Positive Solutions of Urysohn Integral Equations

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Abstract: In this paper we using a new version of compression-expansion fixed point theorems of Krasnoleskii's type for Mönch operators, we establish conditions which ensure the existence of positive solutions of Urysohn integral equations.

Key–Words: Nonlinear integral equations in abstract spaces, Krasnoselskii's fixed point theorem, Mönch's fixed point theorem, Urysohn integral equations

1 Introduction

In analyzing nonlinear phenomena many mathematical models arise to problems for which only nonnegative solutions make sense. Therefore, attention of the researchers is capture by the studies of existence of positive solutions of nonlinear equations and the main tools to do this are the compression-expansion fixed point theorems of Krasnoselskii's type. We remind here this result

Theorem 1 *Let $\mathcal{X}$ be a Banach space, endowed with the norm $|\cdot|$, $\mathcal{K}$ be a cone in $\mathcal{X}$, U_1, U_2 be open subsets of $\mathcal{X}$ with $0_{\mathcal{X}} \in U_1$, $\overline{U}_1 \subset U_2$. Assume that $T : \mathcal{K} \cap \left(\overline{U}_2 \backslash U_1\right) \to \mathcal{K}$ is a completely continuous operator such that either*

$$\begin{cases} |T(u)| \leq |u| \text{ on } \mathcal{K} \cap \partial U_1 \\ |T(u)| \geq |u| \text{ on } \mathcal{K} \cap \partial U_2 \end{cases} \quad (1)$$

or

$$\begin{cases} |T(u)| \geq |u| \text{ on } \mathcal{K} \cap \partial U_1 \\ |T(u)| \leq |u| \text{ on } \mathcal{K} \cap \partial U_2. \end{cases} \quad (2)$$

Then T has at least one fixed point in $\mathcal{K} \cap \left(\overline{U}_2 \backslash U_1\right)$.

This technique has been applied in the literature to scalar equations, when $X = \mathbb{R}$, see [6, 8, 9], and recently to nonlinear equations in Banach spaces, see [3, 12, 4]. In all this works, the nonlinear integral equations were studied assuming that the associated operator is compact ar completely continuous. Our existence result do not require completely continuity of T and are based upon the continuation theorem of Mönch [10] and the corresponding compression theorem.

The aim of this paper is to establish conditions which guarantee the existence of positive solutions of abstract nonlinear integral equation

$$u(t) = \int_0^h G(t, s, u(s))\, ds, \quad t \in [0, h],$$

where the value of G is an element of a Banach space. The main tool in our proof is a new version of compression fixed point theorem of Krasnoselskii's type

Theorem 2 *Let $\mathcal{X}$ be a Banach space, endowed with the norm $|\cdot|$, $\mathcal{K}$ be a cone in $\mathcal{X}$. Assume that the norm $|\cdot|$ is increasing with respect to $\mathcal{K}$, $0 < r < R$ and $T : \mathcal{K} \cap \left(\overline{\Omega}_R \backslash \Omega_r\right) \to \mathcal{K}$ is a Mönch operator such that*

$$\begin{cases} |T(u)| \leq |u| \text{ on } \mathcal{K} \cap \partial \Omega_R \\ |T(u)| \geq |u| \text{ on } \mathcal{K} \cap \partial \Omega_r \end{cases} \quad (3)$$

Then T has at least one fixed point in $\mathcal{K} \cap \left(\overline{\Omega}_R \backslash \Omega_r\right)$.

The proof of this result may be found in [5].

In that follows we present some background results.

1.1 Ordered Banach spaces

Let X be a linear space. By a cone K of X we understand a convex subset of X such that $\lambda K \subset K$ for all $\lambda \geq 0$ and $K \cap (-K) = \{0_X\}$, where 0_X is zero element of X.

There is a one-to-one correspondence between the cones of X and the order relations on X which are compatible with linear structure of X. More precisely, any cone K in X induces a partial order structure on X, i.e.

$$x \leq y \text{ if and only if } y - x \in K.$$

A linear space endowed with a cone is called an ordered linear space, so an ordered Banach space is a Banach space with a closed cone. For an ordered Banach space, we say that the norm $|\cdot|$ is increasing with respect to K if $|x| \leq |y|$ whenever $0 \leq x \leq y$.

Let $C(0, h; X)$ be the Banach space of all continuous functions from $[0, h]$ onto X with the supremum norm $|\cdot|_\infty$ defined by

$$|u|_\infty = \sup_{t \in [0,h]} |u(t)|.$$

Notice $C(0, h; K)$ is a cone of $C(0, h; X)$ and if the norm of X is increasing with respect to K, then so is the norm $|\cdot|_\infty$ with respect to $C(0, h; K)$.

Basic facts about ordered Banach spaces can be found in [2, 11].

1.2 Operators of Mönch type

Let X be a Banach space, endowed with the norm $|\cdot|$, $K \subset X$ be a closed subset of X and $U \subset K$ be an open subset of K. The operator $T : \overline{U} \to K$ is **Mönch operator** whit respect to $x_0 \in U$ if T is continuous and for some $C \subset U$ we have

$$C \subset \overline{cv}(\{x_0\} \cup T(C)) \text{ implies } \overline{C} \text{ compact.}$$

Here, by $\overline{cv}(M)$ we denote the convex, linear encloser of the set M.

In that follow for $M \subset U$ a bounded set, we denote by $\alpha(M)$ the Kuratowski measure of noncompactness on $X$and for $r > 0$ we consider

$$\Omega_r = \{x \in X;\ \ |x| < r\}.$$

Example 1 Let X be a Banach space, endowed with the norm $|\cdot|$, $R > 0$, $D = \Omega_R \times \Omega_R$ and $u, v \in \Omega_R$. We consider the continuous map $f : \overline{D} \to X$ with

$$f(u, y) = v,\ \ y \in \overline{\Omega}_R.$$

and the operator $T : \overline{D} \to X \times X$, defined by

$$T(x, y) = (u, f(x, y)),\ \ (x, y) \in D.$$

Then T is Mönch operator with respect to (u, v).

Indeed, let $C \subset \overline{cv}(\{(u, v)\} \cup T(C))$. Since $(x, y) \in C$, there exists $\lambda \in [0, 1]$ and $(x^\circ, y^\circ) \in C$ such that

$$\begin{aligned} (x, y) &= (1 - \lambda)(u, v) + \lambda T(x^\circ, y^\circ) \\ &= (1 - \lambda)(u, v) + \lambda(u, f(x^\circ, y^\circ)). \end{aligned}$$

Then $x = x^\circ = u$ and

$$y = (1 - \lambda) v + \lambda f(u, y^\circ) = v.$$

Therefore $C = \{(u, v)\}$ is a compact set. So, T is Mönch operator. $\square$

Interest for the operators of Mönch type is given by the fixed point result due to Mönch [10] and the continuation analogue to this one.

Theorem 3 (Mönch's fixed point theorem) *Let X be a Banach space, U be a nonempty, closed, convex subset of X and $T : U \to U$ a continuous map satisfying*

$$C \subset U, \overline{C} = \overline{cv}(\{x_0\} \cup T(C)) \Rightarrow \overline{C} \text{ compact}.$$

for some $x_0 \in U$. Then T has a fixed point.

This result contains, as particular cases, the fixed point theorems of Krasnoselskii, Darbo and Sadovskii for self-maps of closed bounded and convex set. Other details and applications of Theorem 3 can be found in [1, 2, 11, 13].

2 Positive Solutions of Urysohn Integral Equations

Let X be a ordered Banach space, with respect to the cone $K \subset X$ and X are endowed with the increasing norm $|\cdot|$.

In this paragraph we will apply Theorem 2 to establish some hypothesis which guarantee the existence of positive solutions of the Urysohn nonlinear integral equation

$$u(t) = \int_0^h G(t, s, u(s))\, ds,\ \ t \in [0, h], \qquad (4)$$

where $G : [0, h] \times [0, h] \times \Omega_R \to X$.

By positive solution of (4) we mean a function $u \in C(0, h; K)$ satisfying (4).

This part of the paper is inspired by [13], where using Theorem 3, authors studies Volterra and Urysohn integral equations. In order to establish at least one positive solution of (4) we introduce the following conditions:

(U_1) for any $t \in [0,h]$, the map $G_t = G(t,\cdot,\cdot)$ is L^1-Carathéodory and

$$\sup_{t\in[0,h]} \int_0^h \sup_{|x|<R} |G_t(s,x)|\, ds < \infty;$$

(U_2) $\lim_{t\to t^*} \int_0^h \sup_{|x|<R} |G_t(s,x) - G_{t^*}(s,x)|\, ds = 0;$

(U_3) there exists $\omega : [0,h]\times[0,h]\times[0,2R] \to \mathbb{R}$ such that for each $t\in[0,h]$ the map ω_t is L^1-Carathéodory,

$$\alpha(G(t,s,M)) \le \omega(t,s,\alpha(M))$$

for a.e $s\in[0,h]$, $M\subset U$, and the unique map $\phi\in C(0,h;[0,2R])$ satisfying

$$\phi(t) \le 2\int_0^h \omega(t,s,\phi(s))\, ds, \;\; t\in[0,h]$$

is $\phi\equiv 0$;

(U_4) there exist $\mu\in(0,1)$ and $[a,b]\subset[0,1]$ such that for any $t\in[0,h]$, $t'\in[a,b]$

$$\mu G(t,s,x) \le G(t',s,x)$$

for a.e $s\in[0,h]$ and for each $x\in U$;

(U_5) there is $\Xi : K\to K$ such that

$$\Xi(x) \le G(t,s,u(s)), \;\; s\in[a,b], t\in[0,h]$$

for any $u\in C(0,h;K)$ and $x\in K$ with $x < u(s)$, $s\in[a,b]$;

(U_6) there is $\Phi : K\to K$ with

$$\Phi(x) \le \Phi(y) \;\text{ whenever } x\le y$$

and

$$|G(t,s,x)| \le |\Phi(x)|, \;\; t,s\in[0,h], x\in U;$$

(U_7) there exists $0<r<R$ such that

$$\inf_{\substack{x\in K\\ |x|=\mu r}} |\Xi(x)| \ge \frac{r}{b-a},$$

$$\sup_{\substack{x\in C(0,h;K)\\ |x|_\infty=R}} \int_0^h |\Phi(x)| \le R.$$

Theorem 4 *If* (U_1) – (U_7) *are satisfied, then* (4) *has at least one solution* $u\in C(0,h;K)$ *such that*

$$r \le |u|_\infty \le R \tag{5}$$

and

$$\mu u(t) \le u(t') \text{ for } t\in[0,h] \text{ and } t'\in[a,b]. \tag{6}$$

Proof: We will apply Theorem 2 for the Banach space $\mathcal{X} = C(0,h;X)$, the cone

$$\mathcal{K} = \{u\in C(0,h;K); \quad \mu u(t)\le u(t'), \; t\in[0,h], t'\in[a,b]\}$$

and the operator $T : K_{r,R}\to\mathcal{K}$ defined by

$$T(u)(t) = \int_0^h G(t,s,u(s))\, ds, \;\; t\in[0,h],$$

where

$$K_{r,R} = \{u\in\mathcal{K}; \;\; r\le|u|_\infty\le R\} = \mathcal{K}\cap\left(\overline{\Omega}_R\backslash\Omega_r\right).$$

Let $t\in[0,h]$ and $t'\in[a,b]$. By (U_4) we have

$$\begin{aligned}\mu Tu(t) &= \mu\int_0^h G(t,s,u(s))\, ds \\ &= \int_0^h \mu G(t,s,u(s))\, ds \\ &\le \int_0^h G(t',s,u(s))\, ds \\ &= Tu(t').\end{aligned} \tag{7}$$

Therefore $Tu\in\mathcal{K}$. So, T is well defined. Following the line from [13], it can be proof that (U_1) – (U_3) ensure that T is Mönch operator.

Let $u\in\mathcal{K}\cap\partial\Omega_r$, i.e. $u\in C(0,h;\mathcal{K})$, $|u|_\infty = r$ and $\mu u(t)\le u(t')$ for all $t\in[0,h]$, $t'\in[a,b]$. We consider $t^*\in[0,h]$ and we assume that $\mu u(t^*)\le u(t')$ for each $t'\in[a,b]$. By (U_5) results

$$\Xi(\mu u(t^*)) \le G(t,s,u(s))\, ds, \;\; s\in[a,b].$$

We have

$$\begin{aligned}Tu(t) &= \int_0^h G(t,s,u(s))\, ds \\ &\ge \int_a^b G(t,s,u(s))\, ds \\ &\ge \int_a^b \Xi(\mu u(t^*))\, ds \\ &= (b-a)\,\Xi(\mu u(t^*)).\end{aligned}$$

Then

$$|Tu(t)| \geq (b-a) \cdot \inf_{\substack{x \in K \\ |x|=\mu r}} \Xi(x).$$

Now, using the first inequality from (U_7), results $|Tu|_\infty \geq r = |u|_\infty$. Hence the first condition from (3) is satisfy.

Let $u \in \mathcal{K} \cap \partial\Omega_R$, i.e. $u \in C(0,h;\mathcal{K})$ with $|u|_\infty = R$. We have

$$\begin{aligned} |Tu(t)| &= \left| \int_0^h G(t,s,u(s))\,ds \right| \\ &\leq \int_0^h |G(t,s,u(s))|\,ds \\ &\leq \int_0^h |\Phi(u(s))|\,ds. \end{aligned}$$

By second inequality of (U_7), we obtain

$$|Tu|_\infty \leq \sup_{\substack{x \in C(0,h;\mathcal{K}) \\ |x|_\infty = R}} \int_0^h |\Phi(x)| \leq R = |u|_\infty.$$

Hence the second condition from (3) is verify. Thus Theorem 2 applies. □

3 Conclusion

Theorem 4 ensure the existence of positive solutions of (4) and, moreover, it offers the localization of this solutions, i.e. (5) implies that solutions of (4) are lie in the shell $K_{r,R}$ and (6) gives an essential properties of this solutions. This informations can be very useful in numerical approach to solving integral equations.

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