ADAPTIVE FINITE ELEMENT ANALYSIS FOR SOLUTION OF COMPLEX ENGINEERING PROBLEMS

LINA VASILIAUSKIENĖ, SAULIUS VALENTINAVIČIUS, ANTANAS ŠAPALAS
Graphical Systems, Information Technology, Steel and Timber Structures Departments
Vilnius Gediminas Technical University
Saulėtekio av. 11, Vilnius
LITHUANIA

Abstract: - In this article authors present a simple strategy for displacement solution calculation near geometrical singularity points. Solution improvement in singularities is achieved modifying physical coordinate matrix used in least square problem equations: instead of the standard quadratic polynomial a new polynomial with singular elements is taken. Proposed algorithm is incorporated to the adaptive finite element analysis and tested with real engineering structure.

Keywords: - singular points, error estimation, adaptive mesh refinement, finite elements.

1. Introduction

In many applications the primary aim of a finite element (FE) analysis is to obtain few design quantities with a prescribed accuracy. In structural mechanics such quantities as stresses, displacements or surface tractions are often important for design decisions. For this reason special attention should be paid to how our FE model is adapted for computing these quantities and to which techniques are used to extract these quantities from the primary FE results. In order to perform a reliable finite element simulation a lot of works have been made trying to integrate the finite element analysis, error estimation and automatic mesh modification [1–5]. But sometimes even very effective adaptive mesh refinement strategy does not give us a suitable result if our problem domain is caused by re-entrant corners and abrupt changes in material properties. Singularities occur at crack tips or at interface problems and are of great interest from the point of view of failure analysis [6, 7]. Singularities and unbounded domains cause difficulties in standard finite element analysis because in the vicinity of the singular points they depend on the structure of eigen values and eigen functions which may not be known a priori [8, 9]. For this reason an improved method for solution calculation near singular points is presented in this paper. After incorporating this method into *h*-adaptive finite element analysis and testing with different mesh optimality criteria it can be stated that proposed strategy gives us final optimal mesh with minimal element and iteration number. The efficiency of the proposed strategy is demonstrated with real engineering problem.

2. Basic Concepts of the Adaptive Finite Element Analysis

With the help of reliable error estimation procedures available today, a suitable element size distribution can be predicted and new mesh is constructed using adaptive mesh generation algorithms. Prediction of finite element size is based on the results of a previous stage of analysis. In this manner, subsequent meshes of better quality are designed adaptively. After a few trials, solution with an accuracy corresponding to a user specified tolerance is obtained. Adaptive finite element analysis algorithm uses intermediate results for mesh modification in such way, that the final mesh is in some sense optimal. Iterative process is controlled by an error tolerance.

The classical adaptive algorithm starts from the user defined initial mesh and user specified error tolerance. After that FE-solution is computed on a given mesh. Then the resulting error distribution is evaluated and compared with the tolerance – if the error tolerance is not met, a new mesh is generated based on the obtained error distribution. Analysis is terminated if the estimated error is less than or equal to the tolerance.

The most important ingredient of the error estimation is the construction of the new solution of a higher quality since the exact solution for complex-engineering problems is generally unknown. Typically, this new improved solution is obtained by a posteriori procedure, which utilizes the original finite element solution itself. The essence of the postprocessed error estimator is to replace the exact solution with a postprocessed solution of higher quality:

$$\boldsymbol{e}_u \approx \bar{\boldsymbol{e}}_u = \boldsymbol{u}^* - \boldsymbol{u}^h, \quad (1)$$

where $\bar{\boldsymbol{e}}_u$ is the point-wise estimated error. Using the improved solution we have an error estimation:

$$\|\bar{e}\| = \left(\int_{\Omega} \bar{e}_u^T L \bar{e}_u \, d\Omega \right)^{\frac{1}{2}}. \quad (2)$$

In practice we calculate this norm by summing over all elements in whole domain Ω:

$$\|\bar{e}\|^2 = \sum_{i=1}^{nel} \|\bar{e}\|_i^2 = \sum_{i=1}^{nel} \int_{\Omega_i} \bar{\boldsymbol{e}}_u^T \boldsymbol{L} \bar{\boldsymbol{e}}_u \, d\Omega_i \ , \quad (3)$$

where Ω_i is an element domain and *nel* is the total number of elements.

The absolute error defined by an energy norm is not convenient for use in practical computations. The dimensionless forms are favored and are customarily expressed as

$$\eta = \frac{\|e\|}{\|u\|} \ , \qquad \eta_k = \frac{\|e_k\|}{\|u\|} \ , \quad (4)$$

where $\|u\|$ is the strain energy norm, η and η_k are the relative global and relative element error, respectively.

The problem in *h*-adaptivity for finite element methods may be formulated as follows – construct a finite element mesh with as few degrees of freedom, as possible, such as that

$$\bar{\eta} \le \eta_{TOL}, \quad (5)$$

where η_{TOL} is the maximum permissible error.

This is a non-linear minimization problem, which may be solved, approximately in an iterative process. The mesh is redefined using new element size calculated according formula:

$$h_i' = h_i \,/\, \xi_i^{1/p}, \quad (6)$$

where h_i – an old mesh element size, ξ_i – element refinement parameter, based on some mesh optimality criterion, p – degree of shape functions polynomials.

It is usually agreed that a solution is acceptable if the global error in energy is below a specified value of the total strain energy:

$$\|e\| \le \eta \|u\|, \quad (7)$$

there η is the user's specified value of the permissible relative global error.

According this equation the global error parameter ξ_g is defined as

$$\xi_g = \|e\| / \eta \|u\| \quad (8)$$

and the local error indicator $\bar{\xi}_i$ is defined as

$$\bar{\xi}_i = \|e\|_i / \|e\|_{r_i}. \quad (9)$$

There $\|e\|_i$ is the actual error in each element and $\|e\|_{r_i}$ is the required error norm in the element. This definition of the required error in each element $\|e\|_{r_i}$ is a key issue and strongly affects the distribution of element sizes in the mesh.

In general both local and global criterion must be satisfied. This allows the definition of element refinement parameter using both local and global error parameters:

$$\xi_i = \bar{\xi}_i \xi_g = \frac{\|e\|_i}{\|e\|_{r_i}} \cdot \frac{\|e\|}{\eta \|u\|} = \frac{\|e\| \|e\|_i}{\eta \|u\| \|e\|_{r_i}}. \quad (10)$$

This element refinement parameter was first introduced in [10] and since then was used as the basis for defining the new element size in a general adaptive re-meshing strategy.

In the adaptive analysis finite element solution is sought not only to satisfy a prescribed accuracy but to be associated with a reasonably optimal mesh also. A very popular mesh optimality criterion for elliptic problems is that a mesh is said to be optimal if the distribution of the energy norm is equal between all elements [11]. According this definition the required error for the element is defined as the ration between the global error and the total number of mesh elements. Then element refinement parameter is defined as:

$$\xi_i = \bar{\xi}_i \xi_g = \frac{\|e\|_i}{\gamma \|u\| \sqrt{n}}. \quad (11)$$

Parameter ξ_i can now be interpreted as the ratio of the element error and the distributed value of the permissible error over the mesh. Parameter $\xi_i > 1$ will indicate that the element should be further refined and $\xi_i \le 1$ means that both global and local error conditions are satisfied. The new element size is obtained according (6) formula.

3. Postprocessing Error Estimation Accounting Singularity Effect

3.1 Problem statement

Let us suppose we have prismatic planar domain Ω with the interior angle $\omega > \pi$. *C* the singular point of Ω. Radius R is the value defining singular zone boundary for Ω (Fig.1).

In general solution *u* has singular behavior near singular points and can be decomposed into singular and regular parts according formula [12]:

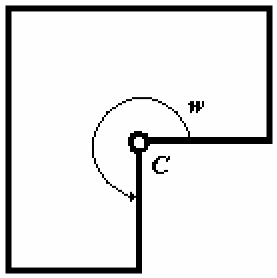

Fig.1 Problem domain

$$u = u_s + u_R = \xi(r) \cdot \gamma(x) \cdot r^{\lambda} \cdot \sin \lambda\varphi + u_R \,, \qquad (12)$$

$$\lambda = \pi/\omega \,. \qquad (13)$$

There r, ϕ are polar coordinates, $\xi(r)$ is a smooth function ($\xi(r) = 1$ for $r < R$, $\xi(r) = 0$ for $r > R$, R is a constant), $\gamma(x)$ is a coefficient

3.2 Displacement solution calculation near singular point

In order to use (12) formula in adaptive finite element analysis we must find regular and singular parts of the solution *u*.

As a regular part we can take finite element solution $u_R = u_{FEM}$. For finding singular part, we will extend superconvergent patch recovery for displacements (SPRD) technique. The idea of the SPRD is to define a new displacement field of *p+1* order over the patch elements [13]. This new field requires to be a least square fit to the original finite element solution at some points where the accuracy of finite element solution is higher. It has been known that the nodal points of the finite element approximation are found to be the exceptional points at which the prime variables (displacements) have higher order accuracy with respect to the global accuracy [14].

The new displacement field over the element mesh is calculated solving equation system for least square problem:

$$\left(\sum_j w_j^2 \left(Q(x_j)\right)^T Q(x_j) \right) b_i = \left(\sum_j w_j^2 \left(Q(x_j)\right)^T \right) \left(u_j^h\right)_i \,. \qquad (14)$$

There $Q = \begin{bmatrix} 1 & x & y & x^2 & xy & y^2 \end{bmatrix}$ is a row matrix containing monomial term of physical coordinates of *p+1* order, and *b* is a set of unknown parameters to be determined. $w(x)$ is a positive weighting function with unity value for the element, defining the patch, and which decreases monotonically with

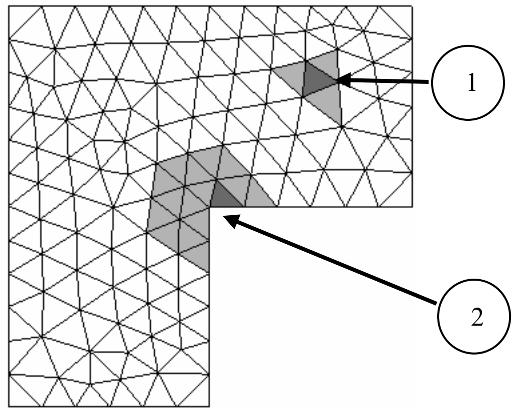

Fig.2 Element patch construction for least square problem: (1) – traditional element patch; (2) – singular element patch.

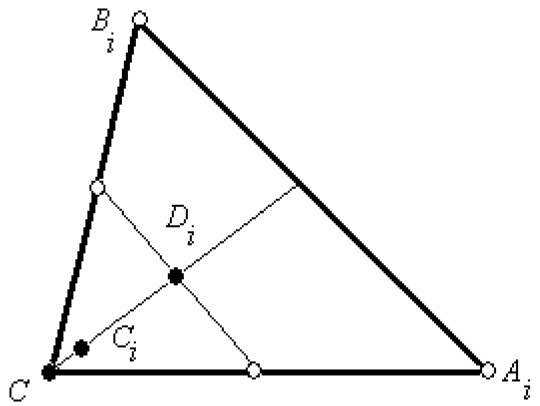

Fig.3 Definition of additional points near singularity

increasing distance away from master element.

Let us suppose domain Ω is covered by *N* triangle elements. Current mesh element is said to be singular if at least one its nodes is placed at the distance less or equal to radius *R*. Set of all singular triangles defines singular zone of Ω and forms the patch for the least square problem.

The main idea of least square method modification is to replace row matrix containing monomial term of physical coordinates with new one containing singular monomials for all nodes of this singular patch. To realize this instead of standard quadratic polynomial we will take only one singular monomial:

$$Q(x) \rightarrow Q_{SING}(r, \phi) = \left\lfloor \xi(r) \cdot r^{\lambda} \cdot \sin \lambda\phi \right\rfloor . \qquad (15)$$

According such modification equation system for least square problem will be solved with two different element patch types (Fig.2) and two different row matrixes depending on whether node belongs to singularity zone or no.

In this place we have one problem left – singular point *C* in polar coordinates is defined as $r = 0$ and $\phi \in [0, 2\pi]$ and singular polynomial can not be defined correctly in point C. In order to solve this conflict for all mesh elements containing singular point C we calculate additional points C_i placed at a very near position from original point C (Fig.3):

$$C_i = (1-t)\cdot C + t\cdot D_i\,, \qquad (16)$$

$$D_i = \frac{1}{2}\cdot\left(\frac{1}{2}\cdot A_i + \frac{1}{2}\cdot C\right) + \frac{1}{2}\cdot\left(\frac{1}{2}\cdot B_i + \frac{1}{2}\cdot C\right), \qquad (17)$$

$$t = \frac{h}{\sqrt{N}}\cdot\frac{const}{\|CD_i\|}\,. \qquad (18)$$

In these equations A_i and B_i are the nodes of singular element, D_i – point on bisector from point C, N – number of mesh elements, h – the size of smallest mesh element, $\|CD_i\|$ – the distance from point C to point D_i, $const \in (0,1]$ – the smoothing value. The definition of parameter t allows us to control positions of new C_i: the ratio $h/\sqrt{N}$ guarantees that new point C_i will be bounded by sides of singular element, and division by $\|CD_i\|$ guarantees that C_i will be placed near original point.

After solving least square method equations with singular patch and singular polynomial we get coefficient vector b_{SING} and for each additional singular point C_i we can calculate displacement solution with included singular part:

$$u(C_j) = u_{FEM}(C_j) + u_S(C_j) = $$
$$= u_{FEM}(C_j) + Q_{SING}(C_j)\cdot b_{SING}(C_j)\,. \qquad (19)$$

General solution for singular point C is obtained using extrapolation from additional singular displacement solutions in points C_i.

After the improved solution is calculated in point C we can take it instead of the exact solution in the posteriori error estimation:

$$\boldsymbol{e}_{u,SING} \approx \bar{\boldsymbol{e}}_{u,SING} = \boldsymbol{u}^*_{SING} - \boldsymbol{u}^h\,, \qquad (20)$$

there $\bar{\boldsymbol{e}}_{u,SING}$ is the point-wise estimated error for singular zone of the problem domain.

4. Numerical Examples

A 2D frame structure (Fig. 4), which has fixed support at the bottom of the two vertical walls, is subjected to uniformly distributed load of intensity 1000 N. Plane stress conditions are assumed with Poison's ratio ν=0.3 and Young's modulus E=10^5 Pa. Permissible error tolerance is η = 15 %.

In structural mechanics problems the primary aim is to obtain final optimal mesh with as small element number as possible according given permissible error. In order to achieve this goal we

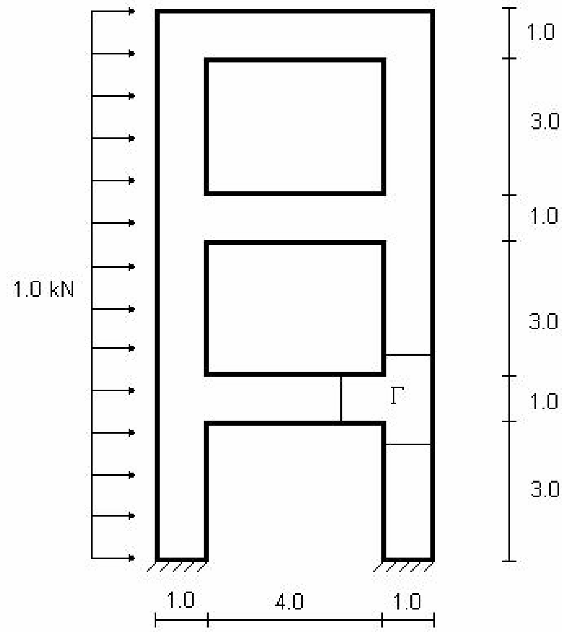

Fig.4 2D-frame structure definition

should find the way to improve solution in the first adaptive mesh refinement strategy steps when total element number is quite small, because after few iterations element number is increased rapidly and obtained solution does not changing considerably.

For this reason two tests were carried out. Starting from the same uniform 302 triangle element mesh adaptive finite element analysis was performed two times: first time adaptive finite element analysis was performed with traditional error estimation (when singularity effect is not accounted) and after that adaptive finite element analysis was performed using proposed algorithm for solution improvement near singular point. In this example 10 singular points are detected. Singular zone is defined around each singular point taking radius $R = 0.1$. Singular element patch for each singular point is combined from all elements having at least one node placed at the distance less or equal to radius $R = 0.1$ from the current singular point. Initial mesh with grey marked singular zones is presented in Fig. 5. All calculation data from both strategies are placed in Table 1.

Table 1. 2D-frame structure analysis results

	Error analysis, η=15 %			
	no singularity effect		with singularity effect	
	η, %	El. Nr.	η, %	El. Nr.
1st iteration	42.54	302	44,39	302
2nd iteration	19.74	2790	**14.90**	**7936**
3rd iteration	15.66	5570	-	-
4th iteration	**13.58**	**9311**	-	-

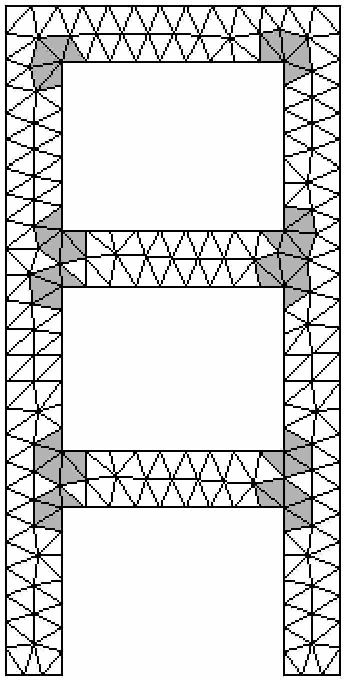

Fig.5 An initial finite element mesh with marked singular zones

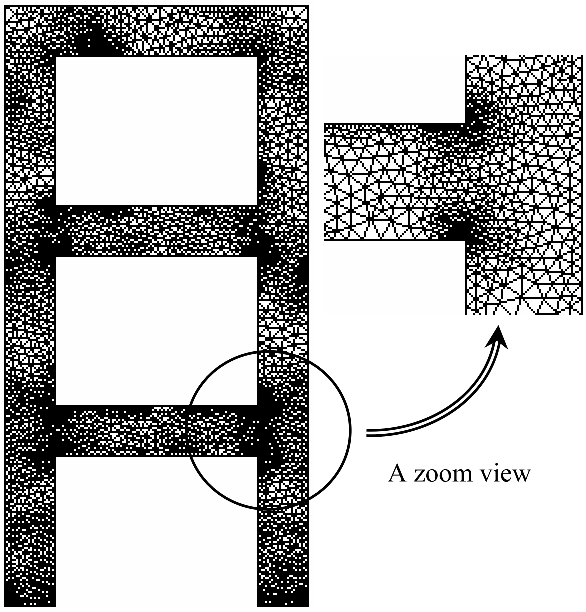

Fig.6 Final finite element mesh, obtained without accounting geometrical singularity influence

According first strategy (when singularity effect was not accounted) final finite element mesh (Fig. 6) according given permissible error η =15 % was obtained through four iterations. Final element number for this mesh is 9311 elements. According second strategy (when singularity influence was accounted) final finite element mesh (Fig. 7) was obtained only by two iterations and with 7936 elements in it. It means that according proposed strategy it is really possible to reduce iteration and

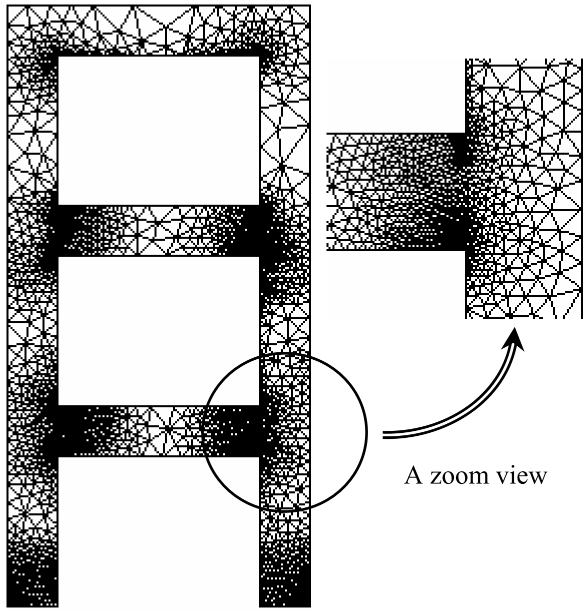

Fig.7 Final finite element mesh, obtained accounting geometrical singularity influence

element number until final mesh is obtained. Besides that we can see that element distribution in both final meshes is different: according first strategy elements are located more smoothly in the whole domain than according second strategy, when more and smaller elements are placed in higher stress concentration zones.

To demonstrate more clearly the advantage of the proposed solution improvement strategy near singularities an additional comparison of the estimated relative errors is performed.

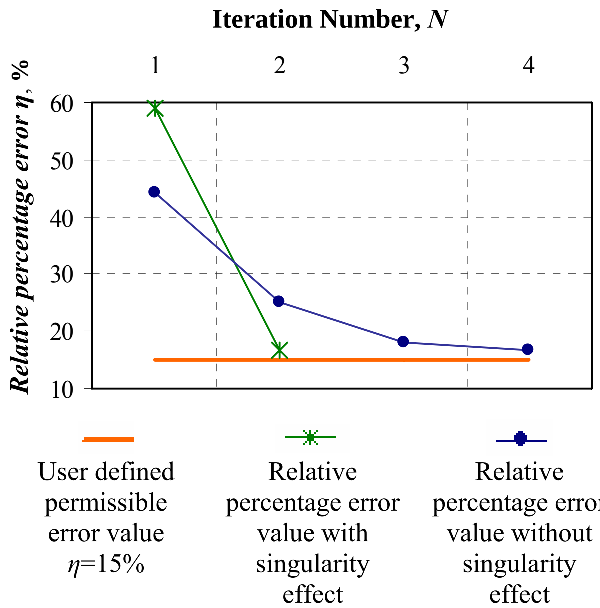

Fig.8 Comparison of the obtained relative percentage error values only in singular zones of the problem domain

Curves in Fig. 8 are calculated using error data only from singular domain (grey color) because summing error values from the whole domain the impact of improved solution vanishes and is not so obvious. Comparing obtained relative percentage error values for the singular zone we can see that in the first adaptive analysis step difference between user defined value and value calculated according proposed solution improvement near singular point is bigger than difference, obtained between user defined value and value, calculated according traditional error estimation algorithm. It means that displacement solution was really improved in singular point zone and obtained error value is calculated more precisely than error value without singularity improvement. It is proved by authors [15] that proposed strategy works in the same way with the other mesh optimality criteria also.

5. Conclusions

In this paper a simple method for postprocessed solution improvement near singularities is presented and tested. The method is well suited, fully automatic and can be used to solve problems with complex geometry. Summarizing can be stated that taking singular polynomial instead of the standard quadratic in the postprocessed solution calculation we can define displacement solution more precisely in domains with different topological incompatibilities and obtain final optimal mesh with the least iteration number and with minimal element number in this final mesh.

References:

[1] L. Y. Li, P. Bettess, Notes on mesh optimality criteria in adaptive finite element computations, *Communications in Numerical Methods in Engineering*, Vol.11, 1995, pp. 911–915.

[2] V. Šapalas, M. Samofalov, V. Šaraškinas, FEM stability analysis of tapered beam-columns, *Journal of civil engineering and management*, Vol. 11(3), 2005, pp. 211-216.

[3] T. Kvamsdal, Variationally consistent postprocessing for adaptive recovery of stresses, *ECCM'99,* München, Germany, 1999, p. 9.

[4] R. Baušys, N.E.Wiberg, Adaptive finite element strategy for acoustic problems, *Journal of Sound and Vibration*, Vol.226, 1999, pp.905–922.

[5] E. Stupak, R. Baušys, Generation of the unstructured FE-grids for complex 2D objects, *Journal of Civil Engineering and Management (Statyba)*, Vol. 6, 2000, pp. 17–24.

[6] Ch. Song, J.P. Wolf, Semi-analytical representation of stress singularities as occurring in cracks in anisotropic multi-materials with the scaled boundary finite-element method, *Computers & Structures*, Vol.80, 2002, pp. 183–197.

[7] V. Žarnovskij, R. Kačianauskas, The finite element investigation of the thickness influence on 3D stress and strain fields in the SENB specimens, *Mechanika*, Vol 4(42), 2003, pp.5–10.

[8] Z.Yosibash, B.A. Szabo, Numerical analysis of singularities in two-dimensions. Part 1: computation of eigenpairs, *Int. J. Num. Meth. Engng.*, Vol.38, 1995, pp. 2055–2082.

[9] B.A. Szabo, Z. Yosibash, Numerical analysis of singularities in two dimensions. Part 2: computation of generalized flux/stress intensity factors, *Int. J. Num. Meth. Engng.*, Vol.39, 1996, pp. 409–434.

[10] O.C. Zienkiewicz, J.Z. Zhu, A Simple Error Estimator and Procedure for Practical Engineering Analysis, *Int. J. Num. Meth. Engng*, Vol. 24, 1987, pp. 337-357.

[11] J.Z. Zhu, O.C. Zienkiewicz, Adaptive Techniques in the Finite Element Method, *Comm. Appl. Numer. Methods*, Vol. 4, 1988, pp. 197-204.

[12] A. Kufner, A.M. Sändig, *Some Applications of Weighted Sobolev Spaces*, Teubner, Leipzig, 1987.

[13] R. Baušys, N.E. Wiberg, Error estimation for eigenfrequencies and eigenmodes in dynamics, *Computing in Civil and Building Engineering*, 1995, pp. 611–616.

[14] R.J.Mackinnon, G.F.Carey, Superconvergence derivatives: A Taylor series analysis, *Int. J. Numer. Methods Eng.*, Vol.28, 1989. pp. 489.

[15] L. Vasiliauskienė, R. Baušys, S. Valentinavičius, Adaptive Finite Element Analysis taking to the Account Singularity Effect, *WSEAS TRANSACTIONS on MATHEMATICS*, Vol. 7(5), 2006, pp. 777–785.