# Multi-Time Euler-Lagrange Dynamics*

CONSTANTIN UDRISTE
University Politehnica of Bucharest
Department of Mathematics
Splaiul Indpendentei 313
ROMANIA

IONEL TEVY
University Politehnica of Bucharest
Department of Mathematics
Splaiul Indpendentei 313
ROMANIA

Abstract: This paper introduces new types of Euler-Lagrange PDEs required by optimal control problems with performance criteria involving curvilinear or multiple integrals subject to evolutions of multidimensional-flow type. Particularly, the anti-trace multi-time Euler-Lagrange PDEs are strongly connected to the multi-time maximum principle. Section 1 comments the limitations of classical multi-variable variational calculus. Sections 2-3 refer to variational calculus with gradient variations and curvilinear or multiple integral functionals. Section 4 is dedicated to the study of the properties of multi-time Euler-Lagrange operator (affine changing of the Lagrangian, anti-trace multi-time Euler-Lagrange PDEs and new conservation laws). Section 5 formulates an application to multi-time rheonomic dynamics. Section 6 underlines the importance of the anti-trace multi-time Euler-Lagrange PDEs.

*Key–Words:* gradient variations, multi-time Euler-Lagrange PDEs, multi-time maximum principle.
Typing manuscripts, LaTeX

## 1 Overview of classical multi-variable variational calculus

The foundations of variational calculus have been built using the classic Lagrange variation of an admissible function, but this determines some limitations that are not suitable for multi-time control theory. The most important limitation comes from the fact that the classical multi-variable variational calculus cannot be applied directly to create a multi-time maximum principle. In fact, the functionals given as multiple integrals, subject to general variation functions produce multi-variable Euler-Lagrange or Hamilton PDEs containing a trace (total divergence), which is not convenient for the conservation of the Hamiltonian. Indeed, the Hamiltonian is not a first integral for the multi-variable Hamilton PDEs, even in the autonomous case.

Our multi-time control theory successfully overcomes the previous limitations [3]-[8]. This theory requires $m$-needle-shaped variations and complete integrability conditions as core issues. Adding new ideas in variational calculus via the gradient variations in curvilinear and multiple integrals and the anti-trace Euler-Lagrange or Hamilton PDEs, we have justified a multi-time maximum principle which is similar to the Pontryaguin maximum principle.

The previous two types of functional variations can be considered as ”celebrities” of the optimization theory, although the use of classical variations is hardly compatible with amplitude constraints, while $m$-needle-shaped variations are barely used in smooth optimization problems.

## 2 Curvilinear integral functional and gradient variations

Let $x^i$, $i = 1, \dots, n$ denote the field variables on the *target space* $R^n$, let $t^\alpha$, $\alpha = 1, \dots, m$ be the multi-time variables on the *source space* $R^m$, and let $x^i_\alpha = \dfrac{\partial x^i}{\partial t^\alpha}$ be the partial velocities. In this context, the jet bundle of order one is the manifold $J^1(R^m, R^n) = \{(t^\alpha, x^i, x^i_\alpha)\}$.

Assume we are given a smooth completely integrable 1-form

$$L = L_\beta(x(t), x_\gamma(t))dt^\beta,\ \beta, \gamma = 1, ..., m,\ t \in R^m_+,$$

called *autonomous Lagrangian 1-form*. The Lagrangian 1-form is determined by the *Lagrange covector field* $L_\beta(x(t), x_\gamma(t))$. The complete integrability conditions are

$$\frac{\partial L_\beta}{\partial x^i}\frac{\partial x^i}{\partial t^\lambda} + \frac{\partial L_\beta}{\partial x^i_\gamma}\frac{\partial x^i_\gamma}{\partial t^\lambda} = \frac{\partial L_\lambda}{\partial x^i}\frac{\partial x^i}{\partial t^\beta} + \frac{\partial L_\lambda}{\partial x^i_\gamma}\frac{\partial x^i_\gamma}{\partial t^\beta}.$$

Let $\Gamma_{0,t_0}$ be an arbitrary piecewise $C^1$ curve joining the points $0$ and $t_0$ in $R^m_+$, and let $\Omega_{0,t_0}$ be a par-

*WSEAS Transactions on Mathematics, ., . (2007), ...-....

allelepiped represented as the closed interval $0 \leq t \leq t_0$, fixed by the diagonal opposite points $0(0,...,0)$ and $t_0 = (t_0^1,...,t_0^m)$ in $R_+^m$. We fix the multi-time $t_0 \in R_+^m$, and two points $x_0, x_1 \in R^n$ and we introduce a new problem of the calculus of variations asking to find an $m$-sheet $x^*(\cdot) : \Omega_{0,t_0} \to R^n$ that minimizes the functional (curvilinear integral)

$$J(x(\cdot)) = \int_{\Gamma_{0,t_0}} L_\beta(x(t), x_\gamma(t))dt^\beta, \ \alpha, \beta = 1,...,m,$$

among all functions $x(\cdot)$ satisfying the conditions $x(0) = x_0, x(t_0) = x_1$, using $C^1$ gradient variation functions constrained by boundary conditions.

**Fundamental question**: how can we characterize that function $x^*(\cdot)$ which is the solution of the previous variational problem?

**Theorem 1 (multi-time non-homogeneous Euler-Lagrange PDEs).** *If the m-sheet $x^*(\cdot)$ minimizes the functional $J(x(\cdot))$ in the previous sense, then $x^*(\cdot)$ is a solution of the multi-time non-homogeneous Euler-Lagrange PDEs*

$$\frac{\partial L_\beta}{\partial x^i} - \frac{\partial}{\partial t^\gamma}\frac{\partial L_\beta}{\partial x_\gamma^i} = c_{\beta i}, \ i = 1,...,n, \ \beta, \gamma = 1,...,m. \quad (E-L)_1$$

Here we have a system of $nm$ second order PDEs with $n$ unknown functions $x^i(\cdot)$. Theorem 1 shows that if we can solve the $(E-L)_1$ PDEs system, then the minimizer of the functional $J^\alpha$ (assuming that it exists) will be among the solutions.

**Proof. Step 1**. Select a smooth gradient variation $y_\alpha : \Omega_{0,t_0} \to R^{nm}$, $y_\alpha^i(0) = 0$, $y_\alpha^i(t_0) = 0$ with the primitive $y : \Omega_{0,t_0} \to R^n$, $y^i(0) = 0$, $y^i(t_0) = 0$. We add the conditions (complete integrability conditions of 1-forms $L$ and of gradient variations)

$$\frac{\partial L_\beta}{\partial x^i}y_\lambda^i + \frac{\partial L_\beta}{\partial x_\gamma^i}\frac{\partial y_\gamma^i}{\partial t^\lambda} = \frac{\partial L_\lambda}{\partial x^i}y_\beta^i + \frac{\partial L_\lambda}{\partial x_\gamma^i}\frac{\partial y_\gamma^i}{\partial t^\beta}, \ \frac{\partial y_\beta^i}{\partial t^\gamma} = \frac{\partial y_\gamma^i}{\partial t^\beta}. \quad (CI)_1$$

Let us define the parameter $\epsilon = (\epsilon^1,...,\epsilon^m)$ and

$$J^\alpha(\epsilon) = J^\alpha(x(\cdot) + \epsilon^\beta y_\beta(\cdot)),$$

for $\epsilon \in R^m$ and we write $x(\cdot) = x^*(\cdot)$, i.e., we omit the superscript $*$.

We remark that the perturbed function $x(\cdot) + \epsilon^\beta y_\beta(\cdot)$ takes the same values as $x(\cdot)$ at the diagonal points (endpoints) 0 and $t_0$. Since $x(\cdot)$ is a minimizer, we can write

$$J(\epsilon) \geq J(x(\cdot)) = J(0).$$

In this way, the function $J(\epsilon)$ has a minimum at the point $\epsilon = 0$, and consequently this must be a critical point, i.e., $\frac{\partial J}{\partial \epsilon^\beta}(0) = 0$.

**Step 2**. We compute the partial derivatives $\frac{\partial J}{\partial \epsilon^\beta}$ of the function

$$J(\epsilon) = \int_{\Gamma_{0,t_0}} L_\sigma\Big(x(t) + \epsilon^\lambda y_\lambda(t), \frac{\partial x}{\partial t^\gamma}(t) + \epsilon^\lambda \frac{\partial y_\lambda}{\partial t^\gamma}(t)\Big)\, dt^\sigma$$

and we write

$$0 = \frac{\partial J}{\partial \epsilon^\beta}(0) = \int_{\Gamma_{0,t_0}} \Big(\frac{\partial L_\sigma}{\partial x^i}(x(t), x_\gamma(t))y_\beta^i(t) + \frac{\partial L_\sigma}{\partial x_\gamma^i}(x(t), x_\gamma(t))\frac{\partial y_\beta^i}{\partial t^\gamma}(t)\Big)\, dt^\sigma.$$

To process this formula we use the conditions for variations, writing

$$0 = \int_{\Gamma_{0,t_0}} \Big(\frac{\partial L_\sigma}{\partial x^i}y_\beta^i + \frac{\partial L_\sigma}{\partial x_\gamma^i}\frac{\partial y_\gamma^i}{\partial t^\beta}\Big)\, dt^\sigma$$
$$= \int_{\Gamma_{0,t_0}} \Big(\frac{\partial L_\beta}{\partial x^i}y_\sigma^i + \frac{\partial L_\beta}{\partial x_\gamma^i}\frac{\partial y_\gamma^i}{\partial t^\sigma}\Big)\, dt^\sigma$$
$$= \int_{\Gamma_{0,t_0}} \Big(\frac{\partial L_\beta}{\partial x^i}\delta_\sigma^\gamma - \frac{\partial}{\partial t^\sigma}\frac{\partial L_\beta}{\partial x_\gamma^i}\Big) y_\gamma^i dt^\sigma + \int_{\Gamma_{0,t_0}} \frac{\partial}{\partial t^\sigma}\Big(\frac{\partial L_\beta}{\partial x_\gamma^i}y_\gamma^i\Big)\, dt^\sigma.$$

**Step 3**. Since the 1-forms

$$\Big(\frac{\partial L_\beta}{\partial x^i}\delta_\sigma^\gamma - \frac{\partial}{\partial t^\sigma}\frac{\partial L_\beta}{\partial x_\gamma^i}\Big)\, dt^\sigma$$

must be pullbacks of some 1-forms $d(A_{\beta i}^\gamma)$, we can evaluate the first curvilinear integral using the formula

$$\frac{\partial}{\partial t^\gamma}(y^i d(A_{\beta i}^\gamma)) = \frac{\partial y^i}{\partial t^\gamma}d(A_{\beta i}^\gamma) + y^i \frac{\partial}{\partial t^\gamma}d(A_{\beta i}^\gamma).$$

One obtains $\int_{\Gamma_{0,t_0}} y^i \frac{\partial}{\partial t^\gamma}d(A_{\beta i}^\gamma) = 0$, for all variations $y_\gamma^i$ with $y^i(0) = 0, y^i(t_0) = 0, y_\alpha^i(0) = 0, y_\alpha^i(t_0) = 0$ satisfying $(CI)_1$ and for all curves $\Gamma_{0,t_0}$ in the curvilinear integrals. Consequently, $\frac{\partial}{\partial t^\gamma}\Big(\frac{\partial L_\beta}{\partial x^i}\delta_\sigma^\gamma - \frac{\partial}{\partial t^\sigma}\frac{\partial L_\beta}{\partial x_\gamma^i}\Big) = 0$. Therefore the $(E-L)_1$ PDEs hold for all multi-times $t \in \Omega_{0,t_0}$.

**Remark**. There are two other ways to process the previous formula, but finally they are inconvenient because of the complete integrability conditions:

1) the creation of a divergence operator, used usually in the variational calculus for multiple integrals,

$$\int_{\Gamma_{0,t_0}} \left( \frac{\partial L_\sigma}{\partial x^i} - \frac{\partial}{\partial t^\gamma} \frac{\partial L_\sigma}{\partial x^i_\gamma} \right) y^i_\beta dt^\sigma$$

$$+ \int_{\Gamma_{0,t_0}} \frac{\partial}{\partial t^\gamma} \left( \frac{\partial L_\sigma}{\partial x^i_\gamma} y^i_\beta \right) dt^\sigma;$$

2) the introduction a procedure suggested by our multi-time maximum principle theory, based on the idea $\frac{\partial y^i_\beta}{\partial t^\gamma} = \frac{\partial y^i_\gamma}{\partial t^\beta}$, namely

$$0 = \int_{\Gamma_{0,t_0}} \left( \frac{\partial L_\sigma}{\partial x^i} \delta^\gamma_\beta y^i_\gamma + \frac{\partial L_\sigma}{\partial x^i_\gamma} \frac{\partial y^i_\beta}{\partial t^\gamma} \right) dt^\sigma$$

$$= \int_{\Gamma_{0,t_0}} \left( \frac{\partial L_\sigma}{\partial x^i} \delta^\gamma_\beta y^i_\gamma + \frac{\partial L_\sigma}{\partial x^i_\gamma} \frac{\partial y^i_\gamma}{\partial t^\beta} \right) dt^\sigma$$

$$= \int_{\Gamma_{0,t_0}} \left( \frac{\partial L_\sigma}{\partial x^i} \delta^\gamma_\beta - \frac{\partial}{\partial t^\beta} \frac{\partial L_\sigma}{\partial x^i_\gamma} \right) y^i_\gamma dt^\sigma$$

$$+ \int_{\Gamma_{0,t_0}} \frac{\partial}{\partial t^\beta} \left( \frac{\partial L_\sigma}{\partial x^i_\gamma} y^i_\gamma \right) dt^\sigma.$$

## 3 Multiple integral functional and gradient variations

Assume we are given a smooth *Lagrangian*

$$L(x(t), x_\gamma(t)),\ t \in R^m_+.$$

We fix the multi-time $t_0 \in R^m_+$, the parallelepiped $\Omega_{0,t_0} \subset R^m_+$ with the diagonal opposite points $0(0,...,0)$ and $t_0 = (t^1_0,...,t^m_0)$, and two points $x_0, x_1 \in R^n$.. We introduce a new problem of the calculus of variations asking to find an $m$-sheet $x^*(\cdot) : \Omega_{0,t_0} \to R^n$ that minimizes the functional (multiple integral)

$$J(x(\cdot)) = \int_{\Omega_{0,t_0}} L(x(t), x_\gamma(t)) dt^1...dt^m,$$

among all functions $x(\cdot)$ satisfying the conditions $x(0) = x_0, x(t_0) = x_1$, using gradient variation functions (satisfying not only usual boundary conditions but also the complete integrability conditions).

**Fundamental question**: how can we characterize that function $x^*(\cdot)$ which is the solution of the previous variational problem?

**Theorem 2 (multi-time non-homogeneous Euler-Lagrange PDEs)**. *If the m-sheet $x^*(\cdot)$ minimizes the functional $J(x(\cdot))$ in the previous sense, then $x^*(\cdot)$ is a solution of the multi-time non-homogeneous Euler-Lagrange PDEs*

$$\frac{\partial L}{\partial x^i} - \frac{\partial}{\partial t^\gamma} \frac{\partial L}{\partial x^i_\gamma} = c_i,\ i = 1,...,n,\ \gamma = 1,...,m. \qquad (E-L)_2$$

Here we have a system of $n$ second order PDEs with $n$ unknown functions $x^i(\cdot)$. Theorem 2 shows that if we can solve the $(E-L)_2$ PDEs system, then the minimizer of the functional $J$ (assuming it exists) will be among the solutions.

**Proof. Step 1**. Select a smooth gradient variation $y_\alpha : \Omega_{0,t_0} \to R^{nm}$, $y^i_\alpha|_{\partial\Omega_{0,t_0}} = 0$ with the primitive $y : \Omega_{0,t_0} \to R^n$, $y^i|_{\partial\Omega_{0,t_0}} = 0$. We add the complete integrability conditions

$$\frac{\partial y^i_\beta}{\partial t^\gamma} = \frac{\partial y^i_\gamma}{\partial t^\beta}. \qquad (CI)_2$$

Define the parameter $\epsilon = (\epsilon^1,...,\epsilon^m)$ and

$$J(\epsilon) = J(x(\cdot) + \epsilon^\beta y_\beta(\cdot)),$$

for $\epsilon \in R^m$ and we write $x(\cdot) = x^*(\cdot)$, i.e., we omit the superscript $*$.

We remark that the perturbed function $x(\cdot) + \epsilon^\beta y_\beta(\cdot)$ takes the same values as $x(\cdot)$ at the boundary of $\Omega_{0,t_0}$. Since $x(\cdot)$ is a minimizer, we can write

$$J(\epsilon) \geq J(x(\cdot)) = J(0).$$

In this way, the function $J(\epsilon)$ has a minimum at the point $\epsilon = 0$, and consequently this must be a critical point, i.e., $\frac{\partial J}{\partial \epsilon^\beta}(0) = 0$.

**Step 2**. We compute the partial derivatives $\frac{\partial J}{\partial \epsilon^\beta}$ of the function

$$J(\epsilon) = \int_{\Omega_{0,t_0}} L\left( x(t) + \epsilon^\lambda y_\lambda(t), \frac{\partial x}{\partial t^\gamma}(t) \right.$$

$$\left. + \epsilon^\lambda \frac{\partial y_\lambda}{\partial t^\gamma}(t) \right) dt^1...dt^m,$$

and we put $\frac{\partial J}{\partial \epsilon^\beta}(0) = 0$ or

$$0 = \int_{\Omega_{0,t_0}} \left( \frac{\partial L}{\partial x^i}(x(t), x_\gamma(t)) y^i_\beta(t) \right.$$

$$\left. + \frac{\partial L}{\partial x^i_\gamma}(x(t), x_\gamma(t)) \frac{\partial y^i_\beta}{\partial t^\gamma}(t) \right) dt^1...dt^m.$$

To process this formula we use the conditions $(CI)_2$ for the variations, writing

$$0 = \int_{\Omega_{0,t_0}} \left( \frac{\partial L}{\partial x^i} y^i_\beta + \frac{\partial L}{\partial x^i_\gamma} \frac{\partial y^i_\gamma}{\partial t^\beta} \right) dt^1 ... dt^m$$

$$= \int_{\Omega_{0,t_0}} \left( \frac{\partial L}{\partial x^i} \delta^\gamma_\beta - \frac{\partial}{\partial t^\beta} \frac{\partial L}{\partial x^i_\gamma} \right) y^i_\gamma dt^1 ... dt^m$$

$$+ \int_{\Omega_{0,t_0}} \frac{\partial}{\partial t^\beta} \left( \frac{\partial L}{\partial x^i_\gamma} y^i_\gamma \right) dt^1 ... dt^m.$$

The last integral is evaluated by the Fubini formula. The first integral can be modified via Gauss formula since

$$\frac{\partial}{\partial t^\gamma} (y^i B^\gamma_{\beta i}) = \frac{\partial y^i}{\partial t^\gamma} B^\gamma_{\beta i} + y^i \frac{\partial}{\partial t^\gamma} B^\gamma_{\beta i}$$

$$B^\gamma_{\beta i} = \frac{\partial L}{\partial x^i} \delta^\gamma_\beta - \frac{\partial}{\partial t^\beta} \frac{\partial L}{\partial x^i_\gamma}.$$

**Step 3**. It remains

$$\int_{\Omega_{0,t_0}} y^i \frac{\partial}{\partial t^\gamma} \left( \frac{\partial L}{\partial x^i} \delta^\gamma_\beta - \frac{\partial}{\partial t^\beta} \frac{\partial L}{\partial x^i_\gamma} \right) dt^1 ... dt^m.$$

This equality holds for all differentiable variations $y^i$ with $y^i|_{\partial \Omega_{0,t_0}} = 0$. Therefore

$$\frac{\partial}{\partial t^\gamma} \left( \frac{\partial L}{\partial x^i} \delta^\gamma_\beta - \frac{\partial}{\partial t^\beta} \frac{\partial L}{\partial x^i_\gamma} \right) = 0$$

and consequently the multi-time $(E-L)_2$ PDEs hold for $t \in \Omega_{0,t_0}$.

# 4 Properties of multi-time Euler-Lagrange operator

## 4.1 Affine changing of the Lagrangian

Suppose that $L(t, x(t), x_\gamma(t))$, $t \in R^m_+$ is a smooth Lagrangian. The following Lemma is well-known.

**Lemma 3**. *The Euler-Lagrange derivative $\mathcal{EL}(L)$ is independent of the partial accelerations $x_{\alpha\beta}$ if and only if the Lagrangian $L$ is an affine function in velocities, i.e., $L(t, x(t), x_\gamma(t)) = W(t, x(t)) + A^\alpha_i(t, x(t)) x^i_\alpha$.*

Let us consider the homogeneous Euler-Lagrange PDEs system

$$\frac{\partial L}{\partial x^i} - \frac{\partial}{\partial t^\gamma} \frac{\partial L}{\partial x^i_\gamma} = 0, \; i = 1, ..., n, \; \gamma = 1, ..., m \quad (1)$$

together with the non-homogeneous Euler-Lagrange PDEs system

$$\frac{\partial L}{\partial x^i} - \frac{\partial}{\partial t^\gamma} \frac{\partial L}{\partial x^i_\gamma} = c_i, \; i = 1, ..., n, \; \gamma = 1, ..., m.$$

The non-homogeneous Euler-Lagrange PDEs system becomes homogeneous if the Lagrangian $L$ is replaced by

$$\hat{L}(t, x(t), x_\gamma(t)) = L(t, x(t), x_\gamma(t)) + W(t, x(t))$$

$$+ A^\alpha_i(t, x(t)) x^i_\alpha,$$

where

$$\left(\frac{\partial A^\gamma_i}{\partial x^j} - \frac{\partial A^\gamma_j}{\partial x^i}\right) x^i_\gamma - \frac{\partial A^\gamma_j}{\partial t^\gamma} + \frac{\partial B}{\partial x^j} = c_j.$$

A particular solution is $W = 0$, $A^\alpha_i = -\frac{1}{m} c_i t^\alpha$.

Combining the previous remarks with some formulas in Sections 2-3, we obtain the following

**Theorem 4**. *The Euler-Lagrange operator $\mathcal{EL}$ has the property*

$$\mathcal{EL}(\hat{L}, h) = \mathcal{EL}(L, \nabla h),$$

*where $h$ stands for standard variations.*

Therefore, the non-homogeneous Euler-Lagrange PDEs system (which originally was obtained from $L$ using gradient variations) is a homogeneous system for the modified Lagrangian $\hat{L}$ and the standard variations. From another point of view, the non-homogeneous Euler-Lagrange PDEs system is a *controlled Lagrangian system* for a constant control [1].

**Open problem**. What is the sense of affine changing of classical Lagrangians? What is the sense of homographic changing of classical Lagrangians? When the gradient variations are adequate?

## 4.2 Anti-trace multi-time Euler-Lagrange PDEs

The statements in Sections 2-3 suggest to introduce the *anti-trace multi-time Euler-Lagrange PDEs*

$$\frac{\partial L}{\partial x^i} \delta^\gamma_\beta - \frac{\partial}{\partial t^\beta} \frac{\partial L}{\partial x^i_\gamma} = 0,$$

as generalizations of classical homogeneous Euler-Lagrange PDEs. But, while the classical homogeneous Euler-Lagrange equations are invariant with respect to changes of variables $(t, x) \to (t', x')$, the anti-trace multi-time Euler-Lagrange PDEs are invariant only with respect to $(t, x) \to (t^{\alpha'} = a^\alpha_\beta t^\beta + b^\alpha, x')$.

Let us consider the *anti-trace procedure* as an algebraic operator applied to Euler-Lagrange operators. Looking for the inverse operator, we discovered two inverses: one algebraic and one differential.

**Theorem 5**. *1) The trace followed by a fibre scaling is an algebraic inverse of the anti-trace procedure.*

*2) The divergence is a differential inverse of the anti-trace procedure.*

**Proof**. 1) If we start with anti-trace multi-time Euler-Lagrange PDEs, by the trace after $\beta$, $\gamma$ and by scaling the partial velocities, in the sense of changing the Lagrangian after the rule

$$L(t, x(t), x_\gamma(t)) = L_1(t, x(t), m x_\gamma(t)),$$

we obtain the classical Euler-Lagrange PDEs associated to the new Lagrangian $L_1$.

2) Applying the divergence operator to anti-trace multi-time Euler-Lagrange PDEs, i.e.,

$$\frac{\partial}{\partial t^\gamma}\left(\frac{\partial L}{\partial x^i}\delta^\gamma_\beta - \frac{\partial}{\partial t^\beta}\frac{\partial L}{\partial x^i_\gamma}\right) = 0,$$

we find

$$\frac{\partial L}{\partial x^i} - \frac{\partial}{\partial t^\gamma}\frac{\partial L}{\partial x^i_\gamma} = c_i.$$

Consequently the divergence and the anti-trace procedure are related as "the primitive with the derivative".

### 4.3 New conservation law in case of multiple integrals

We start with the autonomous Lagrangian $L(x(t), x_\gamma(t))$ and the associated homogeneous Euler-Lagrange PDEs (1). Given the m-sheet $x(\cdot)$, let us introduce the *multi-momentum* $p = (p^\alpha_i)$ by the relations $p^\alpha_i(t) = \frac{\partial L}{\partial x^i_\alpha}(x(t), x_\gamma(t))$. Suppose that these $nm$ equations define $nm$ functions $x^i_\gamma = x^i_\gamma(x, p)$. Sometimes, the variables $x$ and $p$ are called *canonical variables*. We define two tensor fields:

– *anti-trace Euler-Lagrange tensor field*,

$$A^\alpha_{\beta i}(x, x_\gamma) = \frac{\partial L}{\partial x^i}(x, x_\gamma)\delta^\alpha_\beta - \frac{\partial}{\partial t^\beta}\frac{\partial L}{\partial x^i_\alpha}(x, x_\gamma),$$

– *energy-momentum tensor field*

$$T^\alpha_\beta(x, p) = p^\alpha_i x^i_\beta(x, p) - L(x, p)\delta^\alpha_\beta.$$

These tensor fields represent conservation laws for Euler-Lagrange PDEs (1) since

$$\frac{\partial}{\partial t^\alpha}A^\alpha_{\beta i} = 0,\ \ \frac{\partial}{\partial t^\alpha}T^\alpha_\beta = 0$$

along the solutions of (1). While the second law is well-known, the first is new and it appears from the anti-trace idea.

### 4.4 New conservation laws and anti-trace PDEs in case of curvilinear integrals

When we use path independent curvilinear integral functionals, we have similar properties. But, a smooth *Lagrangian* $L(x(t), x_\gamma(t))$, $t \in R^m_+$ produces two smooth completely integrable 1-forms:

- the differential

$$dL = \frac{\partial L}{\partial x^i}dx^i + \frac{\partial L}{\partial x^i_\gamma}dx^i_\gamma$$

of components $(\frac{\partial L}{\partial x^i}, \frac{\partial L}{\partial x^i_\gamma})$, with respect to the basis $(dx^i, dx^i_\gamma)$;

- the restriction of $dL$ to $(x(t), x_\gamma(t))$, i.e., the pullback

$$dL|_{(x(t),x_\gamma(t))} = \left(\frac{\partial L}{\partial x^i}\frac{\partial x^i}{\partial t^\beta} + \frac{\partial L}{\partial x^i_\gamma}\frac{\partial x^i_\gamma}{\partial t^\beta}\right)dt^\beta,$$

of components

$$L_\beta(x(t), x_\gamma(t)) = \frac{\partial L}{\partial x^i}(x(t), x_\gamma(t))\frac{\partial x^i}{\partial t^\beta}(t)$$

$$+\frac{\partial L}{\partial x^i_\gamma}(x(t), x_\gamma(t))\frac{\partial x^i_\gamma}{\partial t^\beta}(t),$$

with respect to the basis $(dt^\beta)$. In this case, we must underline that we have two different anti-trace Euler-Lagrange tensor fields

$$A^\gamma_{\alpha\beta i}(x, x_\lambda) = \frac{\partial L_\alpha}{\partial x^i}(x, x_\lambda)\delta^\gamma_\beta - \frac{\partial}{\partial t^\beta}\frac{\partial L_\alpha}{\partial x^i_\gamma}(x, x_\lambda)$$

$$B^\gamma_{\beta\alpha i}(x, x_\lambda) = \frac{\partial L_\beta}{\partial x^i}(x, x_\lambda)\delta^\gamma_\alpha - \frac{\partial}{\partial t^\beta}\frac{\partial L_\alpha}{\partial x^i_\gamma}(x, x_\lambda),$$

since the following Theorem is true.

**Theorem 6**. *The relations*

$$\frac{\partial}{\partial t^\beta}\frac{\partial L_\alpha}{\partial x} - \frac{\partial}{\partial t^\alpha}\frac{\partial L_\beta}{\partial x} = 0$$

$$\frac{\partial}{\partial t^\beta}\frac{\partial L_\alpha}{\partial x_\gamma} - \frac{\partial}{\partial t^\alpha}\frac{\partial L_\beta}{\partial x_\gamma} = (\delta^\lambda_\alpha\delta^\gamma_\beta - \delta^\lambda_\beta\delta^\gamma_\alpha)\frac{\partial}{\partial t^\lambda}\frac{\partial L}{\partial x}$$

$$\frac{\partial}{\partial t^\gamma}A^\gamma_{\alpha\beta i}(x, x_\lambda) = \frac{\partial}{\partial t^\gamma}B^\gamma_{\beta\alpha i}(x, x_\lambda)$$

*hold true.*

The first relation can be written as a conservation law

$$\frac{\partial}{\partial t^\lambda}(\frac{\partial L_\alpha}{\partial x}\delta^\lambda_\beta - \frac{\partial L_\beta}{\partial x}\delta^\lambda_\alpha) = 0$$

or as

$$\frac{\partial L_\alpha}{\partial x}\delta^\lambda_\beta - \frac{\partial L_\beta}{\partial x}\delta^\lambda_\alpha = (\frac{\partial}{\partial t^\alpha}\delta^\lambda_\beta - \frac{\partial}{\partial t^\beta}\delta^\lambda_\alpha)\frac{\partial L}{\partial x}.$$

It implies the third relation between anti-trace multi-time Euler-Lagrange operators. Furthermore, the tensor fields $A^\gamma_{\alpha\beta i}(x, x_\lambda)$, $B^\gamma_{\beta\alpha i}(x, x_\lambda)$ represent equivalent conservation laws for the homogeneous Euler-Lagrange PDEs

$$\frac{\partial L_\alpha}{\partial x^i} - \frac{\partial}{\partial t^\gamma}\frac{\partial L_\alpha}{\partial x^i_\gamma} = 0. \qquad (2)$$

Indeed, their difference is a tensor of "curl" type and

$$\frac{\partial}{\partial t^\gamma}A^\gamma_{\alpha\beta i}(x, x_\lambda) = 0,\ \frac{\partial}{\partial t^\gamma}B^\gamma_{\beta\alpha i}(x, x_\lambda) = 0$$

along the solutions of (2).

Starting from the Euler-Lagrange PDEs (2), we introduce:

1) the *first kind of anti-trace multi-time Euler-Lagrange PDEs*

$$\frac{\partial L_\alpha}{\partial x^i}(x, x_\lambda)\delta^\gamma_\beta - \frac{\partial}{\partial t^\beta}\frac{\partial L_\alpha}{\partial x^i_\gamma}(x, x_\lambda) = 0;$$

2) the *second kind of anti-trace multi-time Euler-Lagrange PDEs*

$$\frac{\partial L_\beta}{\partial x^i}(x, x_\lambda)\delta^\gamma_\alpha - \frac{\partial}{\partial t^\beta}\frac{\partial L_\alpha}{\partial x^i_\gamma}(x, x_\lambda) = 0.$$

Both anti-trace procedures have the same differential inverse (divergence), but different algebraic inverses (the first, trace followed by a fibre scaling; the second, the trace). Consequently, applying the divergence, both imply the same controlled Euler-Lagrange PDEs system

$$\frac{\partial L_\alpha}{\partial x^i} - \frac{\partial}{\partial t^\gamma}\frac{\partial L_\alpha}{\partial x^i_\gamma} = c_{\alpha i}.$$

That is why, the solutions of anti-trace multi-time Euler-Lagrange PDEs are among the solutions of the controlled Euler-Lagrange PDEs.

# 5 Application in Multi-Time Rheonomic Dynamics

Our theory has applications in Relativistic and Multi-Time Rheonomic Dynamics; the electromagnetic field $E = (E^i),\ H = (H^i),\ i = 1, 2, 3$ determines the *density of electromagnetic deformation energy*

$$L = \frac{1}{2}\delta_{ij}\delta^{\alpha\beta}\left(\frac{\partial E^i}{\partial t^\alpha}\frac{\partial E^j}{\partial t^\beta} + \frac{\partial H^i}{\partial t^\alpha}\frac{\partial H^j}{\partial t^\beta}\right),$$

where $t^1 = x, t^2 = y, t^3 = z, t^4 = it$. The extremals of $L$ under gradient variations are described by the controlled wave PDEs

$$\Delta E^i - \frac{\partial^2 E^i}{\partial t^2} = b^i,\ \Delta H^i - \frac{\partial^2 H^i}{\partial t^2} = c^i,\ i = 1, 2, 3.$$

# 6 Conclusion

The present point of view regarding the multi-time Euler-Lagrange PDEs has the key ideas in the union between [3]-[8] and [1], [2]. Accepting that the evolution is m-dimensional, all the results confirm the possibility of passing from the single-time Pontryaguin's maximum principle to a multi-time maximum principle [4]-[8]. Furthermore, the main results belong to PDEs-constrained optimal control theory.

**Acknowledgements:** Partially supported by Grant CNCSIS 86/ 2007 and by 15-th Italian-Romanian Executive Programme of S&T Co-operation for 2006-2008, University Politehnica of Bucharest.

*References:*


[1] D. E. Chang, A. M. Bloch, N. E. Leonard, J. E. Marsden, C. A. Woolse, The euivalence of controlled Lagrangian and controlled Hamiltonian systems, *ESAIM: Control, Optimisation and Calculus of Variations*, 8 (2002), 393-422.

[2] D. F. M. Torres, On the Noether theorem for optimal control, *European Journal of Control (EJC)*, 8, 1 (2002), pp. 56-63.

[3] C. Udrişte, Geodesic motion in a gyroscopic field of forces, *Tensor, N. S.*, 66, 3 (2005), 215-228.

[4] C. Udrişte, Multi-time maximum principle, *short communication at International Congress of Mathematicians, Madrid*, August 22-30, 2006.

[5] C. Udrişte, Maxwell geometric dynamics, *manuscript*, 2007.

[6] C. Udrişte, M. Ferrara, Multi-time optimal economic growth, *manuscript*, 2007.

[7] C. Udrişte, A. M. Teleman, Hamiltonian approaches of field theory, *IJMMS*, 57 (2004), pp. 3045-3056.

[8] C. Udrişte, I. Ţevy, Multi-Time Euler-Lagrange-Hamilton Theory, *WSEAS Transactions on Mathematics*, 6, 6 (2007), 701-709.