Observability indices and Segre characteristic for multi-input standardizable singular systems

Mª I. GARCÍA-PLANAS, A. DÍAZ
Universitat Politècnica de Catalunya
C. Minería, 1, Esc C. 1-3,
08038 Barcelona, Spain

Abstract:- Let (E, A, C) be a triple of matrices with $E, A \in M_n(\mathbb{C})$, $C \in M_{p\times n}(\mathbb{C})$, representing a singular time-invariant linear system, $E\dot{x} = Ax$, $y = Cx$.

In this paper we present a collection of invariants that we will call observability indices for singular systems in terms of ranks of certain matrices, given a necessary and sufficient condition for observability character.

Key-Words:- Singular systems, output injection, observability, canonical forms.

1 Introduction

Let us consider a finite-dimensional singular linear time invariant system $E\dot{x}(t) = Ax(t)$, $y = Cx(t)$ where $(E, A, C) \in M = M_n(\mathbb{C}) \times M_n(\mathbb{C}) \times M_{p\times n}(\mathbb{C})$.

Different useful and interesting equivalence relations between singular systems have been defined. We deal with the equivalence relation $(E', A', B') = (QEP + F_E CP, QAP + F_A CP, SCP)$. with $P, Q \in Gl(n; \mathbb{C})$, $S \in Gl(p; \mathbb{C})$, $F_A, F_E \in M_{n\times p}(\mathbb{C})$, that is to say, the equivalence relation accepting one or more, of the following standard transformations: basis change in the state space, output space, output injection (proportional and derivative), and pre-multiplication by an invertible matrix.

One of the topics largely studied in control theory for standard systems (systems with $E = I_n$), is the observability, for singular systems L. Dai [1] gives the following definition of observable singular systems.

Definition 1 *A system $(E, A, C) \in M$ is observable if and only if*

$$i) \qquad rank \begin{pmatrix} E \\ C \end{pmatrix} = n$$

$$ii) \qquad rank \begin{pmatrix} sE - A \\ C \end{pmatrix} = n, \text{ for all } s \in \mathbb{C}.$$

It is not difficult to prove that the observability character is invariant under equivalence relation considered.

Systems (E, A, C) verifying condition i), are called standardizable systems, these are systems (E, A, C), for which there exist a derivative output injection F_E making invertible the matrix $E + F_E C$, and as a consequence, the equivalent system

$$(E + F_E C)^{-1} \begin{pmatrix} I_n & & F_E \\ & I_n & \\ & & I_m \end{pmatrix} \begin{pmatrix} E \\ A \\ C \end{pmatrix} = \begin{pmatrix} I_n \\ A_1 \\ C_1 \end{pmatrix}$$

is standard. Then, the triple (E, A, C) can be reduced to (I_n, A_2, C_2), taking $Q' = P^{-1}(E + BF_E)^{-1}$, $P' = P$, $F'_E = F_E P$, $F'_A = F_A$, $S' = S$, where $\begin{pmatrix} A_2 & B_2 \end{pmatrix} = \begin{pmatrix} P^{-1} & F_A \\ 0 & S \end{pmatrix} \begin{pmatrix} A_1 \\ C_1 \end{pmatrix} P$ is in its Kronecker reduced form.

In this paper, we present a collection of structural invariants which we will call *observability indices* of the triple in terms of ranks of certain matrices associated to the triple.

As we say before, we remark that the collection of observability indices of the triple (E, A, C) obtained are not related with the observability indices of the pair (A, C) but related to the observability indices of the unknown pair (A_2, C_2) corresponding to reduced form (I_n, A_2, C_2).

We recall that L. Dai [1], studied the observability character for singular systems but he does not consider output injection in the equivalence relation, considering only, basis change in the state space, output space and pre-multiplication for invertible matrices.

The problem to obtain structural invariants permitting to conclude conditions for observability, was largely studied for standard linear systems under several equivalence relations that can be considered (see [3],[5], [7] for example).

2 Collection of invariants

First of all, we remember the equivalence relation considered over the space M of triples of matrices.

Definition 2 *Two triples (E', A', C') and (E, A, C) in M are called equivalent if, and only if, there exist matrices $P, Q \in Gl(n; \mathbb{C})$, $S \in Gl(p; \mathbb{C})$, $F_E, F_A \in M_{n\times p}(\mathbb{C})$, such that*

$$\begin{pmatrix} E' \\ A' \\ C' \end{pmatrix} = \begin{pmatrix} Q & 0 & F_E \\ 0 & Q & F_A \\ 0 & 0 & S \end{pmatrix} \begin{pmatrix} E \\ A \\ C \end{pmatrix} P.$$

It is easy to check that this relation is an equivalence relation.

Now, we consider a list of ranks of a certain matrices associated to the matrices E, A, C in the triple $(E, A, C) \in M$.

Definition 3 *We consider the following numbers*

1. $r_1 = \operatorname{rank} C$
2. $r_2 = \operatorname{rank} \begin{pmatrix} E \\ C \end{pmatrix}$
3. $r_3 = \operatorname{rank} \begin{pmatrix} A \\ C \end{pmatrix}$
4.
5. $r_4 = (r_4^1, \dots, r_4^\ell, \dots)$, *where*

   1) $r_4^1 = \operatorname{rank} M_4^1$ *with* $M_4^1 = \begin{pmatrix} E & A \\ C & 0 \\ 0 & C \end{pmatrix} \in M_{(n+2p)\times 2n}(\mathbb{C})$

   2) $r_4^2 = \operatorname{rank} M_4^2$ *with* $M_4^2 = \begin{pmatrix} E & A & 0 \\ C & 0 & 0 \\ 0 & E & A \\ 0 & C & 0 \\ 0 & 0 & C \end{pmatrix} \in M_{(2n+3p)\times 3n}(\mathbb{C})$

   $\vdots$

   ℓ) $r_4^\ell = \operatorname{rank} M_4^\ell$ *with* $M_4^\ell = \begin{pmatrix} E & A & & \\ C & 0 & & \\ 0 & E & \ddots & \\ 0 & C & \ddots & \\ & & & E & A \\ & & & C & 0 \\ & & & 0 & C \end{pmatrix} \in M_{s\times t}(\mathbb{C})$ *with* $s = (\ell n + (\ell+1)p)$ *and* $t = (\ell+1)n$.

Proposition 1 *In the set M of singular systems, the r_i numbers are invariant under the equivalence relation considered.*

Proof It is obvious for r_1, r_2, r_3. We are going to prove it for r_4. Let (E, A, C), (E', A', C') be two equivalent triples in M, then, there exist matrices $P, Q \in Gl(n; \mathbb{C})$, $S \in Gl(p; \mathbb{C})$, $F_E, F_A \in M_{n\times p}(\mathbb{C})$ such that

$$\begin{pmatrix} E' \\ A' \\ C' \end{pmatrix} = \begin{pmatrix} Q & 0 & F_E \\ 0 & Q & F_A \\ 0 & 0 & S \end{pmatrix} \begin{pmatrix} E \\ A \\ C \end{pmatrix} P$$

Calling $\mathbf{Q} = \begin{pmatrix} Q & F_E & 0 & F_A & & & \\ 0 & S & 0 & 0 & & & \\ 0 & 0 & Q & F_E & & & \\ & & & & \ddots & & \\ & & & & Q & F_E & F_A \\ & & & & 0 & S & 0 \\ & & & & 0 & 0 & S \end{pmatrix}$

and $\mathbf{P} = \begin{pmatrix} P & & \\ & \ddots & \\ & & P \end{pmatrix}$

$$r'^{\ell}_4 = \operatorname{rank}\begin{pmatrix} \begin{smallmatrix} E' & A' \\ C' & 0 \end{smallmatrix} & & \\ \begin{smallmatrix} 0 & E' \end{smallmatrix} & \ddots & \\ \begin{smallmatrix} 0 & C' \end{smallmatrix} & \ddots & \\ & & \begin{smallmatrix} E' & A' \\ C' & 0 \\ 0 & C' \end{smallmatrix} \end{pmatrix} =$$

$$= \operatorname{rank}\mathbf{Q}\begin{pmatrix} \begin{smallmatrix} E & A \\ C & 0 \end{smallmatrix} & & \\ \begin{smallmatrix} 0 & E \end{smallmatrix} & \ddots & \\ \begin{smallmatrix} 0 & C \end{smallmatrix} & \ddots & \\ & & \begin{smallmatrix} E & A \\ C & 0 \\ 0 & C \end{smallmatrix} \end{pmatrix}\mathbf{P} = r^{\ell}_4.$$

3 Observability condition

It is easy to compute the r^{ℓ}_4-numbers in the case where the system $(E, A, C) \in M$, is standardizable. As we say at the introduction, if a system (E, A, C) is standardizable, there exists a matrix F_E such that $E + F_E C$ is an invertible matrix. Taking into account that the r-numbers are invariant under equivalence relation we can suppose that the system is in the form (I_n, A_1, C_1).

$r^1_4 = \operatorname{rank}\begin{pmatrix} I_n & A_1 \\ C_1 & 0 \\ 0 & C_1 \end{pmatrix}$ making elementary block rows and columns transformations we have

$$r^1_4 = \operatorname{rank}\begin{pmatrix} I_n & 0 \\ 0 & -C_1A_1 \\ 0 & C_1 \end{pmatrix} = n + \operatorname{rank}\begin{pmatrix} C_1A_1 \\ C_1 \end{pmatrix} = n + \bar{r}_1$$

Analogously

$$r^{\ell}_4 = \operatorname{rank}\begin{pmatrix} \begin{smallmatrix} I_n & A_1 \\ C_1 & 0 \end{smallmatrix} & & \\ \begin{smallmatrix} 0 & I_n \end{smallmatrix} & \ddots & \\ \begin{smallmatrix} 0 & C_1 \end{smallmatrix} & \ddots & \\ & & \begin{smallmatrix} I_n & A_1 \\ C_1 & 0 \\ 0 & C_1 \end{smallmatrix} \end{pmatrix}$$

making elementary block rows and columns transformations we have

$$r^{\ell}_4 = \operatorname{rank}\begin{pmatrix} I_n & & & \\ & \ddots & & \\ & & I_n & \\ & & & C_1A_1^{\ell} \\ & & & \vdots \\ & & & C_1A_1 \\ & & & C_1 \end{pmatrix} =$$

$$= \ell n + \operatorname{rank}\begin{pmatrix} C_1A_1^{\ell} \\ \vdots \\ C_1A_1 \\ C_1 \end{pmatrix} = \ell n + \bar{r}_{\ell}.$$

Notice that, if we call $\bar{r}_0 = \operatorname{rank} C$, the numbers $\bar{r}_1 - \bar{r}_0, \dots, \bar{r}_{\ell} - \bar{r}_{\ell-1}, \dots$ are the r-numbers of the pair of matrices (A_1, C_1) defining the triple (I_n, A_1, C_1).

Proposition 2 *Let $(E, A, C) \in M$ a standardizable triple. Then there exists $s \in \mathbb{N}$ such that*

$$\begin{aligned} & r^n_4 + n < r^1_4 \\ & r^1_4 + n < r^2_4 \\ & \vdots \\ & r^{s-2}_4 + n < r^{s-1}_4 \\ & r^{s-1}_4 + n = r^s_4 \\ & \vdots \\ & r^{s+\ell}_4 + n = r^{s+\ell+1}_4,\ \forall \ell \in \mathbb{N}. \end{aligned}$$

Proof Let $(E, A, C) \in M$ a standardizable triple and (I_n, A_1, C_1) an equivalent triple. We consider the pair (A_1, C_1), it is well known (see [4], for example) that there exists $s \in \mathbb{N}$ such that

$$\bar{r}_1 < \dots < \bar{r}_{s-1} < \bar{r}_s = \dots = \bar{r}_{s+\ell} = \dots.$$

So,

$$\begin{aligned} & r^0_4 > 0 \\ & r^1_4 - r^0_5 - n = \bar{r}_1 - \bar{r}_0 > 0 \\ & r^2_4 - r^1_4 - n = \bar{r}_2 - \bar{r}_1 > 0 \\ & \vdots \\ & r^{s-1}_4 - r^{s-2}_4 - n = \bar{r}_{s-1} - \bar{r}_{s-2} > 0 \\ & r^s_4 - r^{s-1}_4 - n = \bar{r}_{s+1} - \bar{r}_s = 0 \\ & \vdots \\ & r^{s+\ell}_4 - r^{s+\ell-1}_4 - n = \bar{r}_{s+\ell} - \bar{r}_{s+\ell-1} = 0 \\ & \vdots \end{aligned}$$

A first application of r^i_4 numbers is a condition for observability of the systems.

Let (E, A, C) be a triple, as we say at the introduction a triple (E, A, C) is observable if and only if

$$\operatorname{rank}\begin{pmatrix} E \\ C \end{pmatrix} = n,$$

$$\operatorname{rank}\begin{pmatrix} sE - A \\ C \end{pmatrix} = n, \quad \forall s \in \mathbb{C}$$

Second condition is not easy to compute. We will go to show that the observability can be obtained computing the rank of a matrix.

Theorem 1 *A triple (E,A,C) is observable if and only if $r_4^{n-1}=n^2$.*

Proof. Suppose that the triple is observable, so verifies the two conditions for observability. The first one implies that the system is standardizable, and we can reduce it to a standard one. So, we can consider the equivalent standard system (I_n,A_1,C_1). Applying definition of observability to (I_n,A_1,C_1) we observe that it is observable if and only if (A_1,C_1) is observable, that is to say if and only if $\bar{r}_{n-1}=n$.

Conversely. It suffices to observe that if $r_4^{n-1}=n^2$, the first block column in the matrix M_4^{n-1} has full rank. So, the first conditio is verified, and we can consider the standard equivalent triple to compute the second condition.

4 Observability indices

We are going to introduce a collection of invariant numbers, which will permit us to deduce the canonical reduced form for a standardizable system.

Now, we call $r_4^0=\operatorname{rank}C$. We define the ρ-numbers in the following manner

$$\begin{aligned}\rho_0 &= r_4^0\\ \rho_1 &= r_4^1-r_4^0-n\\ \rho_2 &= r_4^2-r_4^1-n\\ &\vdots\\ \rho_s &= r_4^{s-1}-r_4^s-n\end{aligned}$$

It is obvious the following proposition.

Proposition 3 *The ρ-numbers are invariant under equivalence relation considered.*

Finally we define the *observability indices* $o_1,\dots,o_p$ for singular systems as the integers $o_1\geq\dots\geq o_p$ such that $[o_1,\dots,o_p]$ is the conjugate partition of $[\rho_0,\rho_1,\dots,\rho_s]$.

We observe that if the triple (E,A,C) is observable then $o_1+\dots+o_p=n$ and $p=\rho_0=\operatorname{rank}C$.

Theorem 2 *Let (E,A,C) be an observable triple with observability indices $(o_1,\dots,o_p)$. Then the triple can be reduced to (I_n,A_1,C_1) with*

$$A_1=\begin{pmatrix}N_1&&\\&\ddots&\\&&N_p\end{pmatrix},\quad C_1=\begin{pmatrix}C_1^1&&\\&\ddots&\\&&C_p^1\end{pmatrix}$$

and $N_i=\begin{pmatrix}0&1&&0\\&\ddots&\ddots&\\&&&1\\&&&0\end{pmatrix}\in M_{k_i}(\mathbb{C})$, $C_i^1=\begin{pmatrix}1&0&\dots&0\end{pmatrix}\in M_{1\times o_i}(\mathbb{C})$.

Proof It suffices to observe that the observability indices for (E,A,C) coincide with observability indices of the pair (A_1,C_1).

Example Let (E,A,C) with $E=\begin{pmatrix}3&1&1&1\\1&3&1&1\\1&1&3&1\\1&1&1&3\end{pmatrix}$, $A=\begin{pmatrix}0&0&0&0\\3&1&1&1\\0&0&0&0\\1&1&3&1\end{pmatrix}$, and $C=\begin{pmatrix}1&3&1&1\\1&1&1&3\end{pmatrix}$.

The observability indices are $o_1=2$, $o_2=2$ the the triple is equivalent to (E_1,A_1,C_1) with $E_1=I_4$, $A_1=\begin{pmatrix}0&1&0&0\\0&0&0&0\\0&0&0&1\\0&0&0&0\end{pmatrix}$ and $C_1=\begin{pmatrix}1&0&0&0\\0&0&1&0\end{pmatrix}$.

5 Conclusion

In this paper a characterization of the observability of a singular system $E\dot{x}=Ax$, $y=Cx$ in terms of a rank of a certain matrix is presented. This characterization simplifies the way to study observability.

References

[1] L. Dai "Singular Control Systems". Springer Verlag. New York (1989).

[2] Mª I. García-Planas, *Controllability indices for multi-input singular systems.* ICM06. European Mathematical Society, (2006), 542 - 542.

[3] Mª I. García-Planas, M.D. Magret, *An alternative System of Structural Invariants for Quadruples of Matrices*, Linear Algebra and its Applications **291**, (1-3), pp. 83-102 (1999).

[4] I. Gohberg, P. Lancaster, L. Rodman, "Invariant Subspaces of matrices with applications," Wiley, (1986).

[5] A.S. Morse, *Structural invariants of linear multivariable systems*, SIAM J. Contr. **11**, pp. 446-465 (1973).

[6] W. Ratz *Canonical forms for linear descriptor systems with variable coefficients*. Treihe der Chemnitzer DFG-Forschergruppe Scientific Parallel Computing, Chemnitz, Germany (1996).

[7] J. S. Thorp, *The Singular Pencil of Linear Dynamical System*, System. Int. J. Control, 18 (3) (1973), pp. 557-596.