2\boldsymbol{u} + \mathbf{curl}\,\boldsymbol{u}\times\boldsymbol{u} + \nabla q \;=\; \boldsymbol{f} \qquad (10)$$

and it is: Given $\boldsymbol{u}_0 \in \boldsymbol{L}^2_\sigma(\Omega)$ and $\boldsymbol{f}$ in $L^2(0,T;\, \boldsymbol{D}^{-1})$. We search for $\boldsymbol{u} \in L^2(0,T;\, \boldsymbol{D}^1) \cap L^\infty(0,T;\, \boldsymbol{L}^2_\sigma(\Omega))$ such that

$$\int_0^T \int_\Omega [-\boldsymbol{u} \cdot \partial_t \boldsymbol{\phi} + \nu \, \mathbf{curl}\, \boldsymbol{u} \cdot \mathbf{curl}\, \boldsymbol{\phi} + (\mathbf{curl}\, \boldsymbol{u} \times \boldsymbol{u}) \cdot \boldsymbol{\phi}] \, \mathrm{d}\boldsymbol{x} \, \mathrm{d}t = \int_\Omega \boldsymbol{u}_0 \cdot \boldsymbol{\phi}(\,.\,,0) \, \mathrm{d}\boldsymbol{x} + \int_0^T \langle \boldsymbol{f}, \boldsymbol{\phi} \rangle_\Omega \, \mathrm{d}t \tag{11}$$

for all $\boldsymbol{\phi} \in C^\infty(0,T;\, \boldsymbol{D}^1)$ such that $\boldsymbol{\phi}(\,.\,,T) = \mathbf{0}$. Here $\boldsymbol{D}^1$ denotes the space $\{\boldsymbol{\varphi} \in \boldsymbol{W}^{1,2}_\sigma(\Omega);\ \mathbf{curl}\, \boldsymbol{\varphi} \cdot \boldsymbol{n}\,|_{\partial\Omega} = 0\}$. It is proved in [1] that each function from $\boldsymbol{D}^1$ coincides with a function of the type $\nabla\psi$ (for some $\psi \in W^{2,2}(\Omega)$) on $\partial\Omega$. The problem (11) has a global in time weak solution (see [1]) and if the given data are smooth then it also has a local in time unique strong solution (see [3]). It seems to be strange at the first sight because while the homogeneous Dirichlet boundary condition (4) in fact represents three scalar conditions and the Navier conditions (5), (7) also together involve three scalar conditions, the conditions (5) and (9), explicitly used in (11), represent only two scalar conditions. However, we can show that the problem (11) implicitly involves in itself the third condition, complementary to (5) and (9), which is

$$\mathbf{curl}^2 \boldsymbol{u} \cdot \boldsymbol{n}\,|_{\partial\Omega} \,=\, 0. \tag{12}$$

Indeed, if we assume that (11) has a solution $\boldsymbol{u}$ "smooth enough" then choosing at first only the test functions $\boldsymbol{\phi}$ which have a compact support in $\Omega \times [0,T)$ and integrating by parts in (11), we deduce that there exists a scalar function q such that the pair $\boldsymbol{u}$, q satisfies the equations (10), (2) in $\Omega \times (0,T)$, the initial condition (3) and the boundary conditions (5), (9). Considering then all admissible test functions $\boldsymbol{\phi}$, integrating again by parts in (11) and using the information that $\boldsymbol{u}$, q satisfy the equation (10), we arrive at the integral identity

$$\int_0^T \int_{\partial\Omega} \nu \, \mathbf{curl}\, \boldsymbol{u} \cdot (\boldsymbol{n} \times \boldsymbol{\phi}) \, \mathrm{d}S \, \mathrm{d}t \,=\, 0.$$

Using the representation $\boldsymbol{\phi} = \nabla\psi$ on $\partial\Omega \times (0,T)$, we obtain

$$\begin{aligned} 0 \,&=\, -\int_0^T \int_{\partial\Omega} \nu \, \boldsymbol{n} \cdot (\mathbf{curl}\, \boldsymbol{u} \times \nabla\psi) \, \mathrm{d}S \, \mathrm{d}t \\ &=\, -\int_0^T \int_\Omega \nu \, \mathrm{div}\, (\mathbf{curl}\, \boldsymbol{u} \times \nabla\psi) \, \mathrm{d}\boldsymbol{x} \, \mathrm{d}t \\ &=\, -\int_0^T \int_\Omega \nu \, \mathbf{curl}^2 \boldsymbol{u} \cdot \nabla\psi \, \mathrm{d}\boldsymbol{x} \, \mathrm{d}t \\ &=\, -\int_0^T \int_{\partial\Omega} \nu \, \mathbf{curl}^2 \boldsymbol{u} \cdot \boldsymbol{n} \, \psi \, \mathrm{d}S \, \mathrm{d}t. \end{aligned}$$

This implies (12).

The boundary conditions (5), (9), (12), since they express the impermeability of $\boldsymbol{u}$, $\mathbf{curl}\, \boldsymbol{u}$ and $\mathbf{curl}^2 \boldsymbol{u}$ through $\partial\Omega$, are called the *generalized impermeability boundary conditions*. (We abbreviate them as GIBC.)

The usage of GIBC in the mathematical theory of the Navier–Stokes equation brings several advantages. (See e.g. [1] and [3].) Let us mention only two of them:

1. If we denote by Π_σ the so called Helmoltz projection, i.e. the orthogonal projection of $\boldsymbol{L}^2(\Omega)$ onto $\boldsymbol{L}^2_\sigma(\Omega)$ then Π_σ commutes with the Laplace operator Δ. This plays an important role especially in the theory of an associated Stokes operator and it also has interesting consequences for the Navier–Stokes equation.
2. While Dirichlet's or Navier's boundary conditions for velocity do not directly induce a well–posed problem for vorticity, the GIBC do: If $\boldsymbol{u}$ is a solution of the problem (10), (2), (3) with GIBC then $\boldsymbol{\omega} = \mathbf{curl}\, \boldsymbol{u}$ satisfies the series of the boundary conditions of the same type as GIBC:

$$\begin{aligned} &\boldsymbol{\omega} \cdot \boldsymbol{n}\,|_{\partial\Omega} = 0, \quad \mathbf{curl}\, \boldsymbol{\omega} \cdot \boldsymbol{n}\,|_{\partial\Omega} = 0, \\ &\nu \, \mathbf{curl}^2 \boldsymbol{\omega} \cdot \boldsymbol{n}\,|_{\partial\Omega} = -\mathbf{curl}\, \boldsymbol{f} \cdot \boldsymbol{n}\,|_{\partial\Omega}\,. \end{aligned} \tag{13}$$

3 The inhomogeneous version of GIBC

The series (13) of the boundary conditions for vorticity, except for many other reasons, is a motivation for the study of the Navier–Stokes equation with an inhomogeneous version of GIBC:

$$\begin{aligned} &\text{(a)} \quad \boldsymbol{u} \cdot \boldsymbol{n}\,|_{\partial\Omega} = \alpha_0, \qquad \text{(b)} \quad \mathbf{curl}\, \boldsymbol{u} \cdot \boldsymbol{n}\,|_{\partial\Omega} = \alpha_1, \\ &\text{(c)} \quad \mathbf{curl}^2 \boldsymbol{u} \cdot \boldsymbol{n}\,|_{\partial\Omega} = \alpha_2. \end{aligned} \tag{14}$$

We further suppose, for simplicity, that α_0, α_1 and α_2 are time independent. The problem can be approached in the following way: at first we find a function $\boldsymbol{a}$, defined a.e. in Ω, which satisfies the first two boundary conditions (14a) and (14b) and we search for the solution in the form $\boldsymbol{u} = \boldsymbol{a} + \boldsymbol{v}$. Then the new unknown function $\boldsymbol{v}$ should satisfy (5), (9). The treatment of the third boundary condition (14c) requires a finer techniques and we shall explain it later. The existence of a function $\boldsymbol{a}$ with the needed properties is guaranteed by the lemma:

Lemma 1 *Let Γ_0, Γ_1, …, Γ_N be the components of $\partial\Omega$ and let $\Omega = \mathrm{Int}\,\Gamma_0 \cup \mathrm{Ext}\,\Gamma_1 \cup \ldots \cup \mathrm{Ext}\,\Gamma_N$. Given functions $\alpha_0 \in W^{1/2,2}(\partial\Omega)$ and $\alpha_1 \in W^{-1/2,2}(\partial\Omega)$*

such that

$$\int_{\partial\Omega} \alpha_0 \, \mathrm{d}S = 0 \quad \textit{and}$$
$$\langle \alpha_1, 1 \rangle_{\Gamma_i} = 0 \quad (i = 0, 1, \dots, N)$$

(where $\langle \, . \, , \, . \, \rangle_{\Gamma_i}$ are dualities in appropriate spaces), there exists a vector function $\boldsymbol{a} \in \boldsymbol{W}^{1,2}(\Omega)$ such that $\operatorname{div} \boldsymbol{a} = 0$ *a.e. in Ω, $\boldsymbol{a}$ is harmonic (in the distributional sense) in some neighborhood of $\partial\Omega$ and*

$$\text{(a) } \boldsymbol{a} \cdot \boldsymbol{n}|_{\partial\Omega} = \alpha_0, \quad \text{(b) } \mathbf{curl}\, \boldsymbol{a} \cdot \boldsymbol{n}|_{\partial\Omega} = \alpha_1. \quad (15)$$

Moreover, there exists a constant $c_1 > 0$, independent of α_0 and α_1, such that

$$\|\boldsymbol{a}\|_{1,2} \ \leq \ c_1 \left(\|\alpha_0\|_{1/2,2;\,\partial\Omega} + \|\alpha_1\|_{-1/2,2;\,\partial\Omega} \right). \quad (16)$$

PROOF. (i) At first we solve the Neumann problem

$$\Delta \psi_1 \ = \ 0 \quad \text{in } \Omega, \qquad \frac{\partial \psi_1}{\partial \boldsymbol{n}} \Big|_{\partial\Omega} \ = \ \alpha_1. \quad (17)$$

There exists a unique (up to an additive constant) weak solution $\psi_1 \in W^{1,2}(\Omega)$ which depends continuously on α_1 in the sense that

$$\|\nabla \psi_1\|_2 \ \leq \ c_2 \, \|\alpha_1\|_{-1/2,2;\,\partial\Omega} \quad (18)$$

where c_2 is independent of α_1.

(ii) Next we consider the problem

$$\mathbf{curl}\, \boldsymbol{\varphi}_0 \ = \ \nabla \psi_1 \quad \text{in } \Omega, \quad \boldsymbol{\varphi}_0|_{\partial\Omega} \ = \ \mathbf{0}. \quad (19)$$

It is solvable in $\boldsymbol{W}_0^{1,2}(\Omega)$ by means of Theorem 2.1 from [2]. Moreover, there exists $c_3 > 0$, independent of $\nabla\psi_1$, such that

$$\|\boldsymbol{\varphi}_0\|_{1,2} \ \leq \ c_3 \, \|\nabla \psi_1\|_2 \, . \quad (20)$$

(iii) Further, we solve the Neumann problem

$$\Delta \psi_0 \ = \ -\operatorname{div} \boldsymbol{\varphi}_0 \quad \text{in } \Omega, \quad \frac{\partial \psi_0}{\partial \boldsymbol{n}} \Big|_{\partial\Omega} \ = \ \alpha_0. \quad (21)$$

It has a unique (up to an additive constant) solution $\psi_0 \in W^{2,2}(\Omega)$ which satisfies the estimate

$$\|\nabla \psi_0\|_{1,2} \ \leq \ c_4 \left(\|\boldsymbol{\varphi}_0\|_{1,2} + \|\alpha_0\|_{1/2,2;\,\partial\Omega} \right) \quad (22)$$

where c_2 is independent of $\boldsymbol{\varphi}_0$ and α_0.

Now we put $\boldsymbol{a} := \nabla\psi_0 + \boldsymbol{\varphi}_0$. The function $\boldsymbol{a}$ is divergence–free because ψ_0 satisfies the equation in (21). It is harmonic because $\mathbf{curl}^2 \boldsymbol{a} = \mathbf{curl}\, \nabla\psi_1 = \mathbf{0}$ in the sense of distributions in Ω. The normal component of $\boldsymbol{a}$ on $\partial\Omega$ equals α_0 because $\boldsymbol{a} \cdot \boldsymbol{n} = \nabla\psi_0 \cdot \boldsymbol{n} = \alpha_0$ on $\partial\Omega$. Since $\mathbf{curl}\, \boldsymbol{a} = \mathbf{curl}\, \boldsymbol{\varphi}_0 = \nabla\psi_1$ and consequently, $\mathbf{curl}\, \boldsymbol{a} \cdot \boldsymbol{n} = \nabla\psi_1 \cdot \boldsymbol{n} = \alpha_1$ on $\partial\Omega$, the function $\boldsymbol{a}$ also satisfies (15b). □

The weak solution $\boldsymbol{u}$ of (10), (2), (3), satisfying the boundary conditions (14), can now be constructed in the form $\boldsymbol{u} = \boldsymbol{a} + \boldsymbol{v}$ where $\boldsymbol{v}$ satisfies in a weak sense the equations

$$\partial_t \boldsymbol{v} + \nu \, \mathbf{curl}^2 \boldsymbol{v} + \mathbf{curl}\, \boldsymbol{a} \times \boldsymbol{v} + \mathbf{curl}\, \boldsymbol{v} \times \boldsymbol{a} + \mathbf{curl}\, \boldsymbol{v} \times \boldsymbol{v} + \nabla q \ = \ \boldsymbol{g} \quad (23)$$

and (2) in $\Omega \times (0,T)$, the homogeneous boundary conditions (5), (9) on $\partial\Omega \times (0,T)$ and the initial condition

$$\boldsymbol{v}(\, . \,, 0) \ = \ \boldsymbol{v}_0 \ := \ \boldsymbol{u}_0 - \boldsymbol{a}(\, . \,, 0) \quad (24)$$

in Ω. (Here $\boldsymbol{g} = \boldsymbol{f} - \nu \, \mathbf{curl}^2 \boldsymbol{a} - \mathbf{curl}\, \boldsymbol{a} \times \boldsymbol{a}$.) This guaranties that $\boldsymbol{u}$ satisfies the conditions (14a) and (14b) on $\partial\Omega \times (0,T)$, but it does not solve the question of validity of (14c). The condition (14c) cannot be treated in the same way as (14a) and (14b) because (14c) involves the second derivatives of $\boldsymbol{u}$ and the construction of the weak solution $\boldsymbol{u}$ in $\boldsymbol{W}^{1,2}(\Omega)$ does not directly provide an opportunity to control $\mathbf{curl}^2 \boldsymbol{u} \cdot \boldsymbol{n}$ on $\partial\Omega$. Thus, the boundary condition (14c) enters the weak formulation through a certain linear functional $\boldsymbol{b}$ which, in the case when the weak solution is "smooth", causes that it satisfies (14c) as a "natural boundary condition".

The weak formulation of the problem (23), (2), (24) with the homogeneous boundary conditions (5), (9) is: Suppose that $\boldsymbol{g} \in L^2(0,T;\, \boldsymbol{D}^{-1})$ and $\boldsymbol{b} \in L^2(0,T;\, \boldsymbol{W}^{-1/2,2}(\partial\Omega))$. We search for $\boldsymbol{v} \in L^2(0,T;\, \boldsymbol{D}^1) \cap L^\infty(0,T;\, \boldsymbol{L}^2_\sigma(\Omega))$ such that

$$\begin{aligned} \int_0^T \int_\Omega & [-\boldsymbol{v} \cdot \partial_t \boldsymbol{\phi} + \nu \, \mathbf{curl}\, \boldsymbol{v} \cdot \mathbf{curl}\, \boldsymbol{\phi} \\ & + \mathbf{curl}\, \boldsymbol{a} \times \boldsymbol{v} \cdot \boldsymbol{\phi} + \mathbf{curl}\, \boldsymbol{v} \times \boldsymbol{a} \cdot \boldsymbol{\phi} \\ & + \mathbf{curl}\, \boldsymbol{v} \times \boldsymbol{v} \cdot \boldsymbol{\phi}] \, \mathrm{d}\boldsymbol{x} \, \mathrm{d}t \\ = & \int_\Omega \boldsymbol{v}_0 \cdot \boldsymbol{\phi}(\, . \,, 0) \, \mathrm{d}\boldsymbol{x} + \int_0^T \langle \boldsymbol{g}, \boldsymbol{\phi} \rangle_\Omega \, \mathrm{d}t \\ & + \int_0^T \langle \boldsymbol{b}, \boldsymbol{\phi} \rangle_{\partial\Omega} \, \mathrm{d}t \end{aligned} \quad (25)$$

for all $\boldsymbol{\phi} \in C^\infty(0,T;\, \boldsymbol{D}^1)$ such that $\boldsymbol{\phi}(\, . \,, T) = \mathbf{0}$.

We do not deal with the question of existence of a solution of this weak problem. Nevertheless, we can state that the existence can be established by essentially the same method as the weak solution to the "classical" Navier–Stokes initial–boundary value problem in a bounded domain, as it was shown by E. Hopf already in 1951.

We will finally discuss the question in which sense the weak formulation involves the boundary condition (14c). Given $\boldsymbol{b} \in \boldsymbol{W}^{-1/2,2}(\partial\Omega)$, we define $\alpha_2 \in W^{-3/2,2}(\partial\Omega)$ by the equation

$$\nu \, \langle \alpha_2, \varphi \rangle^*_{\partial\Omega} \; = \; \langle \boldsymbol{b}, \nabla\varphi \rangle_{\partial\Omega} \tag{26}$$

for all $\varphi \in W^{2,2}(\Omega)$. Here $\langle \, . \, , \, . \, \rangle^*_{\partial\Omega}$ denotes the duality between $W^{-3/2,2}(\partial\Omega)$ and $W^{3/2,2}(\partial\Omega)$.

If $\boldsymbol{g} \in \boldsymbol{H}_0$ and $\boldsymbol{v}$ is a "smooth" solution of (25), then we can at first consider the test functions $\boldsymbol{\phi}$ with a compact support and show that there exists a scalar function q such that $\boldsymbol{v}$, q satisfy the equations (23), (2) a.e. in $\Omega \times (0,T)$. Then, following the standard procedure, we can consider all admissible test functions and show, by means of the integration by parts in (25), that $\boldsymbol{v}$ satisfies

$$\int_0^T \int_{\partial\Omega} \nu \, \mathbf{curl}\, \boldsymbol{v} \cdot (\boldsymbol{n} \times \boldsymbol{\phi}) \, \mathrm{d}S \, \mathrm{d}t \\ = \; \int_0^T \langle \boldsymbol{b}, \boldsymbol{\phi} \rangle_{\partial\Omega} \, \mathrm{d}t. \tag{27}$$

Function $\boldsymbol{\phi}$, as an element of $C^\infty(0,T;\, \boldsymbol{D}^1)$, has the form $\boldsymbol{\phi} = \boldsymbol{\phi}_0 + \nabla\varphi$ where $\boldsymbol{\phi}_0 \in C^\infty(0,T;\, \boldsymbol{W}^{1,2}_0(\Omega))$ and $\varphi \in C^\infty(0,T;\, W^{2,2}(\Omega))$, see [1]. Substituting $\boldsymbol{\phi}$ in this form into the left hand side of (27), we obtain:

$$\begin{aligned} &\int_0^T \int_{\partial\Omega} \mathbf{curl}\, \boldsymbol{v} \cdot (\boldsymbol{n} \times \boldsymbol{\phi}) \, \mathrm{d}S \, \mathrm{d}t \\ &= \; -\int_0^T \int_{\partial\Omega} \boldsymbol{n} \cdot (\mathbf{curl}\, \boldsymbol{v} \times \nabla\varphi) \, \mathrm{d}S \, \mathrm{d}t \\ &= \; -\int_0^T \int_{\Omega} \mathrm{div}\, (\mathbf{curl}\, \boldsymbol{v} \times \nabla\varphi) \, \mathrm{d}\boldsymbol{x} \, \mathrm{d}t \\ &= \; -\int_0^T \int_{\Omega} \mathbf{curl}^2 \boldsymbol{v} \cdot \nabla\varphi \, \mathrm{d}\boldsymbol{x} \, \mathrm{d}t \\ &= \; -\int_0^T \langle \mathbf{curl}^2 \boldsymbol{v} \cdot \boldsymbol{n}, \, \varphi \rangle_{\partial\Omega} \, \mathrm{d}t. \end{aligned}$$

The integrand in the last term can also be expressed as $\langle \mathbf{curl}^2 \boldsymbol{v} \cdot \boldsymbol{n}, \, \varphi \rangle^*_{\partial\Omega}$. Thus, (25) and (27) yield

$$\nu \, \langle \alpha_2 - \mathbf{curl}^2 \boldsymbol{v} \cdot \boldsymbol{n}, \, \varphi \rangle^*_{\partial\Omega} \; = \; 0$$

for a.a. $t \in (0,T)$. This equation shows that $\boldsymbol{v}$ satisfies the boundary condition $\mathbf{curl}^2 \boldsymbol{v} \cdot \boldsymbol{n} = \alpha_2$ in the sense of the equality in $W^{-3/2,2}(\partial\Omega)$ for a.a. $t \in (0,T)$. Since $\boldsymbol{u} = \boldsymbol{a} + \boldsymbol{v}$ and $\mathbf{curl}^2 \boldsymbol{a} = 0$ in the sense of distributions in some neighborhood of $\partial\Omega$, function $\boldsymbol{u}$ also fulfills the boundary condition (14c) in the sense of equality in $W^{-3/2,2}(\partial\Omega)$ for a.a. $t \in (0,T)$.

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