Optimization of Insulation Shape for Minimal Radiative Heat Loss

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Abstract: - Given a prescribed volume of homogeneous insulation material, how should this be distributed around the surface of a body to minimize the rate of heat loss from the body? The answer is to apply the calculus of variations to minimize the functional representing the heat loss, while satisfying the heat conduction equation and the boundary conditions, including the radiation boundary condition on the free surface. The nonlinearity of this boundary condition requires novel treatment. The analytical solution for the heat loss from a spherical body is found by means of a regular perturbation procedure.

Key-Words: - Optimization, Thermal Insulation, Heat Conduction, Calculus of Variations, Variational Principle, Regular Perturbation

1 Introduction

The problem of determining that shape of insulation of given mass which minimizes the rate of heat loss from the body it surrounds has been addressed by the author in previous work [1-3], but never for the case of a radiation boundary condition on the outer surface of the insulation. Yet this case is of prime importance in some applications where minimizing heat loss is critical, such as occur in, for example, the insulation of spacecraft.

We do not review the directly applicable previous work in detail here, as this has already been done in [1-3] and the interested reader is referred to this work. Related recent work on optimisation in problems of heat transfer has been undertaken. In particular Meric and Kul [4] have applied an adjoint problem in a manner that has considerable similarities to that applied in this paper. Razealos [5], and Aziz and Kraus [6] have investigated the optimization of cooling fins. The reader requiring an introduction to heat conduction can do no better than consult the classic text by Carslaw and Jaeger [7].

In fact the approach applied here is analogous to that used in [1], in that the calculus of variations is similarly applied, but there is the important difference, that the nonlinear radiation boundary condition that is fourth-order in the temperature must be addressed. We have adopted the same notation as in [1] to assist the reader.

As in [1] a field variable that is adjoint to the temperature is introduced by formulating a functional defined on the unknown outer boundary based upon the radiation boundary condition. Manipulation of this functional enables the derivation of the first-order variation in the rate of heat loss in an expression that excludes variations in the derivative of the first-order variation of the temperature, thereby allowing an explicit necessary condition for a minimum to be established.

Having thus obtained the optimal boundary value problem, we proceed to solve it approximately for the case where the body is a sphere, where the thickness of the insulation is small compared with the radius of the sphere. We apply a regular perturbation method akin to that deployed in [3], solving to the lowest order, and obtain an explicit analytical solution.

2 Problem Formulation

We define a boundary-value problem representing the heat flow across the layer of insulation D surrounding a body B. Let x_i denote Cartesian co-ordinates in D and let $\theta(x_i)$ be the temperature field within D. This satisfies the classical heat conduction equation for a homogeneous material, namely Laplace's equation,

$$\nabla^2\theta = 0 \ \ in\ D. \qquad (1)$$

Let the inner and outer surfaces of D be S_1 and S_2 respectively. We suppose that on the inner surface, the surface of the body B, a Dirichlet condition applies. The temperature field is a prescribed function of position, so that

$$\theta = \Theta(q_1, r_1) \quad on\ S_1, \tag{2}$$

where q_1, r_1 are curvilinear co-ordinates specifying a point on S_1. We suppose that this function is differentiable in both the co-ordinates. On the outer surface it is assumed that a radiation boundary condition applies, namely

$$\underline{n} \cdot \nabla\theta + \beta\left(\theta^4 - \theta_A^4\right) = 0, \tag{3}$$

where θ_A is the ambient temperature in the region outside the insulation layer, and β is a constant given by

$$\beta = \mu\sigma, \tag{4}$$

where μ is the gray body emissivity and σ is Stefan's Constant.

The Dirichlet condition is believed to be appropriate for the body surface, in that usually the heated domain is kept at constant temperature, but this condition might be changed to either a Robin condition applying Newton's Law of Cooling in linearised form, or another radiation boundary condition, if desired. It is expected that the treatments of [1] and that of here could be extended to address such more complex scenarios.

3 Statement of Optimization Problem

We seek a necessary condition for minimal rate of heat loss from the body B. This is accomplished by variation of the outer surface S_2 of the insulation layer, subject to the isoperimetric constraint

$$\int_D dV = V_0 = const., \tag{5}$$

and to the boundary-value problem (1-3). Let us denote the optimal solution for the temperature field, outer surface of the insulation, and optimal insulation domain by the addition of a subscript '0' in each case, so that the optimal solution is θ_0, S_{20}, D_0, respectively.

Consider a weak variation about the optimal solution as follows:

$$\theta = \theta_0 + \varepsilon\theta_1 + o(\varepsilon) \quad in \quad D \cup D_0, \tag{6}$$

$$x_i^{(2)} = x_{0i}^{(2)}(q_2, r_2) + \varepsilon f(q_2, r_2) n_{0i}(q_2, r_2) + o(\varepsilon), \tag{7}$$

on S_{20}, where $0 < \varepsilon << 1$, and it is assumed that θ and θ_0 may be analytically continued into $D \cup D_0$, while each satisfies the boundary-value problem (1-3) in its own domain. Equation (7) describes a variation of the surface S_2 about the optimal surface S_{20}. The variables q_2, r_2 are curvilinear co-ordinates on S_{20} such that $x_i = x_{0i}^{(2)}(q_2, r_2)$ are Cartesian co-ordinates of a point P on S_{20}. The co-ordinates $x_i^{(2)}$ are the Cartesian co-ordinates of the point on S_2 corresponding to that point P, which is on the normal to S_{20} at P. The function f and the normal components n_{0i} are assumed to be differentiable functions of q_2, r_2.

In terms of f the isoperimetric constraint (5) is

$$\int_{S_{20}} f\, dS = 0. \tag{8}$$

For any solution θ, D given by the variations (6) and (7), the rate of heat loss Q is given by the functional $I[\theta; D]$, where

$$Q = I[\theta, D] = \kappa \int_{S_1} \underline{n} \cdot \nabla\theta\, dS. \tag{9}$$

Substitution of the variations (6) and (7) into (9), and defining

$$\Delta I = I[\theta, D] - I[\theta_0, D_0], \tag{10}$$

the result

$$\Delta I = \varepsilon\kappa \int_{S_1} \underline{n} \cdot \nabla\theta_1\, dS + o(\varepsilon) \tag{11}$$

follows immediately.

If S_{20} exists and satisfies the above smoothness conditions, the coefficient of ε in equation (11) is to vanish for all variations (6) and (7) satisfying condition (8). It remains to manipulate the right-hand side of equation (11) by variational analysis to obtain this necessary condition in terms of θ_0 and S_{20}.

4 Variational Analysis

Let ϕ be any twice differentiable function in D and form an integral functional over S_2 given by

$$M[\phi,\theta]=\int_{S_2}\phi\left\{\underline{n}\cdot\nabla\theta+\beta\left(\theta^4-\theta_A^4\right)\right\}dS=0. \tag{12}$$

Here ϕ is akin to a Lagrange multiplier for the boundary condition (3) viewed as a constraint. Use of the Divergence Theorem enables the first term of (12) to be rewritten as two integrals over the domain D, and the surface S_2. Thus

$$0=\int_D\left(\phi\nabla^2\theta+\nabla\phi\cdot\nabla\theta\right)dV-\int_{S_1}\phi\underline{n}\cdot\nabla\theta dS + \int_{S_2}\beta\phi\left(\theta^4-\theta_A^4\right)dS. \tag{13}$$

It is now convenient to impose the equation

$$\nabla^2\phi=0 \quad in\ D, \tag{14}$$

and the condition

$$\phi=1 \quad on\ S_1, \tag{15}$$

which enable the elimination of certain variations below. Then equation (13) reduces to

$$\int_{S_1}\underline{n}\cdot\nabla\theta dS=\int_D\nabla\phi\cdot\nabla\theta dV+\int_{S_2}\beta\phi\left(\theta^4-\theta_A^4\right)dS. \tag{16}$$

Application of the variations (6) and (7) now gives

$$\begin{aligned}\varepsilon\int_{S_1}\underline{n}\cdot\nabla\theta_1 dS=&-\int_{S_1}\underline{n}\cdot\nabla\theta_0 dS+\int_{D_0}\nabla\phi\cdot\nabla\theta_0\\&+\varepsilon\int_{D_0}\nabla\phi\cdot\nabla\theta_1 dV+\varepsilon\int_{S_{20}}f\nabla\phi\cdot\nabla\theta_0 dS\\&+\int_{S_{20}}\beta\phi\left(\theta_0^4-\theta_A^4\right)dS+\varepsilon\int_{S_{20}}4\beta\phi\theta_0^3\theta_1 dS\\&+\varepsilon\int_{S_{20}}\beta f\left[\frac{\partial\phi}{\partial n_0}\left(\theta_0^4-\theta_A^4\right)+4\phi\theta_0^3\frac{\partial\theta_0}{\partial n_0}\right]dS+o(\varepsilon).\end{aligned} \tag{17}$$

Some straightforward manipulations involving the Divergence Theorem and using (14) and (15) and the governing boundary-value problem yield the relations

$$\int_{D_0}\nabla\phi\cdot\nabla\theta_0 dV=\int_{S_{20}}\phi\underline{n}\cdot\nabla\theta_0 dS+\int_{S_1}\underline{n}\cdot\nabla\theta_0 dS, \tag{18}$$

and

$$\int_{D_0}\nabla\phi\cdot\nabla\theta_1 dV=\int_{S_{20}}\theta_1\underline{n}\cdot\nabla\phi dS. \tag{19}$$

Up to first order in ε the right-hand side of equation (17) comprises a zero-th order term and a first-order term. The zero-th order term reduces to

$$\int_{S_{20}}\phi\left[\underline{n}\cdot\nabla\theta_0+\beta\left(\theta_0^4-\theta_A^4\right)\right]dS=0, \tag{20}$$

vanishing because of condition (3) for the optimal solution. Thus we are left only with the first-order term, so that

$$\begin{aligned}\varepsilon\int_{S_1}\underline{n}\cdot\nabla\theta_1 dS=&\varepsilon\int_{S_{20}}f\nabla\phi\cdot\nabla\theta_0 dS\\&+\varepsilon\int_{S_{20}}\theta_1\left(\underline{n}\cdot\nabla\phi+4\beta\phi\theta_0^3\right)dS\\&+\varepsilon\int_{S_{20}}\beta f\left[\frac{\partial\phi}{\partial n_0}\left(\theta_0^4-\theta_A^4\right)+4\phi\theta_0^3\frac{\partial\theta_0}{\partial n_0}\right]dS.\end{aligned} \tag{21}$$

5 Optimality Conditions

We have now derived an expression for the integral in (11) and since this is to vanish for all variations (6), (7) satisfying (8), it follows that the conditions

$$\underline{n}\cdot\nabla\phi+4\beta\theta_0^3\phi=0, \tag{22}$$

and

$$\nabla\phi\cdot\nabla\theta_0+\beta\left[\frac{\partial\phi}{\partial n_0}\left(\theta_0^4-\theta_A^4\right)+4\phi\theta_0^3\frac{\partial\theta_0}{\partial n_0}\right]=\lambda_0, \tag{23}$$

hold, where λ_0 is a constant. Use of (3) and (22) simplifies this optimality condition to

$$\nabla\phi\cdot\nabla\theta_0-8\beta^2\theta_0^3\phi\left(\theta_0^4-\theta_A^4\right)=\lambda_0. \tag{24}$$

Note the marked nonlinearity of conditions (22) and (24) in contrast with the corresponding conditions in [1].

To sum up, the coupled optimal boundary value problem in θ_0, ϕ to be solved comprises (1)-(3), (14), (15), (22), and (24). We explore simple spherical problems in the next section.

6 Spherical Problem

Consider a spherical body B of radius a. If a spherical optimal solution exists then the optimal surface is a sphere of radius b, with

$$\frac{4}{3}\pi\left(b^3 - a^3\right) = V_0. \tag{25}$$

Let us state the boundary-value problem. Equation (1) is

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2 \frac{d\theta_0}{dr}\right) = 0. \tag{26}$$

The boundary condition (2) for a constant temperature α_1 on S_1 is

$$\theta_0 = \alpha_1, \quad on\ r = a. \tag{27}$$

The radiation boundary condition (3) becomes

$$\frac{d\theta_0}{dr} + \beta\left(\theta_0^4 - \theta_A^4\right) = 0 \ \ on\ r = b. \tag{28}$$

Similarly equation (14) is

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2 \frac{d\phi}{dr}\right) = 0. \tag{29}$$

Condition (15) is

$$\phi = 1 \quad on\ r = a. \tag{30}$$

Condition (22) becomes

$$\frac{d\phi}{dr} + 4\beta\theta_0^3\phi = 0 \quad on\ r = b. \tag{31}$$

Finally, the optimality condition is

$$\frac{d\phi}{dr}\frac{d\theta_0}{dr} - 8\beta^2\theta_0^3\phi\left(\theta_0^4 - \theta_A^4\right) = \lambda_0 \quad on\ r = b. \tag{32}$$

The solutions of equations (26) and (29) are:

$$\theta_0 = -\frac{A}{r} + B, \qquad \phi = -\frac{C}{r} + D, \tag{33}$$

where A, B, C, D are constants to be determined by the boundary conditions.

Conditions (27) and (28) yield

$$\alpha_1 = -\frac{A}{b} + B, \tag{34}$$

and

$$\frac{A}{b^2} + \beta\left\{\left(-\frac{A}{b} + B\right)^4 - \theta_A^4\right\} = 0. \tag{35}$$

Conditions (30) and (31) yield

$$-\frac{C}{a} + D = 1, \tag{36}$$

and

$$\frac{C}{b^2} + 4\beta\left(-\frac{A}{b} + B\right)^3\left(-\frac{C}{b} + D\right) = 0. \tag{37}$$

The final optimality condition (32) becomes

$$\frac{AC}{b^4} - 8\beta^2\left(-\frac{A}{b} + B\right)^3\left(-\frac{C}{b} + D\right) \times\left\{\left(-\frac{A}{b} + B\right)^4 - \theta_A^4\right\} = \lambda_0. \tag{38}$$

Equations (25), (34)-(38) thus constitute six equations to be solved simultaneously for the six unknowns b, A, B, C, D, λ_0. The nonlinearity of the equations requires their solution to be undertaken numerically in most cases.

The rate of heat loss from the body is given by equation (9) as

$$Q = -\kappa\frac{d\theta_0}{dr}\cdot 4\pi a^2 = -4\pi\kappa A. \tag{39}$$

We will seek an analytical perturbation solution for the case of a thin insulation layer.

7. Perturbation Solution for Sphere

To demonstrate that solutions of (25), (34)-(38) may be found, consider the case where the volume V_0 is small and the insulation layer is thin, as may be represented by the relation

$$b = a(1 + \varpi k), \tag{40}$$

where $0 < \varpi << 1$. A regular perturbation solution may be sought by writing:

$$\begin{aligned} A &= A^{(0)} + \varpi A^{(1)} + o(\varpi), \\ B &= B^{(0)} + \varpi B^{(1)} + o(\varpi), \\ C &= C^{(0)} + \varpi C^{(1)} + o(\varpi), \\ D &= D^{(0)} + \varpi D^{(1)} + o(\varpi), \\ \lambda &= \lambda^{(0)} + \varpi \lambda^{(1)} + o(\varpi). \end{aligned} \tag{41}$$

The zero-th order solution is readily shown to be:

$$\begin{aligned} A^{(0)} &= -a^2 \beta(\alpha_1^4 - \theta_A^4), \\ B^{(0)} &= \alpha_1 - a\beta(\alpha_1^4 - \theta_A^4), \\ C^{(0)} &= -4a^2 \beta \alpha_1^3, \\ D^{(0)} &= 1 - 4a\beta\alpha_1^3, \\ \lambda^{(0)} &= -4\beta^2(\alpha_1^4 - \theta_A^4)\alpha_1^3. \end{aligned} \tag{42}$$

The corresponding zero-th order rate of heat loss is thus from (39) simply

$$Q = 4\pi a^2 \kappa \beta (\alpha_1^4 - \theta_A^4), \tag{43}$$

which is, of course, the correct result for zero thickness of insulation. The perturbation method could readily be extended to higher orders.

8. Discussion

In this paper we have shown that the radiation boundary condition (3) has greatly increased the complexity of the optimization problem to be solved over the corresponding linear problem with the Robin boundary condition, as described in [1]. For a general body, where numerical solution will probably be needed, it may even be prudent to undertake some preliminary optimization using the simpler Robin condition, before refining the solution with the exact radiation boundary condition.

In fact we believe that this paper brings to a watershed the series of papers [1]-[3] written over the last twenty-five years approximately. The papers have undertaken the exploration by analytical means of the problem of minimizing the rate of heat loss of a body by arranging the distribution of a given mass or equivalently volume, of thermal insulation. The analysis thus far has been applicable to bodies of smooth shapes and has not addressed discrete layers of insulation, but has allowed only smooth and continuous variation of the insulation shape. The work so far is thus open to the criticism that, while of interest academically, it is not yet widely applicable to many real problems.

However, we believe that we have clearly shown that shape optimization can yield benefits in reducing heat loss and that the problems we have solved are well-posed mathematically. This encourages us to consider the application of numerical methods to these problems for bodies with more awkward geometry, e.g. having edges, such as cuboids, or smooth corners where the radius of curvature is comparable to the thickness of insulation.

We believe, too, that considering the related and perhaps more realistic problem where one has a finite quantity of insulation, typically in a roll, and wishes to know how to deploy it in discretely varying numbers of layers over a surface, such as a loft floor, over which the temperature varies, may be of interest.

Another open area for optimisation work possibly involving shape variation is that of thermal cycling. This is an important problem in extreme environments such as in space, where energy is expended to maintain the interior of a spacecraft at a constant temperature – necessary because fuel would freeze, or electronics malfunction etc. Here the radiation boundary condition is especially relevant.

We see the next step as confirming some of our results thus far with the aid of a suitable heat conduction model. At present we expect to explore the use of finite element methods to this end, but we have interest too in the boundary element method.

The analytical models can clearly serve as very useful verification data for codes that seek to determine the unknown optimal boundary. Indeed, further analytical work may continue to be useful. For example, such tools as asymptotic analysis may allow us to obtain good approximate solutions for some of the more difficult geometrical situations. That would be invaluable in offering more demanding verification examples.

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