Saddle Point Formulation of the Quasistatic Contact Problems with Friction

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Abstract: The paper is concerned with the numerical solution of the quasi-variational inequality modelling a contact problem with Coulomb friction. After discretization of the problem by mixed finite elements and with Lagrangian formulation of the problem by choosing appropriate multipliers, the duality approach is improved by splitting the normal and tangential stresses. The novelty of our approach in the present paper consists in the splitting of the normal stress and tangential stress, which leads to a better convergence of the solution, due to a better conditioned stiffness matrix. This better conditioned matrix is based on the fact that the obtained diagonal blocks matrices, contain coefficients of the same size order. For the saddle point formulation of the problem, using static condensation, we obtain a quadratic programming problem. **Key words**: Contact problem with Coulomb friction, dual mixed formulation, mixed finite element, saddle point problem, quadratic programming, Schur complement.

CLASSICAL AND VARIATIONAL FORMULATION

Let $\Omega \subset \mathbb{R}^d, d = 2$ or 3, the domain occupied by a linear elastic body with a Lipschitz boundary Γ. Let Γ_1, Γ_2 and Γ_C be three open disjoint parts of Γ such that $\Gamma = \overline{\Gamma}_1 \cup \overline{\Gamma}_2 \cup \overline{\Gamma}_C$, $\overline{\Gamma}_1 \cap \overline{\Gamma}_C = \varnothing$ and mes $(\Gamma_1) > 0$. We assume that the body is subjected to volume forces of density $\boldsymbol{f} \in (L^2(\Omega))^d$, to surface traction of density $\boldsymbol{h} \in (L^2(\Gamma_2))^d$ and is held fixed on Γ_1. The Γ_C denotes a contact part of boundary where unilateral contact and Coulomb friction condition between Ω and perfectly rigid foundation are considered. We denote by $\boldsymbol{u} = (u_1, \ldots, u_d)$ the displacement field, $\boldsymbol{\varepsilon} = (\varepsilon_{ij}(\mathbf{u})) = \left(\frac{1}{2}(u_{i,j} + u_{j,i})\right)$ the strain tensor and $\boldsymbol{\sigma} = (\sigma_{ij}(\mathbf{u})) = (a_{ijkl}\varepsilon_{kl}(\mathbf{u}))$ the stress tensor with the usual summation convention, where $i, j, k, l = 1, \ldots, d$. For the normal and tangential components of the displacement vector and stress vector, we use the following notation: $\mathbf{u}_N = u_i \cdot n_i$, $\boldsymbol{u}_T = \boldsymbol{u} - \boldsymbol{u}_N \cdot \mathbf{n}$, $\boldsymbol{\sigma}_N = \boldsymbol{\sigma}_{ij} u_i n_j$, $(\boldsymbol{\sigma}_T)_i = \boldsymbol{\sigma}_{ij} n_j - \boldsymbol{\sigma}_N \cdot n_i$, where $\mathbf{n} = (n_i)$ is the outward unit normal vector to Γ.

We denote by $g \in C(\bar{\Gamma}_C)$, $g \geq 0$ the initial gap between the body and the rigid foundation and lets us denote by $\boldsymbol{f}$ and $\boldsymbol{h}$ the density of body and traction forces, respectively. We assume that $a_{ijkl} \in L^\infty(\Omega)$, $l \leq i, j, k, l \leq d$, with usual condition of symmetry and elasticity, that is

$$a_{ijkl} = a_{jikl} = a_{klij}, \quad 1 \leq i, j, k, l \leq d,$$

$$\text{and} \quad \exists\, m_0 > 0,\ \forall\, \xi = (\xi_{ij}) \in \mathbb{R}^{d^2},$$

$$\xi_{ij} = \xi_{ji},\ 1 \leq i,\ j \leq d,\ a_{ijkl}\, \xi_{ij}\, \xi_{kl} \geq m_0 |\xi|^2\ .$$

In this conditions, the fourth-order tensor $\boldsymbol{a} = (a_{ijkl})$ is invertible a.e., on Ω and if we denote its inverse by $\boldsymbol{b} = (b_{ijkl})$, we have $\boldsymbol{\varepsilon}_{ij}(\boldsymbol{u})) = (b_{ijkl}\sigma_{kl}(\boldsymbol{u}))$, $i, j, k, l = 1, \dots, d$.

The classical contact problem with dry friction in elasticity, in the particular case, is with the normal stress $\sigma_N(u)$ and Γ_C is assumed known and considered as obeying the normal compliance law, is the following

Find $\boldsymbol{u} = \boldsymbol{u}(x, t)$ such that $\boldsymbol{u}(0, \cdot) = \boldsymbol{u}^0(\cdot)$ in Ω and for all $t \in [0, T]$,

$$-\mathrm{div}\,\boldsymbol{\sigma}(\boldsymbol{u}) = \boldsymbol{f}, \quad \text{in } \Omega \tag{1}$$

$$\boldsymbol{\sigma}_{ij}(\boldsymbol{u}) = a_{ijkl} \cdot \varepsilon_{kl}(\boldsymbol{u}), \quad \text{in } \Omega \tag{2}$$

$$\boldsymbol{u} = 0 \quad \text{on } \Gamma_1 \tag{3}$$

$$\boldsymbol{\sigma} \cdot \boldsymbol{n} = \boldsymbol{h} \quad \text{on } \Gamma_2, \tag{4}$$

the contact condition:

$$u_N \leq g,\ \boldsymbol{\sigma}_N(u) \leq 0,\ (u_N - g)\boldsymbol{\sigma}_N(u) = 0 \quad \text{on } \Gamma_C \tag{5}$$

and Coulomb friction on Γ_C:

$$\begin{aligned} &\|\sigma_T(u)\| \leq \mu_F |\sigma_N(u)|, \text{ such that :} \\ &- \text{if } \|\sigma_T(u)\| < \mu_F |\sigma_N(u)| \Rightarrow u_T = 0 \\ &- \text{if } \|\sigma_T(u)\| = \mu_F |\sigma_N(u)| \Rightarrow\ \exists\, \alpha \geq 0, \end{aligned} \tag{6}$$

such that $\dot{u}_T = -\alpha\sigma_T$ where $\boldsymbol{u}^0$ denotes the initial displacement of the body. Supposing that a positive coefficient $\mu_F \in L^\infty(\Gamma_C)$, $\mu_F \geq \mu_0$ a.e. on Γ_C of Coulomb friction is given, we introduce the space of *virtual displacements*

$$V = \left\{ v \in (H^1(\Omega))^2 | v = 0 \text{ on } \Gamma_1 \right\}$$

and its convex subset of *kinematically admissible displacements*

$$K = \{ v_N \in V | v_N \equiv v \cdot n \leq g \text{ on } \Gamma_C \}.$$

We assume that the normal force on Γ_C is known (as normal compliance) so that one can evaluate the non-negative slip bound $p \in L^\infty(\Gamma_C)$ as a product of the friction coefficient and the normal stress, i.e. $p = \mu_F\lambda_1$, when λ_1 is the normal stress. We assume that normal interface response (the normal compliance law) is:

$$\sigma_N(u) = -c_N(u_N - g)^{m_N}$$

where c_N and m_N are material constant depending on interface properties.

(P_1) *Find* $u \in K$ *such that* $J(u) = \min_{v \in K} J(v)$.

The minimized functional representing the total potential energy of the body has the form:

$$J(v) = \frac{1}{2} a(v, v) - L(v) + \overline{j}(v)$$

where: the bilinear form a is given by

$$a(v, w) = \int_\Omega a_{ijkl}\varepsilon_{ij}(v)\varepsilon_{kl}(w)dx$$

linear functional L is given by:

$$L(v) = \int_\Omega fvdx + \int_{\Gamma_2} hvds;$$

the sublinear functional $\overline{j}$ is given by:

$$\overline{j}(v) = \int_{\Gamma_C} p\,|v_T|\,ds + \int_{\Gamma_C} c_N(u - g)^{m_n} v_N ds$$

where $v_T \in (L^\infty(\Gamma_C))^2$ denotes the tangent vector to boundary Γ. It is known that the problem (P_1) is non-differentiable due to the sublinear term $\overline{j}$, and has a unique solution [9].

The variational formulation, in the quasi-static case, is equivalent to the quasi-variational inequality:
(P_2) *Find* $u(x, t) \in K \times [0, T]$ s. t. $a(u, v - \dot{u}) + \overline{j}(v - \dot{u}) \geq (L, u - \dot{v})\ \ \forall v \in K, \forall t \in [0, T], T > 0$, with initial conditions $u(x, 0) = u_0, \dot{u}(x, 0) = u_1$.

The existence and uniqueness of the solution of this quasi-variational inequality are proven under the assumption that μ_F is sufficiently small and $mes(\Gamma_0) > 0$ [16].

The *Lagrangian formulation* of the problem (P_1) is given by introducing
$L : V \times \Lambda_1 \times \Lambda_2 \to \mathbb{R}$, with

$$L(v, \mu_1, \mu_2) = \frac{1}{2} a(v, v) - L(v) + \\ + \langle \mu_1, v_N - g \rangle \int_{\Gamma_C} \mu_2 v_T ds$$

where $\Lambda_1 = \{\mu_1 \in H^{-\frac{1}{2}}(\Gamma_C) | \mu_1 \geq 0\}$, $\Lambda_2 = \{\mu_2 \in L^\infty(\Gamma_C) |\ \ |\mu_2| \leq p \text{ on } \Gamma_C\}$.

The space $H^{-\frac{1}{2}}(\Gamma_C)$ is the dual of

$$H^{\frac{1}{2}}(\Gamma_C) = \\ = \{\gamma \in L^2(\Gamma_C) |\, \exists v \in V\ s.t.\ \gamma = v_N \text{ on } \Gamma_C\}$$

and the ordering $\mu_1 \geq 0$ means, in the variational form, that $\langle \mu_1, v_N - g \rangle \leq 0, \ \forall v \in K$, where $\langle \cdot, \cdot \rangle$ denotes the duality pairing between $H^{-\frac{1}{2}}(\Gamma_C)$ and $H^{\frac{1}{2}}(\Gamma_C)$. Since $L^2(\Gamma_C)$ is dense in $H^{-\frac{1}{2}}(\Gamma_C)$, the duality pairing $\langle \cdot, \cdot \rangle$ is represented by a scalar product in $L^2(\Gamma_C)$.

The Lagrange multipliers $\mu_1, \ \mu_2$ are considered as functionals on the contact part of the boundary Γ. It is important that the Lagrange multipliers do have mechanical significance: while the first one is related to the non-penetration conditions and represents the normal stress, the second one removes the non-differentiability of the sublinear functional

$$j_2(v) = \sup_{\mu_2 \in \Lambda_2} \int_{\Gamma_C} \mu_2 v_T ds$$

and represents the tangential stress.

The equivalence between the problem (P_1) and the lagrangian formulation is given by:

$$\inf_{v \in K} J(v) = \inf_{v \in V} \sup_{\mu_1 \in \Lambda_1, \mu_2 \in \Lambda_2} L(v, \mu_1, \mu_2).$$

By the mixed variational formulation of the problem (P_1) we mean a saddle point problem:

$$(P_3) \textit{find } (w, \lambda_1, \lambda_2) \in V \times \Lambda_1 \times \Lambda_2 \textit{ such that}$$

$$L(w, \mu_1, \mu_2) \leq L(w, \lambda_1, \lambda_2) \leq L(v, \lambda_1, \lambda_2),$$
$$\forall (v, \mu_1, \mu_2) \in V \times \Lambda_1 \times \Lambda_2.$$

It is known that (P_3) has a unique solution [2] and its first component $w = u \in K$ solves (P_1) and the Lagrange multipliers $\lambda_1, \ \lambda_2$ represent the normal and tangential contact stress on the contact part of the boundary, respectively.

Remarks.

1^0. For the contact problem with Coulomb friction, we use the formula $p \equiv \mu_F \lambda_1$, for the slip bound on the contact boundary Γ_C , where $\lambda_1 \equiv \lambda_1(p)$ is the normal stress on Γ_C and μ_F is the coefficient of friction. Unfortunately this problem cannot be solved as a convex quadratic programming problem because p is an a priori parameter in (P_3), while λ_1 is an a posteriori one.

2^0. Because we can consider the mapping $\Psi : \Lambda_1 \to \Lambda_1, \ \Psi : p \to \lambda_1 \equiv \lambda_1(p)$ defined by the second component of the solution for the contact problem with given friction (P_3), the solution of the contact problem with Coulomb friction will be defined as a fixed point of this mapping in Λ_1. Results concerning the existence of fixed points for sufficiently small friction coefficients may be found in [17].

THE TIME DISCRETIZATION AND FINITE ELEMENT APPROXIMATIONS OF THE CONTACT PROBLEMS WITH COULOMB FRICTION

Let us consider a partition $(t^0, t^1, \ldots, t^N)$ of time interval $[0, T]$ and also the incremental formulation obtained by using the backward finite difference approximation of the time derivative of u.

If we use $u_h^k = u_h(x, t^k)$, $\Delta u_h^k = u_h^{k+1} - u_h^k$, $\Delta t^k = t^{k+1} - t^k$, $\dot{u}_h(t^{k+1}) = \Delta u_h^k / \Delta t$, $f_h^k = f_h(k\Delta t)$, for $k = 0, 1, \ldots, N-1$ where $\Delta t = \dfrac{T}{N}$, we obtain, at each time t^k, the following quasi-variational inequality

$$\textit{find } \Delta u_h^k \in V_h \text{ s.t.} \tag{7}$$
$$a(\Delta u_h^k, v_h - \Delta u_h^k) + \overline{j}(u_h^k + \Delta u_h^k, v_h - \Delta u_h^k) \geq$$
$$\geq \Delta L^k(v_h - \Delta u_h^k) - F(u_h^k, v_h - \Delta u_h^k),$$

$\forall \, v_h \in K_h$, where $F(u_h^k, v_h - \Delta u_h^k) = a(u_h^k, v_h - \Delta u_h^k) - L^k(v_h - \Delta u_h^k)$.

The time discretization of the problem (P_2) follows. For a given load history the quasi-static problem is approximated by a sequence of incremental problems (7); although every problem (7) is a static one, it requires appropriate updating of the displacements, so loads for each increment and so we obtain the following sequence:

(P_2^{ht}) *Find* $\mathbf{u} \in \mathbf{K_h}$, *for each time* t^k *such that* $\mathbf{J}(\mathbf{u}) = \min_{\mathbf{v} \in \mathbf{K_h}} \mathbf{J}(\mathbf{v})$,

where $\mathbf{u} \equiv \Delta u_h^k$, $\mathbf{v} \equiv v_h$, $J(\boldsymbol{v}) = \frac{1}{2} \boldsymbol{v}^T \boldsymbol{K} \boldsymbol{v} - \boldsymbol{v}^T \boldsymbol{f} + \boldsymbol{p}^T |\boldsymbol{T} \boldsymbol{v}|$ and $K_h = \{\boldsymbol{v} \in \mathbb{R}^n | \boldsymbol{N} \boldsymbol{v} \leq \boldsymbol{g}\}$. Here, we by denote $\boldsymbol{K} \in \mathbb{R}^{n \times n}$ the positive definite stiffness matrix, $\boldsymbol{f} \in \mathbb{R}^n$ is the load vector,

$\boldsymbol{p} \in \mathbb{R}^m$ is the nodal slip bounds vector for contact nodes. The matrices $\boldsymbol{N}, \boldsymbol{T} \in \mathbb{R}^{m\times n}$ contain the rows of the normal and tangential vectors in the contact nodes, respectively, and $\boldsymbol{g} \in \mathbb{R}^m$ is the vector of distances between the contact nodes and the rigid foundation.

The *matrix form of the Lagrangian* for the problem (P_2^{ht}), at each time t^k is:

$$L(\boldsymbol{v}, \boldsymbol{\mu}_1, \boldsymbol{\mu}_2) = \frac{1}{2}\boldsymbol{v}^T\boldsymbol{K}\mathbf{v} - \boldsymbol{f}^T\mathbf{v} + \boldsymbol{\mu}_2^T\boldsymbol{T}\boldsymbol{v} + \boldsymbol{\mu}_1^T(\boldsymbol{N}\boldsymbol{v} - \mathbf{g})$$

where $\boldsymbol{\mu}_1 \in \boldsymbol{\Lambda}_1$, $\boldsymbol{\mu}_2 \in \boldsymbol{\Lambda}_2$ are the Lagrange multipliers and $\boldsymbol{\Lambda}_1 = \{\boldsymbol{\mu}_1 \in \mathbb{R}^m | \boldsymbol{\mu}_1 \geq \boldsymbol{0}\}$, $\boldsymbol{\Lambda}_2 = \{\boldsymbol{\mu}_2 \in \mathbb{R}^m | |\boldsymbol{\mu}_2| \leq \boldsymbol{p}\}$.

The algebraic mixed formulation of (P_2^{ht}) is:

Find $(\boldsymbol{v}, \boldsymbol{\mu}_1, \boldsymbol{\mu}_2) \in \mathbb{R}^n \times \boldsymbol{\Lambda}_1 \times \boldsymbol{\Lambda}_2$ such that

$$\boldsymbol{K}\boldsymbol{u} = \boldsymbol{f} - \boldsymbol{N}^T\boldsymbol{\lambda}_1 - \boldsymbol{T}^T\boldsymbol{\lambda}_2 \qquad (8)$$

$$(\boldsymbol{N}\boldsymbol{u} - \boldsymbol{g})^T(\boldsymbol{\lambda}_1 - \boldsymbol{\mu}_1) + \boldsymbol{u}^T\boldsymbol{T}^T(\boldsymbol{\lambda}_2 - \boldsymbol{\mu}_2) \geq 0, \qquad (9)$$

$(\boldsymbol{\mu}_1, \boldsymbol{\mu}_2) \in \boldsymbol{\Lambda}_1 \times \boldsymbol{\Lambda}_2$.

After computing $\mathbf{u}$ from (8) and substituting u into (9), we obtain the *algebraic dual formulation*, for each time t^k, i.e.,

$$\min\left\{\frac{1}{2}\boldsymbol{\lambda}^T\boldsymbol{A}\boldsymbol{\lambda} - \boldsymbol{\lambda}^T\boldsymbol{B}\right\} \; s.t. \qquad (10)$$

$$\boldsymbol{\lambda}_1 \geq 0, \;\; |\boldsymbol{\lambda}_1| \leq \boldsymbol{g}, \; \boldsymbol{\lambda} = (\boldsymbol{\lambda}_1^T, \boldsymbol{\lambda}_2^T)^T,$$

where

$$\boldsymbol{A} = \begin{pmatrix} \boldsymbol{N}\boldsymbol{K}^{-1}\boldsymbol{N}^T & \boldsymbol{N}\boldsymbol{K}^{-1}\boldsymbol{T}^T \\ \boldsymbol{T}\boldsymbol{K}^{-1}\boldsymbol{N}^T & \boldsymbol{T}\boldsymbol{K}^{-1}\boldsymbol{T}^T \end{pmatrix} \text{ and}$$

$$\boldsymbol{B} = \begin{pmatrix} \boldsymbol{N}\boldsymbol{K}^{-1}\boldsymbol{f} - \boldsymbol{g} \\ \boldsymbol{T}\boldsymbol{K}^{-1}\boldsymbol{f} \end{pmatrix}.$$

The problem (10) is a *quadratic programming* problem that can be solved by several efficient algorithms.

ALGORITHM FOR SOLVING THE ALGEBRAIC DUAL FORMULATION

It is known that the matrix $\mathbf{A}$ is ill conditioned, and its diagonal blocks corresponding to the normal and tangential stress are closely related to the dual Schur complement whose spectrum is not so ill conditioned.

The performance of duality algorithms may be improved if we split the normal and tangential stress. To exploit this fact, let us introduce a new notation for the natural block structure of the dual Hessian $\mathbf{A}$ and for the matrices $\mathbf{B}$, corresponding to normal stress, and $\boldsymbol{\lambda}_N$, corresponding to tangential stress $\boldsymbol{\lambda}_T$:

$$\boldsymbol{A} = \begin{pmatrix} \boldsymbol{A}_{11} & \boldsymbol{A}_{12} \\ \boldsymbol{A}_{21} & \boldsymbol{A}_{22} \end{pmatrix}, \;\; \boldsymbol{B} = \begin{pmatrix} \boldsymbol{B}_1 \\ \boldsymbol{B}_2 \end{pmatrix}, \;\; \boldsymbol{\lambda} = \begin{pmatrix} \boldsymbol{\lambda}_1 \\ \boldsymbol{\lambda}_2 \end{pmatrix}.$$

The Gauss-Seidel algorithm for problem (10), leads to a sequence of approximations of $\boldsymbol{\lambda}_{\boldsymbol{N}}^{(i)}$ and $\boldsymbol{\lambda}_{\boldsymbol{T}}^{(0)}$ as follows:

Initialize $\boldsymbol{\lambda}_{\boldsymbol{N}}^{(0)} := \boldsymbol{g}^{(0)}$; $\boldsymbol{\lambda}_{\boldsymbol{T}}^{(0)} := 0$; $\boldsymbol{i} := 0$; $\boldsymbol{t}^0 := 0$;

repeat

$\boldsymbol{i} := \boldsymbol{i} + 1, \quad \boldsymbol{t}^{k+1} = \Delta t^{k+1} + t^k$;

$\lambda_{\boldsymbol{T}}^{(\boldsymbol{i})} := (\boldsymbol{D} - \boldsymbol{L})^{-1} \cdot (\boldsymbol{U} \cdot \lambda_{\boldsymbol{T}}^{(\boldsymbol{i}-1)} + \boldsymbol{B}_2)$ such that $|\lambda_{\boldsymbol{T}}| \leq \mu_F \cdot \lambda_{\boldsymbol{N}}^{(\boldsymbol{i}-1)}$;

$\lambda_{\boldsymbol{N}}^{(\boldsymbol{i})} := \boldsymbol{A}_{11}^{-1} \cdot (\boldsymbol{B}_1 - \boldsymbol{A}_{12} \cdot \lambda_{\boldsymbol{T}}^{(\boldsymbol{i})})$ such that $\lambda_{\boldsymbol{N}} \geq 0$

until $|\boldsymbol{\lambda}^{(i)} - \boldsymbol{\lambda}^{(i-1)}| \leq$ Tol, and $t^{k+1} \leq T$;

where Tol is the chosen tolerance, the matrices $\boldsymbol{D}$, $-\boldsymbol{L}$ and $-\boldsymbol{U}$ representing the diagonal, strictly lower triangular and strictly upper triangular parts of $\boldsymbol{S}$, respectively, with $\boldsymbol{S} = \boldsymbol{A}_{21}\boldsymbol{A}_{11}^{-1}\boldsymbol{A}_{12} - \boldsymbol{A}_{22}$ being the Schur complement of the matrix $\boldsymbol{A}$.

Conclusions. The novelty of our approach in the present paper consists in the splitting, within the known algorithm, of the normal stress and tangential stress, which leads to a better convergence of the solution, due to a better conditioned stiffness matrix. This better conditioned matrix is got due to the fact that the obtained diagonal blocks matrices, contain coefficients of the same size order.

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