An Implementation for Stability In Hybrid Systems

LAVINIU BEJENARU
Department of Automatic Control,
University of Craiova
Decebal Str., No. 107, 200440, Craiova, Dolj, Tel.: +40.251.438198
ROMANIA

.

Abstract: This work treats hybrid systems that concern some aspects of the discrete states and some aspects of stability. I show a way through we can determinate automatically a fundamental cycle in a hybrid system, which is useful for hybrid systems with many states. Another aspect from this paper is to present a way to determinate a switching sequence which assures the stability in a LTI switched systems, a special case of hybrid systems. Also, I give some examples.

Keywords: Hybrid systems, limit cycles, stability, graph theory, algorithm.

1 Introduction

A switched systems is a hybrid systems which consist of several subsystems and a rule that switch control the switching among them. The problem of switching stabilization is to find a switching rule which to make the switching system stable or, better, asimptotically stable. The stability which is considerated is in the sense of Lyapunov. In literature the problem of stability was extensively studied. The reader can consult recent developments in [7], [11]. There are several constructive design procedures in [16], [17]. An another tool is given by Lyapunov-like functions which were introduced to cope with intrinsic discontinuous nature of switched systems and it can be consulted [4], [14], [18].

The obtain one sequence of Lyapunov functions is a priority in hybrid systems field. Different results for stability with Lyapunov functions can be found in [5]. An hybrid automaton is a mode to represent a hybrid system. Some details can be found in [12], [15]. More, the idea to use graph theory in hybrid system was adopted by Hogan and Homer in [9]. The contribution of this paper is the finding one switching scheme for stabilization of a switched linear systems. I introduce the notion of hybrid graph and I use an algorithm to find a fundamental cycle in a hybrid systems.

2 Preliminaries

Let $\mathbf{R}^n$ denote the nth-dimentional Euclidian real space and let the system be

$$\begin{cases} \dot{x}(t) = f(x(t), q(t)) \equiv f_{q(t)}(x(t)), \\ q(t) = s(x(t), q(t^-)) \end{cases}, \quad (1)$$

with initial conditions

$$x(t_0) = x_0, \quad q(t_0) = i_0, \quad (2)$$

where $x(t) \in \mathbf{R}^n$ is the continuous state vector and $q(t) \in Q = \{1, 2, ..., n_Q\}$ is the discrete state. The hybrid state space is $H = \mathbf{R}^n \times Q$, being a combination between continuous and discrete spaces. Sometimes, $f_{q(t)}(x(t)) = A_{q(t)} \cdot x(t)$, where if the matrixes A are constants, we have linear time-invariant (LTI) switched systems.

2.1 Fundamental cycles

A *fundamental cycle* is a closed sequence of discrete states where each occurs once, i.e.

$$i_1, i_2, ..., i_d, i_1 \text{ where } i_p \neq i_q \text{ for } p \neq q.$$

In [8] it is shown that fundamental cycles is a means to describe the switch structure of a hybrid system. This notion is a step to find limit cycles. So, an isolated closed orbit is referred to as a *limit cycle*, where by *closed orbit* we mean that a state system maps a certain state into itself. It will appear a set of points on the switch sets, named *switch points* and using them can be determined the stability of the limit cycle. The stability can be found by discrete-time Lyapunov theory and using LMI. All techniques are presented in [13].

2.2 Stabilizability

We are interested to find a switched sequence to stabilize the system. This means in the hybrid field that we search a discrete sequence of commutation for that the state of the hybrid systems is stable.

Lyapunov-like functions When we considerate a system $\dot{x} = f_i(x)$ and an equilibrium point $\bar{x} \in \Omega_i \subset \mathbf{R}^n$, a *Lyapunov-like function* is a real-valued function $V_i(x)$, with continuous partial derivatives, defined over the region Ω_i and satisfying the condition:

a.positive definitess: $V_i(\bar{x}) = 0$ and $V_i(x) > 0$ for $\bar{x} \neq x$, $x \in \Omega_i$ (often $\bar{x} = 0$)

b.negative definite derivative: for $x \in \Omega_i$

$$\dot{V}_i(x) = \frac{\partial V_i}{\partial x} \cdot f_i(x) \leq 0 \tag{3}$$

We define a Lyapunov-like family $\{V_i, i = 1, 2, \ldots, n_Q\}$. So, we have from [7]:

Theorem 1. Giving the *M*-switched nonlinear $\dot{x} = f_{q(t)}(x(t))$, suppose each vector field f_i has an associated Lyapunov-like function V_i in region Ω_i, each with equilibrium point $\bar{x} = 0$, and suppose $\bigcup_i \Omega_i = R^n$. Let $q(t)$ be a given switching sequence such that $q(t)$ can take on the value i only if $x(t) \in \Omega_I$, and in addition:

$$V_i(x(t_{i,k})) \leq V_i(x(t_{i,k-1})) . \tag{4}$$

where $t_{i,k}$ denotes the kth time that vector field f_i is "switched in" . Then the (1) is Lyapunov stable.

2.3 Switch sets

The function $s: H(=\mathbf{R}^n \times Q) \to Q$ by $s(x, i) = j$, where $x \in \mathbf{R}^n$ and $i,j \in Q$, means that we have *a change* of discrete state from state i to state j. To put together all continuous states which are involved in this change, we define *switch set*. One image of the switch sets is in [13]. In [5] the notion of *switch set* is replaced by *switch surface*. So, the switch set $S_{i,j}$ is defined by

$$S_{i,j} = \{x \in R^n \mid s(x,i) = j\} \tag{5}$$

For every $i \in Q$, the vector field $f(\cdot,i): R^n \to R^n$ is assumed to be locally Lipschitz continuous.

The switch set can be given by *switch functions*. So, if a switch function is a map $s_{i,j}: \mathbf{R}^n \to \mathbf{R}^n$, then the switch set can be defined as $S_{i,j} = \{x \mid s_{i,j}(x) = 0\}$. Generally, the switch functions represent hyperplanes in the extended state space, i.e. $s_{i,j}(x) = C_{i,j}x + D_{i,j}$. Let assume that we have for our system *m* switch sets.

3 Problem Solution

I introduce the notion of hybrid graph in order to use graph theory for a hybrid system. This use us to determinate automatically the fundamental cycles.

I propose an algorithm that will assure the stability and a right choose of the switching sequence. We use the representation of the hybrid system by hybrid graph.

The technique is the next: we start from a node of the hybrid graph and we search for the corresponding Lyapunov-like function another node for that we'll have decreasing values in the sense of. Theorem 1 More, we can search for every switching another Lyapunov-like function for our conditions.

3.1 Hybrid graph

Definition 1. A *hybrid node* is an ensemble compose from a discrete state *i* and a motion trajectory corresponding to the state *i*, $\dot{x} = f(x,i)$.

So, for a discrete state *i* we add the trajectory for discrete state and in this way we guarantee move of the continuous state from discrete state *i* to next discrete state *j*. The evolution of the hybrid dynamically system's trajectory from discrete state *i* to discrete state *j* is conform with $\dot{x}(t) = f(x,i)$, or, write as $\dot{x}(t) = f_i(x)$. This trajectory reaches the *switch set* $S_{i,j}$ from where it evolves conform with $\dot{x}(t) = .f_j(x)$.

Definition 2. A *hybrid graph* is a graph where the nodes are composed from *hybrid nodes* and the edges of the hybrid graph are the switch sets corresponding for two discrete state.

So, if we denote a hybrid node (i, $\dot{x} = f(x,i)$) with h_i, then the hybrid graph is a set of *n* nodes h_1, h_2, …, h_n, where *n* is the number of the *branches* of the hybrid system. If we have *m* switch sets for the systems, then the hybrid graph will have *m* edges. The hybrid graph is a kindred notion of the hybrid automatons about which you can read in [12]. You can see an example of hybrid graph in *Figure 1*.

There is a aspect for fundamental cycles: to find them. Here we can help by the hybrid graph associated with a hybrid system (see *Definition* 2).

Theorem 2. For the hybrid system (1) it can be determined if there are fundamental cycles and which are there the fundamental cycles automatically.

Proof: The proof will be make in steps.

Step 1. For the hybrid system (1) we find number of branches and associate the hybrid graph corresponding. In Section 4 we have an example with *4* number of branches (the same as discrete states) and the hybrid graph is shown in *Figure* 1. There is possible to add some switch sets.

Step 2. For the hybrid graph, we associate a directed graph which is formed by the discrete states of the hybrid graph as nodes and the edges are corresponding with switch sets $S_{i,\ j}$. In this way we have in example the directed graph from *Figure* 2.

Step 3. We associate to the graph a matrix named *adjacent matrix* which is a *n*-dimensional quadratic matrix and it is binary, where *n* is number of nodes. $A[i,j] = 1$ in the adjacent matrix *A* there is the switch set $S_{i,\ j}$ (and not $S_{j,\ i}$); otherwise it is equals to zero. For our example, we have the adjacent matrix *A*=(0,1,1,0; 0,0,1,0; 1,0,0,1; 0,1,1,0). Using the adjacent matrix we can give some information for such a directed graph using specific techniques and algorithms. First, through the depth cross algorithm (or *depth first* algorithm) we know if there are cycles or not.

An *elementary cycle* for a directed graph is a cycle in which every nodes are different, except first and last. An example of not elementary cycle is 1,3,4,2,3,1 because the inner node 3 is twice, but 1,2,3,1 is on. To determinate possible fundamental cycles in the hybrid system is the same with to determinate elementary cycles in the associated directed graph. For our case the elementary cycles, and so the fundamental cycles, are: a)1,3,1; b)1,2,3,1; c)2,3,4,2; d)3,4,3 because the cycles 1,3,1 and 3,1,3 are considered the same. To determine elementary cycles in the graph we can use a specific *algorithm* and a computer program and all can be done automatically. These algorithms are well-studied and they can be found, by example, in [6].

3.2 Stabilization algorithm

The idea to use algorithms in hybrid systems can be found in other papers, like in [1], [2], [3]. Another way in the stability of hybrid systems is through linear matrix inequality (LMI) and solve them in by LMI or SeDuMi Toolboxes from Matlab.

I propose an algorithm which finds the discrete sequence of the switching in a hybrid system to assure the stability of this. The idea: we start from a node, let it say i_1. Begging from this node and considerate the hybrid graph, we search a Lyapunov-like function for this branch and another node that satisfies condition (4). The form of the Lyapunov-like function is a quadratic form $V(x) = x^t P x$, where $P > 0$. For a computational proper form of the *V* we use the definition: a matrix *A* is positive-defined if it is symmetric and it has the dominant diagonal and $a_{ii}>0$ for $1<=i<=n$. The algorithm is the next:

Step **1** we associate the hybrid graph for the system (1) + the switch sets

Step **2** we extract all discrete nodes from the hybrid graph and we form the directed graph

Step **3** we search a fundamental cycles in the directed graph (using specific algorithms like *depth first*). Let it be obtained in the *I*. Let I_C be the set of choose points. Initial $I_C = \phi$. We associate adjacent matrix *A*.

Step **4** FOR every $i1 \in I$ we make the next actions:

4.**1** WHILE $I_C \neq I$ AND process is running: pr=1

4.1.**1** $I_C = I_C \cup \{i1\}$.

4.1.**2** FOR all switch set which has *i*1 the first index

4.1.2.**1** we choose a point $x_1 \in$ switch set and we store the value $val1 = V(x_1)$

4.1.2.**2** FOR every $i2 \in I - I_C$ (an unchoice node)

4.1.2.2.**1** search a Lyapunov-like function (using definition)

4.1.2.2.**2** find x_2 in a proper mode $x_{i2}=\{ x \in R^n \mid \dot{x} = A_{i1}x + b_{i1} \} \cap S_{i1,\ i2}$

4.1.2.2.**3** we store $val2 = V(x_2)$

4.1.2.2.**4** pr=0 (we stop the process)

4.1.2.2.**5** IF we have condition (4) using *val*2 and *val*1

4.1.2.2.5.**1** $I_C = I_C \cup \{i1\}$

4.1.2.2.5.**2** store Lyapunov-like function

4.1.2.2.5.**3** $val1 = val2$

4.1.2.2.5.**4** $i1 = i2$

4.1.2.2.5.**5** pr=1

4.**2** IF we have pr=0, then the process was stopped: and we haven't stability

4.**3** IF pr=1, THEN we have a switched sequence for all subsystems which stabilizes the hybrid system

Step **5** We put again $I_C = \phi$ and we go to the *Step* **4**.

Theorem 3. The solution I_C of the above algorithm stabilize the hybrid systems.

Proof: The discrete sequence stored into I_C is composed from all nodes of the hybrid graph which means all nodes of the hybrid system. By the construction of the algorithm, for every node *i* which is chosen to add to I_C we store the Lyapunov-like function that satisfies the condition (4). This way assures decreasing values for the trajectory of the hybrid system in every switching. The point x_{i2} from the algorithm is the point on trajectory that intersect the switch set, so $x_{i2}=\{ x \in R^n \mid \dot{x} = A_{i1}x + b_{i1} \} \cap S_{i1,\ i2}$. The right choise of this point is made using the $A[i1][i2] = 1$ value from adjacent matrix and the equation of the switch set $\{ x \in R^n \mid C_{i2} \cdot x + d_{i2} = 0\}$. So, we have a discrete sequence i_1, i_2,…, i_n and a corresponding decreasing value sequence $V_{i1}(x_{i1})$, $V_{i2}(x_{i2})$,…, $V_{in}(x_{in})$ along the trajectory of the hybrid system that assures by Theorem 1 the stability.

4 Examples

Example 1. The first example is an example in which we present results for the fundamental cycles. We have a system which consist of two tanks and two on/off valves. The first valve adds to the inflow in tank one and the second valve is a drain valve from tank two; there is a constant outflow from tank two, e.g. caused by a pump. The tow discrete states of the valves result in four discrete modes with different continuous dynamics

Without changing the settings of the valves, the tanks will either be flooded or drained. Switching is based on the water-levels x_1 and x_2 in the tanks. The notation x_i is used for elements in the state vector.

For each initial continuous state there is an associated unique initial discrete mode: $q_1=q(t)=1$ (*off* / *off*); $q_2=q(t)=2$ (*on* / *off*); $q_3=q(t)=3$ (*off* / *on*); $q_4=q(t)=4$ (*on* / *on*). The above relations shows the discrete modes and state space regions where they are used as initial modes. Here, the systems is two dimensional.

The modelling of the tow-tank system gives the hybrid system: $H = \boldsymbol{R}^2 \times Q$ is hybrid state space; $Q = \{q_1,q_2,q_3,q_4\}$is discrete state space (and it is identical with {1,2,3,4} by a bijection). The continuous dynamics on the continuous state space $\boldsymbol{R}^2$ are $\dot{x}(t) = A_q \cdot x(t) + B_q$, where $q \in Q$.

For the hybrid graph we associate the directed graph. The hybrid graph can be seen in Fig.1 and the directed graph can be seen in Fig.2.The elementary cycles for the directed graph are the fundamental cycles for the hybrid graph. So, using a proper algorithm for the graphs and his implementation in a software, we have: a)1,3,1; b)1,2,3,1; c)2,3,4,2; d)3,4,3 because the cycles 1,3,1 and 3,1,3 are considered the same.

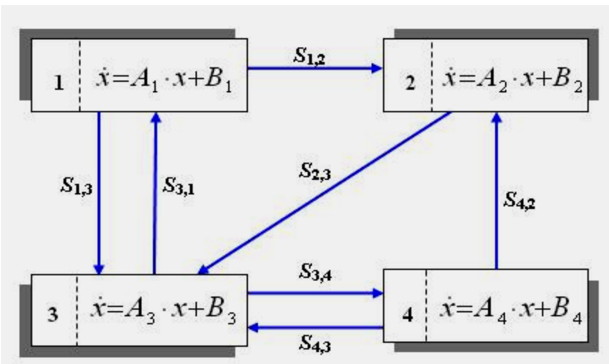

Fig.1. The hybrid graph associated to the given hybrid systems. It consists of hybrid nodes. A hybrid none has discrete and continuous state.

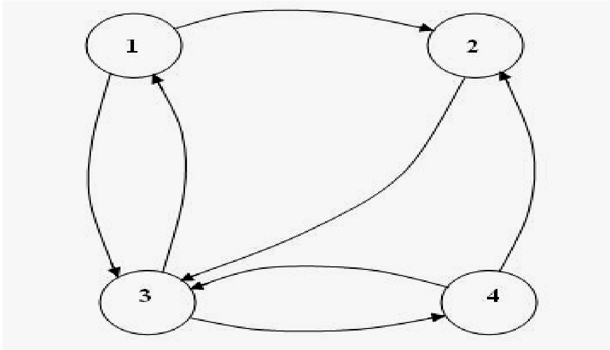

Fig.2. The directed graph associated to the hybrid graph. In this graph we can use the algorithms for cross and in this way we can determinate the cycles

The importance of that theory to determinate automatically the fundamental cycles is better if the systems are more complex, with many nodes. In a same way, only the computer can determinates automatically and rapid the solutions.

Example 2. In this example I give the result of algorithm from Section 3 about discrete sequence.

For a two dimensional case, and $f_{q(t)}(x(t)) = A_{q(t)}x(t) + B_{q(t)}$, with values:
$A_1 = \begin{pmatrix} -1 & 0 \\ 0 & -3 \end{pmatrix}$; $B_1 = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$; $A_2 = \begin{pmatrix} -2 & 0 \\ 0 & -0.5 \end{pmatrix}$;
$B_2 = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$ $A_3 = \begin{pmatrix} -1 & 0 \\ 0 & -1.5 \end{pmatrix}$; $B_3 = \begin{pmatrix} -2 \\ -3 \end{pmatrix}$ we obtain first switch points P_1 = (-1500 -2.1200)'; P_2 = (0.1040 0.4268)'; P_3 = (1.2119 -1.0159). This points belong to the three switch sets. The sequence for switching is 1 (with initial value P_1), 2 (with initial value P_2), 3(with initial value P_3). In Fig.3 we can see an aspect of the hybrid system.

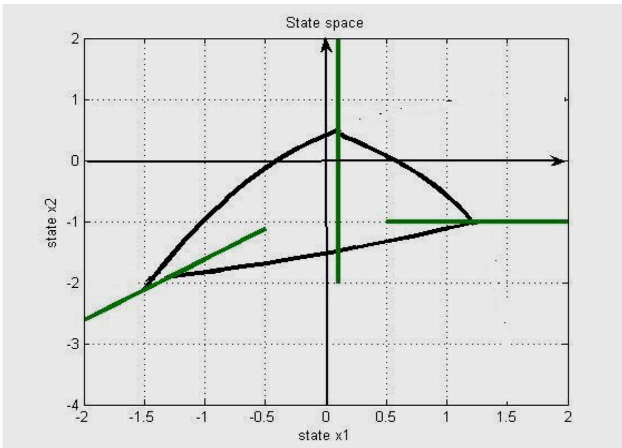

Fig.3 The trajectory and switch sets of the hybrid systems. P_1, P_2 and P_3 are the points where trajectory intersects the switch sets.

5 Conclusions

It has been introduced the hybrid graph notion, which is in relation with the graph theory. It has been show how to find fundamental cycles in a hybrid systems in a automatically way. The efficiency of this method is bigger for the big hybrid systems, with many nodes. Also, it was been presented an algorithm for finding the switching sequence in a LTI switched system which assure the stability.

References:

[1] Bemporad, A., G. Ferrari-Trecate and M. Morari, Observability and Controllability of Piecewise Affine and Hybrid Systems, *Proceedings of the 38 Conference on Decision & Control*, Phoenix, Arizona, USA, 1999.

[2] Berovic, D.P. and R.B. Vinter, The Application of Dynamic Programming to Optimal Inventory Control, *IEEE Transaction On Automatic Control*, Vol.49, No.5, 2004, pp.676-685.

[3] Branicky, M. and G. Zhang, Solving Hybrid Control Problems: Level Sets and Behavioral Proramming, *Proceedings of the American Control Conference*. Chicago, IL, USA, 2000.

[4] Branicky, M. S., Multiple Lyapunov functions and other analysis tools for switched and hybrid systems, *IEEE Transactions on Automatic Control*, Vol.43, No.X, 1998, pp. 475-482.

[5] Branicky, M.S., Studies in hybrid systems: Modeling, analysis and control, *Ph.D. dissertation,* Dept. of electrical and Computer Engineering, MIT, USA, 1995.

[6] Cormen, T.H., C.E. Leiserson and R.R. Rivest, *Introduction to Algorithms,* MIT Press, USA, 1990.

[7] DeCarlo, R.A., M.S. Branicky, S. Petterson, and B.Lennartson, Perspective and results on the stability and stabilizability of hybrid systems. *Proc.IEEE*, Vol.88, No.X, 2000, pp. 1069-1082.

[8] He, K.X. and M.D. Lemmon, Lyapunov stability of continuous-valued systems under the supervision of discrete-event transition systems, *Hybrid Systems: Computation and Control, LNCS 1386*, 1998, pp. 175-189, Springer-Verlag.

[9] Hogan, S.J. and M.E. Homer, Graph theory and piecewise smooth dynamical systems of arbitrary dimension, *Chaos Solitons Fractals*, 10, 1999, pp. 1869-1880.

[10] Homer, M.E., Bifurcations and dynamics of piecewise smooth dynamical systems of arbitrary dimension, *Ph.D. thesis*, University of Bristol, 1999.

[11] Liberzon, D. and A. S. Morse, Basic problem in stability and design of switched systems, *IEEE Control Systems Magazine*, Vol.19, No.X, 1999, pp. 59-70.

[12] Lunze, J., R. Drath and J. Raisch, other, *Modelling, Analysis, and Design of Hybrid Systems,* (S. Engell, G. Frehse and E. Schnieder (Ed)), *Lecture Notes in Control and Information Science (LNCIS)*, Vol. 279, Heidelberg: Springer-Verlag,. Berlin, 2002.

[13] Rubensson, M. and B. Lennartson, Stability of Limit Cycles in Hybrid Systems using Discrete-Time Lyapunov Techniques, *Proc. Conf. Decision Control*, 2000, pp. 1397–1402, Sydney, Australia.

[14] Savkin, A.V. and R.J. Evans, *Hybrid dynamical Systems: Controller and Sensor Switching Problems,* MA:Birkhauser, Boston, 2002.

[15] van der Schaft, A. and H. Schumacher, An Introduction to Hybrid Dynamical Systems, *Lecture Notes in Control and Information Sciences,* 251, Springer; 1999.

[16] Wicks, M.A. , P.Peleties and R.A. DeCarlo, Switched controller synthesis for the quadratic stabilization of a pair of unstable linear systems, *European Journal of Control*, Vol.4, No.2, 1998, pp. 140-147.

[17] Xu, X. and P.Antsaklis, Stabilization of second order LTI switched systems, *International Journal of Control*, Vol.73, No.14, 2000, pp. 1261- 1279.

[18] Ye, H., A.N. Michel and L. Hou, Stability theory for hybrid dynamical systems, *IEEE Transactions on Automatic Control*, Vol.43, No.4, 1998, pp. 461-474.