Reachability, Observability and Minimality for a Class of 2D Continuous-Discrete Systems*

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Abstract: Reachability and observability criteria are obtained for 2D continuous-discrete time-variable Attasi type systems by using suitable 2D reachability and observability Gramians. Necessary and sufficient conditions of reachability and observability are derived for time-invariant systems. The duality between the two concepts is emphasized as well as their connection with the minimality of these systems.

Key–Words: 2D continuous-discrete linear systems, controllability, reachability, observability, duality principle. Typing manuscripts, LaTeX

1 Introduction

The concepts of controllability and observability, introduced by Kalman for 1D systems were extended to 2D systems for Roesser [9], Fornasini and Marchesini [2], and Attasi [1] models; in order to keep their relationship with minimality, new concepts of modal controllability and modal observability were introduced in [6].

In this paper a class of 2D continuous-discrete time-variable linear systems is studied, which is related to Attasi's 2D discrete model and represents the extension to time-variable framework of the hybrid systems introduced in [8]. Such systems can appear in various problems as signal and image processing, seismology and geophysics, control of multipass processes, iterative learning control synthesis [5] or repetitive processes [3].

The state and output formulæ for these systems are established in Section 2 and the notions of complete reachability and complete observability are defined. These properties are characterized by means of the full ranks of suitable 2D reachability and observability Gramians.

Section 4 is devoted to time-invariant 2D continuous-discrete systems and a list of criteria of reachability and observability is provided. The duality between the two concepts is emphasized.

In Section 5 the relation between reachability, observability and minimality is established.

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2 The state space representation of the 2D continuous-discrete systems

We consider the linear spaces $X = \mathbf{R}^n$, $U = \mathbf{R}^m$ and $Y = \mathbf{R}^p$, called respectively the *state*, *input* and *output spaces*. The time set is $T = \mathbf{R} \times \mathbf{Z}$.

Definition 1 *A <u>two-dimensional continuous-discrete linear system</u> (2Dcd) is a quintuplet* $\Sigma = (A_1(t,k), A_2(t,k), B(t,k), C(t,k), D(t,k)) \in \mathbf{R}^{n\times n} \times \mathbf{R}^{n\times n} \times \mathbf{R}^{n\times m} \times \mathbf{R}^{p\times n} \times \mathbf{R}^{p\times m}$ *with* $A_1(t,k)A_2(t,k) = A_2(t,k)A_1(t,k)$ $\forall (t,k) \in T$, *where all matrices are continuous with respect to* $t \in \mathbf{R}$ *for any* $k \in \mathbf{Z}$*; the state space representation of* Σ *is given by the state and output equations*

$$\dot{x}(t,k+1) = A_1(t,k+1)x(t,k+1) + A_2(t,k)\dot{x}(t,k) - A_1(t,k)A_2(t,k)x(t,k) + B(t,k)u(t,k) \quad (1)$$

$$y(t,k) = C(t,k)x(t,k) + D(t,k)u(t,k) \quad (2)$$

where $\dot{x}(t,k) = \frac{\partial x}{\partial t}(t,k)$.

Let us denote by $\Phi(t,t_0;k)$ or $\Phi_{A_1}(t,t_0;k)$ the (continuous) fundamental matrix of $A_1(t,k)$ with respect to $t \in \mathbf{R}$, for any fixed $k \in \mathbf{Z}$. $\Phi(t,t_0;k)$ has the following properties, for any $t, t_0, t_1 \in \mathbf{R}$:

$$\begin{aligned} i) &\quad \frac{\mathrm{d}}{\mathrm{d}t}\Phi(t,t_0;k) = A_1(t,k)\Phi(t,t_0;k), \\ ii) &\quad \Phi(t_0,t_0;k) = I_n, \\ iii) &\quad \Phi(t,t_1;k)\Phi(t_1,t_0;k) = \Phi(t,t_0;k), \\ iv) &\quad \Phi(t,t_0;k)^{-1} = \Phi(t_0,t;k). \end{aligned} \quad (3)$$

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If A_1 is a constant matrix, then $\Phi(t, t_0; k) = e^{A_1(t-t_0)}$.

The *discrete fundamental matrix* $F(t; k, k_0)$ of the matrix $A_2(t, k)$ is defined by $F(t; k, k_0) =$

$$= \begin{cases} A_2(t,k-1)A_2(t,k-2)\cdots A_2(t,k_0) & \text{for} \quad k > k_0 \\ I_n & \text{for} \quad k = k_0 \end{cases}$$

for any fixed $t \in \mathbf{R}$.

If A_2 is a constant matrix, then $F(t; k, k_0) = A_2^{k-k_0}$.

$\Phi(t, t_0; k)$ and $F(s; l, l_0)$ are commutative matrices for any $t, t_0, s \in \mathbf{R}$ and $k, l, l_0 \in \mathbf{Z}$ since $A_1(t, k)$ and $A_2(t, k)$ are commutative matrices.

Definition 2 *A vector $x_0 \in X$ is said to be the initial state of Σ at the moment $(t_0, k_0) \in T$ if, for any $(t, k) \in T$ with $(t, k) \geq (t_0, k_0)$ the following conditions hold:*

$$x(t,k_0) = \Phi(t, t_0;k_0)x_0, \; x(t_0,k) = F(t_0; k,k_0)x_0. \quad (4)$$

In [7] it was proved:

Proposition 3 *The state of the system Σ at the moment $(t, k) \in T$ determined by the control $u(\cdot, \cdot)$ and by the initial state $x_0 \in X$ is*

$$x(t,k) = \Phi(t,t_0;k)F(t_0;k,k_0)x_0 + \int_{t_0}^{t} \sum_{l=k_0}^{k-1} \Phi(t, s; k)F(s; k, l + 1)B(s, l)u(s, l)\mathrm{d}s. \quad (5)$$

By replacing the state $x(t, k)$ given by (5) in the output equation (2) we obtain

Proposition 4 *The input-output map of the system Σ is given by the formula*

$$y(t,k) = C(t,k)\Phi(t,t_0;k)F(t_0;k,k_0)x_0 + \int_{t_0}^{t} \sum_{l=k_0}^{k-1} C(t,k) \Phi(t, s; k)F(s; k,l+1)B(s,l)u(s,l)\mathrm{d}s + D(t,k)u(t,k). \quad (6)$$

3 Reachability and observability of time-variable 2D continuous-discrete systems

For the concept of reachability we need only the state equation (1), hence a 2Dcd system can be reduced to the triplet $\Sigma = (A_1(t, k), A_2(t, k), B(t, k))$. For observability the system can be reduced to the triplet $\Sigma = (A_1(t, k), A_2(t, k), C(t, k))$. For both notions a system will be a quadruplet $\Sigma = (A_1(t, k), A_2(t, k), B(t, k), C(t, k))$.

By $(s, l) < (t, k)$ we mean $s \leq t, l \leq k$ and $(s, l) \neq (t, k)$.

A triplet $(t, k, x) \in \mathbf{R} \times \mathbf{Z} \times X$ is said to be a *phase* of Σ if x is the state of Σ at the moment (t, k) (i.e. $x = x(t, k)$, where $x(t, k)$ is given by (5)).

Definition 5 *A phase (t, k, x) of Σ is said to be reachable if there exist $(t_0, k_0) \in T$, $(t_0, k_0) < (t, k)$ and a control $u(\cdot, \cdot)$ which transfers the phase $(t_0, k_0, 0)$ to (t, k, x).*

A phase (t, k, x) is said to be controllable if there exist $(t_1, k_1) \in T$, $(t_1, k_1) > (t, k)$ and a control $u(\cdot, \cdot)$ which transfers the phase (t, k, x) to $(t_1, k_1, 0)$.

If for some fixed $(\tau, \chi) \in T$ every phase (τ, χ, x) is reachable (controllable), the system Σ is said to be completely reachable (completely controllable) at the moment (τ, χ).

Definition 6 *A phase (τ, χ, x) is said to be unobservable if for any control u it provides the same output $y(s, l)$ for $(s, l) \geq (\tau, \chi)$ as the phase $(\tau, \chi, 0)$. In this case the state $x \in X$ is said to be unobservable at (τ, χ).*

The system Σ is said to be completely observable at (τ, χ) if there is no state unobservable at (τ, χ).

Definition 7 *The matrices*

$$\mathcal{R}_\Sigma(t, \tau; k, \chi) = \int_{t}^{\tau} \sum_{l=k}^{\chi-1} \Phi(\tau, s; \chi)F(s; \chi, l+1) \times B(s, l)B(s, l)^T F(s; \chi, l + 1)^T \Phi(\tau, s; \chi)^T \mathrm{d}s, \quad (7)$$

$$\mathcal{O}_\Sigma(\tau, t; \chi, k) = \int_{\tau}^{t} \sum_{l=\chi}^{k} \Phi(s, \tau; l)^T F(\tau; l, \chi)^T \times C(s, l)^T C(s, l)F(\tau; l, \chi)\Phi(s, \tau; l)\mathrm{d}s. \quad (8)$$

are called respectively the reachability Gramian and the observability Gramian of Σ.

We have proved in [8]:

Theorem 8 *Σ is completely reachable at (τ, χ) if and only if*

$$rank \mathcal{R}_\Sigma(t, \tau; k, \chi) = n$$

for some $(t, k) \leq (\tau, \chi)$.

Theorem 9 *The system $\Sigma = (A_1(t, k), A_2(t, k), C(t, k))$ is completely observable at (τ, χ) if and only if*

$$rank \mathcal{O}_\Sigma(\tau, t; \chi, k) = n. \quad (9)$$

for some $(t, k) \geq (\tau, \chi)$.

4 Reachability and observability of time-invariant 2D continuous-discrete systems

Let us consider a time invariant system $\Sigma = (A_1, A_2, B, C)$, i.e. a system with A_1, A_2, B and C constant matrices. In this case we can consider the initial moment $(t_0, k_0) = (0, 0)$ and the time set $T = \mathbf{R}^+ \times \mathbf{Z}^+$. Then the state formula (5) and the input-output map (6) become

$$x(t,k) = e^{A_1 t} A_2^k x_0 + \int_0^t \sum_{l=0}^{k-1} e^{A_1(t-s)} A_2^{k-l-1} Bu(s,l)\mathrm{d}s \tag{10}$$

$$y(t,k) = Ce^{A_1 t} A_2^k x_0 + \int_0^t \sum_{l=0}^{k-1} Ce^{A_1(t-s)} A_2^{k-l-1} Bu(s,l)\mathrm{d}s. \tag{11}$$

Definition 10 *The system $\Sigma^d = (A_1^d, A_2^d, B^d, C)$ is called the dual of Σ if $A_1^d = A_1^T, A_2^d = A_2^T, B^d = C^T, C^d = B^T$.*

We can prove (see [8]) the Duality Principle:

Theorem 11 *The system Σ^d is completely observable if and only if Σ is completely reachable.*

We associate to Σ the *reachability matrix* $C_\Sigma = [B\ A_1B\ ...\ A_1^{n-1}B\ A_2B\ A_1A_2B\ ...\ A_1^{n-1}A_2B\ ...\ A_2^{n-1}B\ \ A_1A_2^{n-1}B\ \ A_1^{n-1}A_2^{n-1}B]$

and the *observability matrix* $O_\Sigma = [C^T\ \ A_1^TC^T\ \ ...\ \ (A_1^T)^{n-1}C^T\ \ A_2^TC^T\ \ A_1^TA_2^TC^T\ \ ...\ \ (A_1^T)^{n-1}A_2^TC^T\ \ ...\ \ (A_2^T)^{n-1}C^T\ \ A_1^T(A_2^T)^{n-1}C^T\ \ ...\ \ (A_1^T)^{n-1}(A_2^T)^{n-1}C^T\,]^T$.

Theorem 8 gives (see [7],Theorem 4.2)

Theorem 12 *$\Sigma = (A_1, A_2, B)$ is completely reachable if and only if*

$$rank C_\Sigma = n. \tag{12}$$

We can prove

Proposition 13 *The set of all reachable states of Σ is $X_r = Im C_\Sigma$.*

Proposition 14 *The set of all reachable states of Σ is the smallest subspace of X which is (A_1, A_2)-invariant and contains the columns of B.*

An immediate consequence of Proposition 14 is the following

Theorem 15 *$\Sigma = (A_1, A_2, B)$ is completely reachable if and only if X is the smallest subspace of X which is (A_1, A_2)-invariant and contains the columns of B.*

By duality (Theorem 11) and Theorem 12 we obtain

Theorem 16 *The system $\Sigma = (A_1, A_2, C)$ is completely observable if and only if*

$$rank O_\Sigma = n. \tag{13}$$

From (10) and Definition 7 we obtain

Proposition 17 *The set of all unobservable states of Σ is $X_{uo} = Ker O_\Sigma$.*

Proof: By using Hamilton-Cayley Theorem for A_1 and A_2 we obtain $X_{uo} = \{x \in X \mid CA_1^l A_2^k x = 0, \forall l, k = \overline{0, n-1}\} = \mathrm{Ker} O_\Sigma$.

Proposition 18 *The set X_{uo} of all unobservable states of Σ is the greatest subspace of X which is $(A_1, A_2)-$invariant and is contained in $Ker C$.*

We obtain from Proposition 18:

Theorem 19 *The system $\Sigma = (A_1, A_2, C)$ is completely observable if and only if $\{0\}$ is the greatest subspace of X which is $(A_1, A_2)-$invariant and is contained in $Ker C$.*

Definition 20 *Two systems $\Sigma = (A_1, A_2, B, C)$ and $\tilde{\Sigma} = (\tilde{A}_1, \tilde{A}_2, \tilde{B}, \tilde{C})$ are said to be isomorphic if there exists a nonsingular matrix $T \in \mathbf{R}^{n\times n}$ such that*

$$\tilde{A}_i = T^{-1}A_iT,\ \ i = 1, 2;\ \ \tilde{B} = T^{-1}B,\ \ \tilde{C} = CT. \tag{14}$$

Theorem 21 *Any system $\Sigma = (A_1, A_2, B)$ is isomorphic to a system $\tilde{\Sigma} = (\tilde{A}_1, \tilde{A}_2, \tilde{B})$ of the form*

$$\tilde{A}_1 = \begin{bmatrix} A_{111} & A_{121} \\ 0 & A_{221} \end{bmatrix},\ \tilde{A}_2 = \begin{bmatrix} A_{112} & A_{122} \\ 0 & A_{222} \end{bmatrix},\ \tilde{B} = \begin{bmatrix} B_1 \\ 0 \end{bmatrix}, \tag{15}$$

with $A_{111}, A_{112} \in \mathbf{R}^{q\times q}, B_1 \in \mathbf{R}^{q\times m}, q \le n$. The triplet $\Sigma_1 = (A_{111}, A_{112}, B_1)$ is completely reachable.

Proof: We consider the direct sum decomposition of the state space $X = \mathbf{R}^n$ as $X = X_1 \oplus X_2$ where $X_r = X_1$. The partitions of the matrices in (15) are obtained with respect to this decomposition, since by Proposition 14 X_r is (A_1, A_2)-invariant and contains the columns of B; q is the dimension of the subspace X_r.

We can derive other criteria of reachability.

Theorem 22 $\Sigma = (A_1, A_2, B)$ *is completely reachable if and only if there is no common left eigenvector of matrices* A_1 *and* A_2*, orthogonal on the columns of* B.

Proof: Let us assume that there exists $v \in \mathbf{R}^n \setminus \{0\}$ such that $\exists \lambda, \mu \in \mathbf{C}$ with $v^T A_1 = \lambda v^T, v^T A_2 = \mu v^T$ and $v^T B = 0$. Then $v^T A_1^i A_2^j B = \lambda^i \mu^j v^T B = 0 \ \ \forall i,j \geq 0$, hence $v^T C_\Sigma = 0$, i.e. $\Sigma = (A_1, A_2, B)$ is not completely reachable.

Conversely, if Σ is not completely reachable, then there exists $v \in \mathbf{R}^n \setminus \{0\}$ such that $v^T C_\Sigma = 0$, hence the subspace $S_1 = \{x \in \mathbf{R}^n | x^T C_\Sigma = 0\}$ contains a vector $v \neq 0$. If $x \in S_1$, then $x^T A_1^i A_2^j B = 0$ for any $i, j = \overline{0, n-1}$ and by Hamilton-Cayley Theorem this equality is true for any $i, j \geq 0$. Then, for any $x \in S_1$, $(A_1^T x)^T A_1^i A_2^j B = x^T A_1^{i+1} A_2^j B = 0, \ \forall i, j \geq 0$, hence $A_1^T x \in S_1$, i.e. S_1 is A_1^T-invariant; analogously, S_1 is A_2^T-invariant. It results that S_1 contains an eigenvector x of A_1^T; let λ be the corresponding eigenvalue. Let us consider the subspace $S_2 = \{x \in X | A_1^T x = \lambda x\}$. If $x \in S_2$ then $A_1^T(A_2^T x) = A_2^T A_1^T x = \lambda A_2^T x$, hence $A_2^T x \in S_2$, that is S_2 is A_2^T-invariant and so is $S_3 = S_1 \cap S_2$. Then S_3 contains an eigenvector w of A_2^T and since $S_3 \subset S_2$, w is an eigenvector of A_1^T too. Moreover, since $S_3 \subset S_1$, we have $w^T C_\Sigma = 0$ and particularly $w^T B = 0$, hence w is a common left eigenvector of A_1 and A_2 orthogonal on the columns of B.

The following theorem is an extension to 2Dcd systems of the Popov-Hautus-Belevitch criterion of reachability.

Theorem 23 $\Sigma = (A_1, A_2, B)$ *is completely reachable if and only if for any* $\lambda_1, \lambda_2 \in \mathbf{C}$

$$\mathit{rank}\begin{bmatrix} B & \lambda_1 I - A_1 & \lambda_2 I - A_2 \end{bmatrix} = n.$$

Proof: Obviously, the existence of $\lambda_1, \lambda_2 \in \mathbf{C}$ such that rank$\begin{bmatrix} B & \lambda_1 I - A_1 & \lambda_2 I - A_2 \end{bmatrix} < n$ is equivalent to the existence of $v \in \mathbf{R}^n \setminus \{0\}$ such that $v^T \begin{bmatrix} B & \lambda_1 I - A_1 & \lambda_2 I - A_2 \end{bmatrix} = 0$ which means $v^T B = 0, v^T A_1 = \lambda_1 v^T, v^T A_2 = \lambda_2 v^T$ that is, by Theorem 22, to the fact that Σ is not completely reachable.

By duality we obtain the following results concerning observability:

Theorem 24 *The system* $\Sigma = (A_1, A_2, C)$ *is completely observable if and only if there is no common eigenvector of the matrices* A_1 *and* A_2 *belonging to* *Ker*C.

Theorem 25 *The system* $\Sigma = (A_1, A_2, C)$ *is completely observable if and only if for any* $\lambda_1, \lambda_2 \in \mathbf{C}$

$$\mathit{rank}\begin{bmatrix} C \\ \lambda_1 I - A_1 \\ \lambda_2 I - A_2 \end{bmatrix} = n.$$

5 Reachability, observability and minimality

Definition 26 *The matrix*

$$T_\Sigma(s,z) = C(sI - A_1)^{-1}(zI - A_2)^{-1}B \qquad (16)$$

is called the *transfer matrix* *of the time-invariant system* $\Sigma = (A_1, A_2, B, C)$.

Obviously, $T_\Sigma(s,z)$ is a $p \times m$ rational strictly proper (in both variables s and z) matrix with separable denominator, since it has the form $T_\Sigma(s,z) = \frac{1}{\det(sI - A_1)\det(zI - A_2)} C(sI - A_1)^*(zI - A_2)^* B$.

Definition 27 *Given a strictly proper matrix* $T(s,z)$*, a system* $\Sigma = (A_1, A_2, B, C)$ *is said to be a* *realization* *of* Σ *if* $T(s,z) = T_\Sigma(s,z)$*, that is if*

$$T(s,z) = C(sI - A_1)^{-1}(zI - A_2)^{-1}B. \qquad (17)$$

A realization Σ *of* $T(s,z)$ *is* *minimal* *if dim*$\Sigma \leq$*dim*$\tilde{\Sigma}$ *for any realization* $\tilde{\Sigma}$ *of* $T(s,z)$.

Now let us consider the Laurent series expansion of $T(s,z)$ about $s = \infty, z = \infty$

$$T(s,z) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} M_{i,j} s^{-i-1} z^{-j-1}. \qquad (18)$$

The matrices $M_{i,j} \in \mathbf{R}^{p \times m}$ are called the *Markov parameters* of $T(s,z)$.

Proposition 28 $\Sigma = (A_1, A_2, B, C)$ *is a realization of* $T(s,z)$ *if and only if, for any* $i, j \in \mathbf{N}$,

$$M_{i,j} = C A_1^i A_2^j B. \qquad (19)$$

Proof: By (17) we have $T(s,z) = C(sI - A_1)^{-1}(zI - A_2)^{-1}B = C(\sum_{i=0}^{\infty} A_1^i s^{-i-1})(\sum_{j=0}^{\infty} A_2^j z^{-j-1})B = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} C A_1^i A_2^j s^{-i-1} z^{-j-1}$. Since (18) holds and two equal Laurent series have equal corresponding coefficients, (19) is true.

The following theorem establishes the connection between the concepts of reachability, observability and minimality.

Theorem 29 *A system $\Sigma = (A_1, A_2, B, C)$ is a minimal realization of some strictly proper matrix $T(s,z)$ if and only if Σ is completely reachable and completely observable.*

Proof: Necessity. By negation, let us assume that Σ is not completely reachable. Then Σ is isomorphic to a system $\widetilde{\Sigma}$ as in Theorem 21 with $q < n$. If we partition $\widetilde{C}$ as $\widetilde{C} = [\ C_1 \quad C_2]$ with $C_1 \in \mathbf{R}^{p\times q}$, since $(sI_n - \widetilde{A}_k)^{-1} = \begin{bmatrix} (sI_q - A_{11k}) & -A_{12k} \\ 0 & (sI_{n-q} - A_{22k}) \end{bmatrix}^{-1} = \begin{bmatrix} (sI_q - A_{11k})^{-1} & ? \\ 0 & (sI_{n-q} - A_{22k})^{-1} \end{bmatrix}, k = 1,2$ we obtain $T(s,z) = T_\Sigma(s,z) = T_{\widetilde{\Sigma}}(s,z) = C_1(sI_q - A_{111})^{-1}(zI_q - A_{112})^{-1}B_1 = T_{\Sigma_1}(s,z)$, hence $\Sigma_1 = (A_{111}, A_{112}, B_1, C_1)$ is a realization of $T(s,z)$ of dimension $q < n$, i.e. Σ is not minimal. The case Σ not completely observable is similar.

Sufficiency. Let us assume that Σ is completely reachable and completely observable. If Σ is not minimal, let $\widehat{\Sigma} = (\widehat{A}_1, \widehat{A}_2, \widehat{B}, \widehat{C})$ be a realization of $T(s,z)$ with dim$\widehat{\Sigma} = \hat{n} < n =$ dimΣ. Let us consider the controllability and observability matrices C_Σ and O_Σ and let us denote by $C_{\widehat{\Sigma}}$ and $O_{\widehat{\Sigma}}$ the matrices obtained from C_Σ and O_Σ by replacing the matrices A_1, A_2, B and C by $\widehat{A}_1, \widehat{A}_2, \widehat{B}$ and $\widehat{C}$.

By Proposition 28, since Σ and $\widehat{\Sigma}$ are realizations of the same matrix $T(s,z)$ we have (see (19)): $CA_1^iA_2^jB = \widehat{C}\widehat{A}_1^i\widehat{A}_2^j\widehat{B}, \forall i,j \geq 0$; these matrices are the block elements of the product matrices $O_\Sigma C_\Sigma$ and $\widehat{O}_\Sigma \widehat{C}_\Sigma$ hence $O_\Sigma C_\Sigma = \widehat{O}_\Sigma \widehat{C}_\Sigma$. Then, by Sylvester Inequalities, we have rankO_Σ + rank$C_\Sigma - n \leq$ rank$O_\Sigma C_\Sigma$ = rank$\widehat{O}_\Sigma \widehat{C}_\Sigma \leq$ min(rank$\widehat{O}_\Sigma$, rank$\widehat{C}_\Sigma$). By hypothesis rank$O_\Sigma = n$, rank$C_\Sigma = n$ and rank$\widehat{O}_\Sigma \leq \hat{n} < n$, rank$\widehat{C}_\Sigma \leq \hat{n} < n$; we get $n \leq \hat{n}$, contradiction, hence Σ is minimal.

From Theorems 23, 25 and 29 we get

Theorem 30 *$\Sigma = (A_1, A_2, B, C)$ is a minimal realization if and only if*

$$\text{rank}[\ B \quad \lambda_1 I - A_1 \quad \lambda_2 I - A_2\ ] = n, \tag{20}$$

and

$$\text{rank}\begin{bmatrix} C \\ \lambda_1 I - A_1 \\ \lambda_2 I - A_2 \end{bmatrix} = n \tag{21}$$

for any $\lambda_1, \lambda_2 \in \mathbf{C}$.

Since for any $n \times n$ matrix A det$(sI - A) = 0$ if and only if $s \in \sigma(A)$ (where $\sigma(A)$ denotes the spectrum of A, rank$(sI - A) = n$ for any $s \notin \sigma(A)$ and Theorem 30 gives

Corollary 31 *$\Sigma = (A_1, A_2, B, C)$ is a minimal realization if and only if equalities (20) and (21) hold for any $(s,z) \in \sigma(A_1) \times \sigma(A_2)$.*

Application Let us consider the 2Dcd system $\Sigma = (A_1, A_2, B, C)$, where $A_1 = \begin{bmatrix} a & 0 & 0 \\ 0 & b & c \\ 0 & c & b \end{bmatrix}$, $A_2 = \begin{bmatrix} d & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ and $C = \begin{bmatrix} 0 & 1 & -1 \end{bmatrix}$. The characteristic polynomials are det$(sI - A_1) = (s-a)(s-b-c)(s-b+c)$ and det$(sI - A_2) = (z-a)(z-1)(z+1)$. The matrices in (20) and (21) are respectively $\mathcal{C} = \begin{bmatrix} s-a & 0 & 0 & z-d & 0 & 0 & 1 \\ 0 & s-b & -c & 0 & z & -1 & 0 \\ 0 & -c & s-b & 0 & -1 & z & 1 \end{bmatrix}$ and $\mathcal{O}^T = \begin{bmatrix} s-a & 0 & 0 & z-d & 0 & 0 & 0 \\ 0 & s-b & -c & 0 & z & -1 & 1 \\ 0 & -c & s-b & 0 & -1 & z & -1 \end{bmatrix}$. The spectra are respectively $\sigma(A_1) = \{a, b-c, b+c\}$ and $\sigma(A_2) = \{d, 1, -1\}$ Then the 3^{rd} order minors of $\mathcal{C}$ have respectively the values 0, $(a-b)^2 - c^2$,$d^2 - 1$ and $-a + b + dc$ for $(s,z) = (a,d)$, $c(b+c-a)$, $a - b + c\, c(d-1)$ and $d - 1$ for $(s,z) = (b+c, 1)$, $c(a-b+c)$, $-2c(a-b+c)$, and $c(d+1)$ for $(s,z) = (b-c,-1)$ and there are similar expressions for the other values of $(s,z) \in \sigma(A_1) \times \sigma(A_2)$. We can conclude that Σ is unreachable in the following four cases:$(a - b = c, d = 1)$, $(c = 0, d = 1)$, $(a - b = -c, d = -1)$ and $(c = 0, d = -1)$; otherwise Σ is completely reachable. Obviously, for $s = a$ and $z = d$ rank$\mathcal{O} \leq 2$, hence Σ is unobservable. By Corollary 31 Σ is not minimal.

Now, the transfer matrix of Σ is $T_\Sigma(s,z) = C(sI - A_1)^{-1}(zI - A_2)^{-1}B = \frac{(s-a)(s-b-c)(z-d)(1-z)}{(s-a)(s-b-c)(s-b+c)(z-d)(z-1)(z+1)} = -\frac{1}{(s-b+c)(z+1)}$ hence a minimal realization of Σ is $\tilde{A}_1 = b - c$, $\tilde{A}_2 = -1$, $\tilde{B} = 1$ and $\tilde{C} = -1$.

6 Conclusion

Reachability and observability criteria were obtained for 2D continuous-discrete time-variable systems by using suitable 2D reachability and observability Gramians. In the case of time-invariant systems, necessary and sufficient conditions of reachability and observability were obtained by means of duality. The relationship between reachability, observability and minimality was emphasized. This research can be developed and extended to other topics of the Systems

Theory such as stability, stabilizability, detectability, feedback and observers and optimal control for 2D continuous-discrete systems.

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