Controllability matrix of second order generalized linear systems

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Abstract:- Let (E, A_1, A_2, B) be a quadruple of matrices representing a two-order generalized time-invariant linear system, $E\ddot{x} = A_1\dot{x} + A_2x + Bu$.

In this paper we study controllability of second order generalized systems by means the rank of a certain matrix that we will call controllability matrix of second order generalized linear systems.

Key-Words:- Linear systems, Feedback equivalence.

1 Introduction

The study of generalized linear systems have a great interest in recent years. First order are applied in engineering for example they are used in modelling a three-link planar manipulator by M. Hou [10]. Second order generalized systems are applied to power systems by Campbell and Rose [1].

A second order generalized linear system is described by the following state space equation

$$E\ddot{x} = A_1\dot{x} + A_2x + Bu, \qquad (1)$$

where A_i are n-square complex matrices and B a $n \times m$-rectangular complex matrix in adequate size. We denote this kind of systems by quadruples of matrices (E, A_1, A_2, B), and the space of all quadruples by $\mathcal{M}_{n,m}$:

$$\mathcal{M}_{n,m} = \{(E, A_1, A_2, B) \mid E, A_1, A_2 \in M_n(\mathbb{C}), B \in M_{n\times m}(\mathbb{C})\}$$

One of the problems for a control theory is to maintain stability and controllability of the system. If the system is not stable and/or not controllable then ones would like to chose the control variables u in such a way that the resulting system is stable and controllable. Suppose that the control variables are chosen as $u = -F_3\ddot{x} + F_1\dot{x} + F_2 + v$, then the system becomes $(E + BF_3)\ddot{x} = (A_1 + BF_1)\dot{x} + (A_2 + BF_2)x + v$. This system is called close-loop system while the system (1) is called open-loop system.

Controllability is a qualitative property of second order linear dynamical systems largely studied (see [9], [11], [5], [6], [8], [12] for example). In this paper we will go to study the controllability property for second order generalized linear systems.

2 Controllability

A second order generalized linear system is called controllable if, for any $t_1 > 0$, $x(0), \dot{x}(0) \in \mathbb{C}^n$ and $w, w_1 \in \mathbb{C}^n$, there exists a control $u(t)$ such that $x(t_1) = w$, $\dot{x}(t_1) = w_1$.

It is well known (see [4], for example), the following result.

Proposition 2 *A second order general-*

ized linear system (E, A_1, A_2, B), is controllable if and only if

$i)$ $\operatorname{rank}\begin{pmatrix} E & B \end{pmatrix} = n$
$ii)$ $\operatorname{rank}\begin{pmatrix} s^2E - sA_1 - A_2 & B \end{pmatrix} = n \quad \forall s \in \mathbb{C}$.

Remark 1 *Condition i) ensures that there exists a second order derivative feedback F such that $E+BF$ is regular and premultiplying the system by $(E+BF)^{-1}$ the new system is standard. We will call standardizable the systems verifying this property.*

In this paper we show that we can study controllability computing the rank of a certain matrix.

We consider the following $2n^2 \times ((2n-2)n + 2nm)$-matrix that we will call controllability matrix.

$$\mathcal{C} = \underbrace{\begin{pmatrix} -E & 0 & \dots & 0 \\ -A_1 & -E & \dots & 0 \\ A_2 & -A_1 & \dots & 0 \\ & & \ddots & \\ 0 & 0 & \dots & -E \\ 0 & 0 & \dots & -A_1 \\ 0 & 0 & \dots & A_2 \end{pmatrix}}_{(2n-2)n} \underbrace{\begin{pmatrix} B & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & B & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & B & \dots & 0 & 0 & 0 \\ & & & \ddots & & & \\ 0 & 0 & 0 & \dots & B & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & B & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 & B \end{pmatrix}}_{2nm}$$

Remark 2

i) If $n = 1$, $\mathcal{C} = \begin{pmatrix} B & \\ & B \end{pmatrix} \in M_{2\times 2m}(\mathbb{C})$,

ii) If $n = 2$, $\mathcal{C} = \begin{pmatrix} -E & 0 & B & 0 & 0 & 0 \\ -A_1 & -E & 0 & B & 0 & 0 \\ A_2 & -A_1 & 0 & 0 & B & 0 \\ 0 & A_2 & 0 & 0 & 0 & B \end{pmatrix} \in M_{8\times(4+4m)}(\mathbb{C})$,

iii) If $m = 1$, the matrix $\mathcal{C}$ is square.

The controllability character of a system is related to the rank of this matrix, as we can see in the following theorem.

Theorem 1 *A second order generalized linear system $(E, A_1, A_2, B) \in \mathcal{M}_{n,m}$, is controllable if and only if the controllability matrix $\mathcal{C}$, has full rank:*

$$\operatorname{rank} \mathcal{C} = 2n^2.$$

For the proof we need some lemmas

Considering $X = \begin{pmatrix} x \\ \dot{x} \end{pmatrix}$, we can rewrite the second order generalized linear system (1) as a generalized linear system

$$\mathbb{E}\dot{X} = \mathbb{A}X + \mathbb{B}u, \qquad (2)$$

with $\mathbb{E} = \begin{pmatrix} I_n & 0 \\ 0 & E \end{pmatrix}$, $\mathbb{A} = \begin{pmatrix} 0 & I_n \\ A_2 & A_1 \end{pmatrix}$ and $\mathbb{B} = \begin{pmatrix} 0 \\ B \end{pmatrix}$, we will call this system reduced system.

Lemma 1 *Controllability of the second order generalized system (1) is equivalent to the controllability of the reduced system (2).*

Proof.

i) $\operatorname{rank}\begin{pmatrix} \mathbb{E} & \mathbb{B} \end{pmatrix} = n + \operatorname{rank}\begin{pmatrix} E & B \end{pmatrix}$.

ii) $\operatorname{rank}\begin{pmatrix} s\mathbb{E} - \mathbb{A} & \mathbb{B} \end{pmatrix} =$
$\operatorname{rank}\left(s\begin{pmatrix} I & 0 \\ 0 & E \end{pmatrix} - \begin{pmatrix} 0 & I \\ A_2 & A_1 \end{pmatrix} \quad \begin{pmatrix} 0 \\ B \end{pmatrix} \right) =$
$\operatorname{rank}\begin{pmatrix} 0 & I & 0 \\ s^2E - sA_1 - A_2 & 0 & B \end{pmatrix} =$
$n + \operatorname{rank}\begin{pmatrix} s^2E - sA_1 - A_2 & B \end{pmatrix}$.

Lemma 2 *The generalized linear system (2), is controllable if and only if the generalized controllability matrix for generalized linear systems $M_{2n-1} \in M_{2n^2\times(2n-1)(n+2m)}(\mathbb{C})$,*

$$M_{2n-1} = \begin{pmatrix} \mathbb{E} & \mathbb{B} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \\ \mathbb{A} & 0 & \mathbb{B} & \mathbb{E} & \mathbb{B} & 0 & 0 & 0 & 0 & \\ 0 & 0 & 0 & \mathbb{A} & 0 & \mathbb{B} & \mathbb{E} & \mathbb{B} & 0 & \\ 0 & 0 & 0 & 0 & 0 & 0 & \mathbb{A} & 0 & \mathbb{B} & \\ & & & & & & & & & \ddots \end{pmatrix},$$

has full rank: $\operatorname{rk} M_{2n-1} = 2n^2$.

See [7] for a proof.

Now we can prove Theorem 1.

Proof. For our case the matrix M_{2n-1} has the form

$$\begin{pmatrix} I & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & E & \dots & 0 & 0 & B & 0 & \dots & 0 & 0 \\ 0 & I & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ A_2 & A_1 & \dots & 0 & 0 & 0 & B & \dots & 0 & 0 \\ & & \ddots & & & & & \ddots & & \\ 0 & 0 & & I & 0 & & & & 0 & \\ 0 & 0 & & 0 & E & & & & B & 0 \\ 0 & 0 & & 0 & I & & & & 0 & 0 \\ 0 & 0 & & A_2 & A_1 & & & & 0 & B \end{pmatrix}$$

Making block-elementary row transformations to the matrix, we obtain that it is rank equivalent to

$$\begin{pmatrix} I & & & & & & & \\ & \ddots^{(2n} & & & & & & \\ & & I & & & & & \\ & & & -E & & & B & & \\ & & & -A_1 & \ddots & & & B & \\ & & & A_2 & & & & & \ddots \\ & & & & & -E & & & \\ & & & & & -A_1 & & & \\ & & & & & A_2 & & & & B \end{pmatrix}$$

The rank of this matrix is

$$= 2n^2 + \text{rk}\begin{pmatrix} -E & 0 & \dots & 0 & B & 0 & 0 & 0 & 0 & 0 \\ -A_1 & -E & \dots & 0 & 0 & B & 0 & 0 & 0 & 0 \\ A_2 & -A_1 & \dots & 0 & 0 & 0 & B & 0 & 0 & 0 \\ & & \ddots & & & & & & & \\ 0 & 0 & \dots & -E & \dots & 0 & 0 & B & 0 & 0 \\ 0 & 0 & \dots & -A_1 & \dots & 0 & 0 & 0 & B & 0 \\ 0 & 0 & \dots & A_2 & \dots & 0 & 0 & 0 & 0 & B \end{pmatrix} = r$$

and $r = 4n^2$if and only if

$$\text{rk}\begin{pmatrix} -E & 0 & \dots & 0 & B & 0 & 0 & 0 & 0 & 0 \\ -A_1 & -E & \dots & 0 & 0 & B & 0 & 0 & 0 & 0 \\ A_2 & -A_1 & \dots & 0 & 0 & 0 & B & 0 & 0 & 0 \\ & & \ddots & & & & & & & \\ 0 & 0 & \dots & -E & \dots & 0 & 0 & B & 0 & 0 \\ 0 & 0 & \dots & -A_1 & \dots & 0 & 0 & 0 & B & 0 \\ 0 & 0 & \dots & A_2 & \dots & 0 & 0 & 0 & 0 & B \end{pmatrix} = 2n^2$$

Now, we present some examples,

Example 1

Let (E, A_1, A_2, B) be a quadruple with $E = \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}$, $A_1 = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$, $A_2 = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$,

$$\text{rk}\begin{pmatrix} -1 & -2 & 0 & 0 & 1 & 0 & 0 & 0 \\ -1 & -2 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & -1 & -1 & -2 & 0 & 1 & 0 & 0 \\ -1 & 0 & -1 & -2 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & -1 & 0 & 0 & 1 & 0 \\ 1 & 1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \end{pmatrix} = 8.$$

Then, the system is controllable.

Example 2 Let (E, A_1, A_2, B) be a quadruple with $E = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $A_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $A_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$,

$$\text{rk}\begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & -1 & 0 & 1 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{pmatrix} = 6 < 8.$$

Then, the system is not controllable.

Example 3

Let $(E, A_1, A_2, B) \in \mathcal{M}_{n,m}$ be a two-parametric family of quadruples of matrices with $E = \begin{pmatrix} 1 & 3 & 1 \\ 3 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$, $A_1 = \begin{pmatrix} 1 & 1 & 3 \\ 1 & 3 & 1 \\ 0 & 0 & 0 \end{pmatrix}$, $A_2 = \begin{pmatrix} \lambda & 3\lambda & \lambda \\ 3\lambda + \mu & \lambda + \mu & \lambda + 3\mu \\ 0 & 0 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$.

The matrix

$$\mathcal{C} = \begin{pmatrix} -E & 0 & 0 & 0 & B & 0 & 0 & 0 & 0 & 0 \\ -A_1 & -E & 0 & 0 & 0 & B & 0 & 0 & 0 & 0 \\ A_2 & -A_1 & -E & 0 & 0 & 0 & B & 0 & 0 & 0 \\ 0 & A_2 & -A_1 & -E & 0 & 0 & 0 & B & 0 & 0 \\ 0 & 0 & A_2 & -A_1 & 0 & 0 & 0 & 0 & B & 0 \\ 0 & 0 & 0 & A_2 & 0 & 0 & 0 & 0 & 0 & B \end{pmatrix},$$

has full rank if and only if $\lambda \neq 0$. That is to say, the quadruples of the given family are controllable if and only if $\lambda \neq 0$.

Observe that it is easier to compute rank $\mathcal{C}$ than rank $\begin{pmatrix} s^2E - sA_1 - A_2 & B \end{pmatrix}$ for all $s \in \mathbb{C}$.

3 Controllability indices

As in the generalized systems we can study the degree of controllability. We consider the following matrices

$\mathcal{C}_0 = \begin{pmatrix} B & \\ & B \end{pmatrix}$,

$\mathcal{C}_1 = \begin{pmatrix} -E & B & & \\ -A_1 & 0 & B & \\ A_2 & 0 & 0 & B \end{pmatrix}$,

$\mathcal{C}_2 = \begin{pmatrix} -E & 0 & B & & & \\ -A_1 & -E & 0 & B & & \\ A_2 & -A_1 & 0 & 0 & B & \\ 0 & A_2 & 0 & 0 & 0 & B \end{pmatrix}$

$\vdots$
$\mathcal{C}_{2n-2} = \mathcal{C}$
and the following numbers
$r_{-1} = \text{rank}\, B$
$r_0 = \text{rank}\, \mathcal{C}_0,$
$r_1 = \text{rank}\, \mathcal{C}_1,$
$r_2 = \text{rank}\, \mathcal{C}_2,$
$\vdots$
$r_{2n-2} = \text{rank}\, \mathcal{C}_{2n-2}.$
measuring the controllability degree.

It is not difficult to prove that the conjugate partition of $r_0 - r_{-1}, r_1 - r_0, \ldots, r_{2n-2} - r_{2n-3}$, is the collection of controllability indices of the triple $(\mathbb{E}, \mathbb{A}, \mathbb{B})$.

5 Conclusion

In this work a controllability matrix for second order generalized linear systems in the form $E\ddot{x} = A_1\dot{x} + A_2x + Bu$, is presented. This matrix depends only on the matrices E, A_1, A_2 and B defining the system and gives us a simple method to analyze the controllability of the system.

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