Solution of Some Problems of Thermodynamical Systems

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Abstract: The analytical solutions of the systems of Laplace's differential equations of transfer laws in the body with n binding degrees of freedom are presented. It is suggested that potential fields are one-, two- and three-dimensional. Laplace's differential equations are analysed in Cartesian, cylindrical and spherical coordinates taking into account various boundary conditions. There are two specific problems solved in the paper. The solutions presented in the paper increase the possibility of employing these systems in practice.

Key-Words: thermodynamics, Laplace's equations, degrees of freedom, heat transfer.

1 Introduction

It is known that the potential $u = u(x,y,z)$ satisfies Laplace's equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$$

if u is a temperature potential, the potential of the stationary electromagnetic field, a material filtration potential, the potential of the speed of non-vortex non-compressible liquid flow, the potential of the gravitational force in all space points not being in the masses created space, the potential of the electrical charges interaction in all points of charge-free region of space, the potential of the definition of castings quality, and so on.

Therefore the solutions of Laplace's equations with the corresponding boundary conditions attract attention of many researchers [1-8]. In this paper, it is presented a method of the solution of Laplace's differential equations system expressed in the form:

$$\sum_{j=1}^{n} L_{ij}\left(\frac{\partial^2 P_j}{\partial x^2} + \frac{\partial^2 P_j}{\partial y^2} + \frac{\partial^2 P_j}{\partial z^2}\right) = 0, \qquad (1)$$

$$i = 1,2,\ldots,n$$

or corresponding form in cylindrical and spherical coordinates under different boundary conditions.

The system of equations (1) describes the law of transfer for a nonequilibrum system (or body) with n by connected degrees of inner freedom and three-dimensional fields of potentials $P_j = P_j(x,y,z)$, where $P_j = \frac{\partial U}{\partial E_j}$ is a generalized potential; $U = f(E_1, E_2, \ldots, E_n)$ is an internal energy of a system, J; L_{ij} – a coefficient of transfer and $L_{ij} = L_{ji}$. The coefficient L_{ii} is called a principal coefficient of transfer. It characterises conductivity of a thermodynamic system in relation to a charge integrated with potential P_i. Coefficient L_{ij} when $i \neq j$ is called a cross-coefficient. It characterises the influence of j-th charge on potential P_i integrated with it [9].

2 Method of the solution

A system of equations (1) after some transformations:

$$\sum_{j=1}^{n} L_{ij}\left(\frac{\partial^2 P_j}{\partial x^2} + \frac{\partial^2 P_j}{\partial y^2} + \frac{\partial^2 P_j}{\partial z^2}\right) =$$

$$= \sum_{j=1}^{n} L_{ij}\frac{\partial^2 P_j}{\partial x^2} + \sum_{j=1}^{n} L_{ij}\frac{\partial^2 P_j}{\partial y^2} + \sum_{j=1}^{n} L_{ij}\frac{\partial^2 P_j}{\partial z^2} =$$

$$= \frac{\partial^2}{\partial x^2}\left(\sum_{j=1}^{n} L_{ij} P_j\right) + \frac{\partial^2}{\partial y^2}\left(\sum_{j=1}^{n} L_{ij} P_j\right) + \frac{\partial^2}{\partial z^2}\left(\sum_{j=1}^{n} L_{ij} P_j\right)$$

and

$$u_i = \sum_{j=1}^{n} L_{ij} P_j, i = 1,2,3,\ldots n$$

is expressed as

$$\sum_{j=1}^{n} L_{ij}\left(\frac{\partial^2 P_j}{\partial x^2} + \frac{\partial^2 P_j}{\partial y^2} + \frac{\partial^2 P_j}{\partial z^2}\right) =$$

$$= \frac{\partial^2 u_i}{\partial x^2} + \frac{\partial^2 u_i}{\partial y^2} + \frac{\partial^2 u_i}{\partial z^2}$$

So equation (1) in Cartesian coordinates can be written in the form:

$$\frac{\partial^2 u_i}{\partial x^2} + \frac{\partial^2 u_i}{\partial y^2} + \frac{\partial^2 u_i}{\partial z^2} = 0 \tag{2}$$

in cylindrical coordinates - in the form:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_i}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u_i}{\partial \varphi^2} + \frac{\partial^2 u_i}{\partial z^2} = 0 \tag{3}$$

or in spherical coordinates - in the form:

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u_i}{\partial r}\right) + \frac{1}{r^2 \sin\Theta}\frac{\partial}{\partial \Theta}\left(\sin\Theta\frac{\partial u_i}{\partial \Theta}\right) + \\ + \frac{1}{r^2 \sin^2\Theta}\frac{\partial^2 u_i}{\partial \varphi^2} = 0 \tag{4}$$

where

$$u_i = \sum_{j=1}^{n} L_{ij} P_j \tag{5}$$

$i = 1,2,...,n.$

Thus, the procedure of theoretical solutions of transfer differential equations of a thermodynamic system with n binding degrees of freedom is the following:

1. Find the solution u_i of equations Laplace's of a kind (2), (3) or (4) under appropriate boundary conditions. The well-known formulas indicated in works on equations of mathematical physics can be used as the basis for this purpose.

2. After determination of the free members (functions u_i) find generalized potentials P_j of a thermodynamic system. The system of linear (concerning potentials P_j) of equations can be solved by various ways, for example, Cramer's rule can be applied [10]:

$$P_j = \frac{\Delta_{P_j}}{\Delta}, \; j = 1,2,...,n$$

3 Examples of the solving procedures

In one dimension, the equations (2), (3) and (4) have the forms:

$$\sum_{j=1}^{n} L_{ij}\frac{d^2 P_j}{dx^2} = 0 \text{ (in rectangular coordinates)},$$

$$\sum_{j=1}^{n} L_{ij}\frac{d}{dr}\left(r\frac{dP_j}{dr}\right) = 0 \text{ (in cyclical coordinates)},$$

$$\sum_{j=1}^{n} L_{ij}\frac{d}{dr}\left(r^2\frac{dP_j}{dr}\right) = 0 \text{ (in spherical coordinates)}.$$

In two dimensions, the thermodynamic potentials are defined by the following systems of differential equations:

in Cartesian coordinates:

$$\sum_{j=1}^{n} L_{ij}\left(\frac{\partial^2 P_j}{\partial x^2} + \frac{\partial^2 P_j}{\partial y^2}\right) = 0, \tag{6}$$

in cylindrical coordinates:

$$\sum_{j=1}^{n} L_{ij}\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial P_j}{\partial r}\right) + \frac{\partial^2 P_j}{\partial z^2}\right] = 0 \tag{7}$$

or

$$\sum_{j=1}^{n} L_{ij}\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial P_j}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 P_j}{\partial \varphi^2}\right] = 0, \tag{8}$$

and in spherical coordinates:

$$\sum_{j=1}^{n} L_{ij}\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial P_j}{\partial r}\right) + \\ + \frac{1}{r^2 \sin\Theta}\frac{\partial}{\partial \Theta}\left(\sin\Theta\frac{\partial P_j}{\partial \Theta}\right)\right] = 0 \tag{9}$$

The equation (7), noting (5), can be expressed as

$$\frac{\partial^2 u_i}{\partial x^2} + \frac{\partial^2 u_i}{\partial y^2} = 0 \tag{10}$$

$i = 1,2,...,n$

The equations (8) and (9), noting (5), can be expressed analogously. Consequently, the calculation of two-dimensional thermodynamic potentials $P_j = P_j(x,y)$ consists of two steps. In the first step, Laplace's equation (10) is solved using respective boundary conditions. In the second step, the system of equations (5) is solved.

Some problems solved using the recommendations of the work [11] are presented below.

Example 1. The thermodynamic potentials $P = P(r,z)$ in a solid and finite dimensions body ($0 \le r \le a$, $0 \le z \le l$, axis z is symmetry axis of the cylinder) satisfy the following boundary conditions:

$$P_j = 0, z = l, \qquad 0 < r < a;$$

$$P_j = F_j(r), z = 0, \qquad 0 < r < a;$$

$$\frac{\partial P_j}{\partial r} + h_i P_j = 0, r = a, \quad 0 < z < l$$

where $F_j(r)$ *is* a bounded function; $i = j = 1,2,...,n$.

In this case Laplace's equation has the form

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_i}{\partial r}\right)+\frac{\partial^2 u_i}{\partial z^2}=0 \quad (11)$$

where

$$u_i=u_i(r,z)=\sum_{j=1}^{n}L_{ij}P_j(r,z) \quad (12)$$

$i=1,2,...,n$.

The thermodynamic potentials are calculated from the equations (12) using known functions u_i. In this case, it is necessary to solve the equation (11) under the following conditions:

$$u_i=0; z=l; \qquad 0<r<a$$

$$u_i=\sum_{j=1}^{n}L_{ij}F_j(r), z=0, \quad 0<r<a$$

$$\frac{\partial u_i}{\partial r}+h_i u_i=0, r=a, \qquad 0<z<l$$

$i=1,2,...,n.$

Functions u_i are defined by

$$u_i=\sum_{k=1}^{\infty}\frac{2\alpha_{ik}^2 J_0(r\alpha_{ik})sh[(l-z)\alpha_{ik}]}{a^2\left(h_i^2+\alpha_{ik}^2\right)J_0^2(a\alpha_{ik})sh(l\alpha_{ik})}C_{ik} \quad (13)$$

where α_{ik} are positive roots of equation:

$$h_i J_0(a\alpha_i)-\alpha_i J_1(a\alpha_i)=0 \quad (14)$$

$J_0(x)$, $J_1(x)$ are first-order cylindrical functions,

$$C_{ik}=\int_0^a rJ_0(r\alpha_{ik})\sum_{j=1}^{n}L_{ij}F_j(r)dr$$

$i=1,2,...,n.$

If $F_j(r) = P_{j0} = const$, then the equation (13) can be written as

$$u_i=\sum_{j=1}^{n}L_{ij}P_{j0}\sum_{k=1}^{\infty}\frac{2h_i J_0(r\alpha_{ik})sh[(l-z)\alpha_{ik}]}{a\left(h_i^2+\alpha_{ik}^2\right)J_0(a\alpha_{ik})sh(l\alpha_{ik})} \quad (15)$$

The solutions of this equation are used in the equation (12) for obtaining potentials $P_j(r,z)$.

If $P_j = P_j(r,z)$ satisfy the following boundary conditions

$$P_j=f_j(r), z=0, \qquad 0<r<a;$$

$$\frac{\partial P_j}{\partial z}+h_i P_j=0, z=l, \qquad 0<r<a;$$

$$\frac{\partial P_j}{\partial r}+h_i P_j=0, r=a, \qquad 0<z<l,$$

$i=1,2,...,n$

then u_i is calculated:

$$u_i=\sum_{k=1}^{\infty}\frac{A_{ik}}{B_{ik}}\int_0^a rJ_0(r\alpha_{ik})\sum_{j=1}^{n}L_{ij}f_j(r)dr, \quad (16)$$

where α_{ik} are roots of equation (14), $f_j(r)$ are defined functions,

$$A_{ik}=2\alpha_{ik}J_0(r\alpha_{ik})\alpha_{ik}ch[(l-z)\alpha_{ik}]+h_i sh[(l-z)\alpha_{ik}]$$

$$B_{ik}=a^2\left(h_i^2+\alpha_{ik}^2\right)J_0^2(a\alpha_{ik})[\alpha_{ik}ch(l\alpha_{ik})+h_i sh(l\alpha_{ik})]$$

$i=1,2,...n$.

Using the values of u_i, found above, the thermodynamic potentials are calculated from the equations (12).

In three dimensions, the thermodynamic potentials are defined by the following systems of differential equations:

$$\sum_{j=1}^{n}L_{ij}\left(\frac{\partial^2 P_j}{\partial x^2}+\frac{\partial^2 P_j}{\partial y^2}+\frac{\partial^2 P_j}{\partial z^2}\right)=0 \quad (17)$$

By introducing the designation (5), the equation (17) assumes the form of the equation (2).

A specific problem in a case of three-dimensional thermodynamic potentials is given below.

Example 2. The three-dimensional system is given as $0 \le x \le a$, $-b \le y \le b$, $-c \le z \le c$. The potentials $P_j = P_j(x,y,z)$ satisfy the following boundary conditions

$$P_j=P_{j0}=const, x=0$$

$$\frac{\partial P_j}{\partial x}+h_i P_j=0, x=a$$

$$\frac{\partial P_j}{\partial y}-h_i P_j=0, y=-b$$

$$\frac{\partial P_j}{\partial y}+h_i P_j=0, y=b$$

$$\frac{\partial P_j}{\partial z}-h_i P_j=0, z=-c$$

$$\frac{\partial P_j}{\partial z}+h_i P_j=0, z=c$$

$i=1,2,...,n$

In this case the functions u_i are calculated from the equations (2) under the following conditions:

$$u_i=\sum_{j=1}^{n}L_{ij}P_{j0}=V_{i0}=const, x=0$$

$$\frac{\partial u_i}{\partial x}+h_i u_i=0, x=a$$

$$\frac{\partial u_i}{\partial y}-h_i u_i=0, y=-b$$

$$\frac{\partial u_i}{\partial y} + h_i u_i = 0, y = b$$

$$\frac{\partial u_i}{\partial z} - h_i u_i = 0, z = -c$$

$$\frac{\partial u_i}{\partial z} + h_i u_i = 0, z = c$$

$$i = 1,2,...,n$$

The solution is expressed as follows:

$$u_i = \sum_{r=1}^{\infty}\sum_{s=1}^{\infty} \frac{4h_i^2 V_{i0} \cos(\alpha_{ir} y)\cos(\beta_{is} z)\varphi_i(x)}{K_{ir}\left[\left(\beta_{is}^2 + h_i^2\right)c + h_i\right]\cos(\alpha_{ir} b)\cos(\beta_{is} c)} \quad (18)$$

where

$$\varphi_i(x) = \frac{h_i sh[l_i(a-x)] + l_i ch[l_i(a-x)]}{h_i sh(al_i) + l_i ch(al_i)}$$

$$K_{ir} = \left(\alpha_{ir}^2 + h_i^2\right)b + h_i$$

$$l_i^2 = \alpha_{ir}^2 + \beta_{ir}^2$$

α_{ir} and β_{is} are positive roots of equations:

$$\alpha_i tg(b\alpha_i) = h_i$$

$$\beta_i tg(c\beta_i) = h_i$$

$$i = 1,2,...,n .$$

The solutions considerably facilitate the numerical methods [12-14] put into solutions of the thermodynamics systems with n binding degrees of freedom and increase the possibility of employing these systems in practice.

4 Conclusions

A procedure for the solution of the systems differential equations of transfer laws in the body with n binding degrees of freedom is presented. The one-, two- and three-dimensional potential fields are analyzed. Laplace's differential equations are analysed in Cartesian, cylindrical and spherical coordinates taking into account various boundary conditions. The solutions considerably increase the possibility of employing Laplace's equations in thermodynamics.

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