Chart of Long Period Solutions of the Two-Degree of Freedom Dynamical System of the Coupled Non-Linear Double Oscillator with Third Order Polynomial Potential

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Abstract: - This paper presents the results of a remarkable effort to approximate the general solution of the axi-symmetric, two-degree of freedom dynamical system of the non-linear double oscillator corresponding to a third order polynomial potential. The general solution is approximated in the (x, C) plane through a set of initial conditions that generate symmetric periodic solutions of long period, i.e. solutions of a large number of intersections with the x-axis and a set of initial conditions that generate escape solutions of a large number of intersections with the x-axis, too.

Key-Words: - Non-linear double oscillator, escape solutions, third order potential, symmetric periodic solutions.

1 Introduction

The problem which we have treated here is that of the coupled non-linear double oscillator. The potential V of this problem is given by the expression:

$$V = \frac{1}{2}\left(Ax^2 + By^2\right) - bxy^2 , \qquad (1)$$

with A, B and b being constants.
The equations governing the motion are the following ones:

$$\frac{d^2x}{dt^2} = -Ax + by^2 , \quad \frac{d^2y}{dt^2} = -By + 2bxy , \qquad (2)$$

with the expression

$$\frac{1}{2}\left[\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + Ax^2 + By^2\right] - bxy^2 = C \qquad (3)$$

being the integral of its energy.

2 Chart of Solutions

On the plane of the initial conditions (x, C), $x \in [-2, 2]$, $C \in [0, 3]$, we present the chart of families of symmetric periodic solutions, for values of the constants A=1, B=2 and b=0.5, as it is shown in Figure 1. We have chosen to search solutions of a great multiplicity, in a way that their families to be dense on the plane (x, C). We aimed at finding the solutions which are 1000ple, i.e. they intersect perpendicularly the Ox axis at the 1000^{th} intersection or otherwise speaking they occur 1000 intersections at half period of time. This means that we proceed to find each periodic solution the multiplicity of which is a divisor of 1000. The search of such solutions has been undertaken by using a grid-search method [1]. According to this method we create an orthogonal grid of fine-partition of the plane (x, C), with a step $h_1 = 0.001$ along Ox axis and with a step $h_2 = 0.001$ along the axis of C and we search successive pairs of nodes, along Ox axis of the grid, corresponding to solutions of the equations of motion with opposite signs at the point of the their 1000^{th} intersection with Ox axis. In between two successive pairs characterized by the above propriety there obviously exists a point where $\dot{x}$ is reduced to zero, therefore the solution is periodic taking for granted that it has two different points at which it intersects perpendicularly Ox axis.

The search on the plane (x, C) of initial conditions conducting to escape solutions after 1000 intersections with Ox axis is attained by using the method of an orthogonal grid of fine-partition, with a step $h_1 = 0.001$ along Ox axis and with a step $h_2 = 0.01$ along the axis of C. The initial conditions of the solutions for 2.01<C<3, marked as red colored points in Figure 1, extend up to the space of order of the problem [2]. This space is constituted countably densely by linearly stable periodic solutions [3]. All solutions into the space of stable solutions of the plane (x, C) give the appearance corresponding to order on the plane $(x, \dot{x})$ [4].

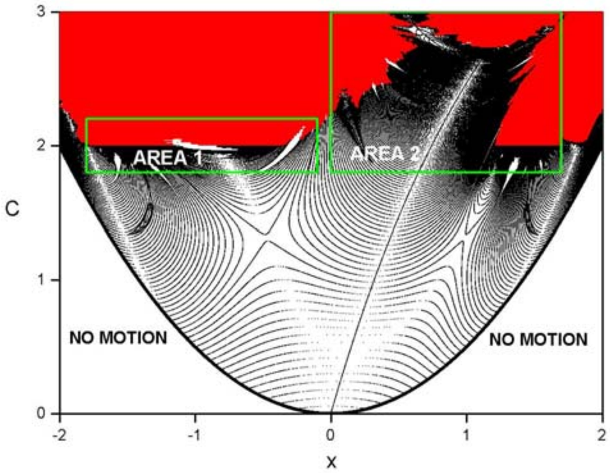

Fig. 1: Families of periodic solutions and area of escape solutions.

The content of the AREA 1 in Figure 1, up to the value C=2, is presented in more detail in Figure 2. There families of periodic solutions appear of three intersections (in magenta color), of seven intersections (in cyan color), of thirteen intersections (in blue color) and of nineteen intersections (in green color), members of which are found in regions of (x, C) non possessing solutions of 1000 intersections. Thus in Figure 3 and to the regions A, B, C, D a complementary search of solutions has been undertaken of 999, 1001, 1001, 1007 intersections, respectively.

The region characterized by the existence of a remarkable density of families of periodic solutions is named CHAOS, as the invariant curves in this region present an image of chaos. We swept Ox axis from x = -1.8 up to x = -0.1 with a step $h_3 = 0.02$ along the axis of x and by integrating the equations of motion we computed 50000 intersections with Ox axis, for each value of x. The results of the computation of invariant curves for a specific value of C, i.e. C=1.9, are given in Figure 4. In the position interval -1.62 < x < -1.12 the invariant curves appear an image of chaos with the exception of the small position interval -1.58 < x < -1.56 into which

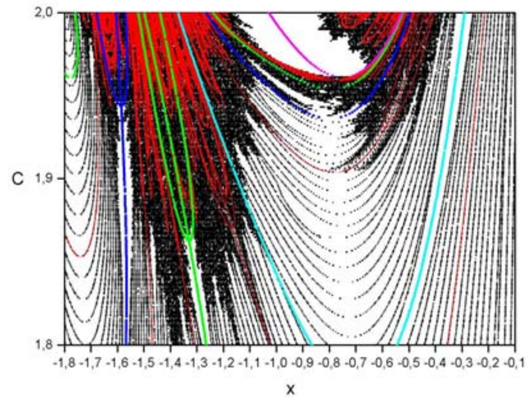

Fig. 2: Families of periodic solutions of 3, 7, 13, 19, 50 and 1000 intersections

the invariant curves appear an image of order. In the plane (x, C) in Figure 3 for the same value of

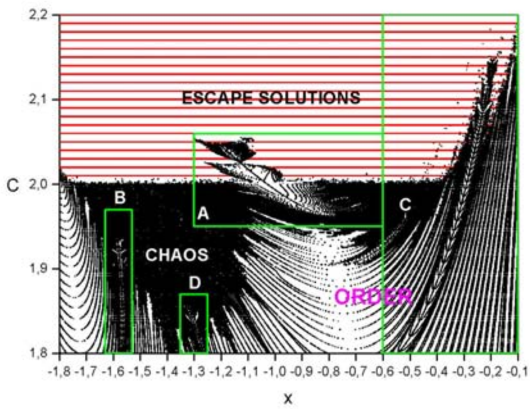

Fig. 3: Content of the AREA 1 in Figure

C=1.9 we remark the existence of an important number of families of periodic solutions within exactly the same interval -1.62 < x < -1.12 with the exception of the exactly the same small position

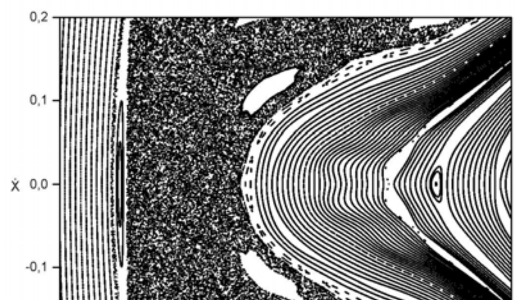

Fig. 4: Invariant curves for C=1.9.

interval -1.58< x < -1.56 into which distinguished periodic solutions there exist around the periodic solution of 13 intersections for x=-1.573, C=1.9, as it is seen in Figure 2, too. However, in the plane (x, C) we have the possibility to give an idea about the content of the region of CHAOS in Figure 3, by searching solutions of smaller multiplicity, for instance of 50 intersections (in red color), as it is seen in Figure 2. Finally, in Figure 5 the content of the AREA 2 of Figure 1 appears.

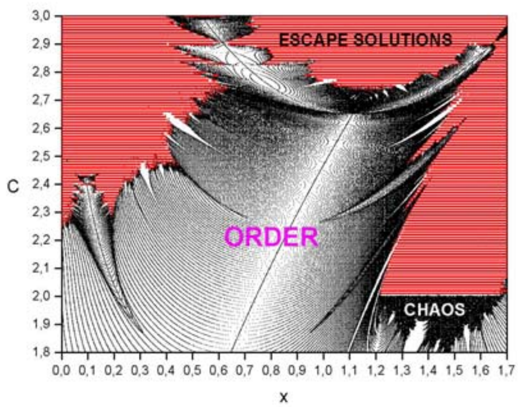

Figure 5: Content of the AREA 2 in Figure 1.

In the Tables 1-4 we give the results of computation for some members of families appearing in Fig. 2. X01 is the initial position, CINT is the initial constant of energy, VOUT(1) is the final position at half period, CT is final constant of energy at half period, TEND is the time at half period, and JTOM is the number of intersections at half period.

Table 1

X01	CINT	VOUT(1)	CT	TEND	JTOM
-0,989	1,991	-0,98913	1,991	8,3755	3
-0,551	1,991	-0,55079	1,991	7,9455	3
-0,993	1,992	-0,9936	1,992	8,3832	3
-0,548	1,992	-0,54782	1,992	7,9448	3
-0,998	1,993	-0,99821	1,993	8,3919	3
-0,545	1,993	-0,54482	1,993	7,9441	3
-1,002	1,994	-1,00262	1,994	8,3996	3
-0,542	1,994	-0,5418	1,994	7,9434	3
-1,007	1,995	-1,00713	1,995	8,4084	3
-0,539	1,995	-0,53875	1,995	7,9427	3
-1,011	1,996	-1,01148	1,996	8,4163	3
-0,536	1,996	-0,53568	1,996	7,942	3
-1,015	1,997	-1,01581	1,997	8,4241	3
-0,533	1,997	-0,53258	1,997	7,9413	3
-1,02	1,998	-1,02019	1,998	8,433	3
-0,531	1,998	-0,53097	1,998	7,9413	3
-1,024	1,999	-1,02446	1,999	8,441	3
-0,528	1,999	-0,52783	1,999	7,9406	3
-1,028	2	-1,02871	2	8,449	3
-0,525	2	-0,52467	2	7,9399	3

Table 2

X01	CINT	VOUT(1)	CT	TEND	JTOM
-0,865	1,8	1,23132	1,8	18,0281	7
-0,544	1,8	1,17096	1,8	17,8793	7
-0,869	1,801	1,23226	1,801	18,0327	7
-0,541	1,801	1,17071	1,801	17,8804	7
-0,873	1,802	1,23319	1,802	18,0373	7
-0,538	1,802	1,17047	1,802	17,8814	7
-0,876	1,803	1,23389	1,803	18,0413	7
-0,536	1,803	1,1704	1,803	17,8826	7
-0,88	1,804	1,23482	1,804	18,046	7
-0,533	1,804	1,17017	1,804	17,8837	7
-0,883	1,805	1,23551	1,805	18,0499	7
-0,531	1,805	1,17011	1,805	17,8849	7
-0,887	1,806	1,23644	1,806	18,0547	7
-0,528	1,806	1,16988	1,806	17,886	7
-0,89	1,807	1,23713	1,807	18,0587	7
-0,526	1,807	1,16982	1,807	17,8872	7
-0,894	1,808	1,23806	1,808	18,0635	7
-0,523	1,808	1,1696	1,808	17,8884	7
-0,897	1,809	1,23874	1,809	18,0675	7
-0,521	1,809	1,16955	1,809	17,8896	7

Table 3

X01	CINT	VOUT(1)	CT	TEND	JTOM
-1,568	1,8	-1,568	1,8	33,4769	13
-1,567	1,801	-1,56701	1,801	33,4873	13
-1,567	1,802	-1,56701	1,802	33,4966	13

-1,567	1,803	-1,56701	1,803	33,5059	13
-1,567	1,804	-1,56701	1,804	33,5152	13
-1,567	1,805	-1,56701	1,805	33,5245	13
-1,567	1,806	-1,56701	1,806	33,5338	13
-1,567	1,807	-1,567	1,807	33,5432	13
-1,566	1,808	-1,56601	1,808	33,5534	13
-1,566	1,809	-1,56602	1,809	33,5628	13
-1,566	1,81	-1,56602	1,81	33,5722	13
-1,566	1,811	-1,56602	1,811	33,5815	13
-1,566	1,812	-1,56602	1,812	33,591	13
-1,566	1,813	-1,56601	1,813	33,6004	13
-1,566	1,814	-1,56601	1,814	33,6098	13
-1,566	1,815	-1,56601	1,815	33,6193	13
-1,565	1,817	-1,56502	1,817	33,6388	13
-1,565	1,818	-1,56502	1,818	33,6483	13
-1,565	1,819	-1,56503	1,819	33,6578	13
-1,565	1,82	-1,56503	1,82	33,6673	13

Table 4

X01	CINT	VOUT(1)	CT	TEND	JTOM
-1,264	1,8	-1,26406	1,8	49,4345	19
-1,265	1,801	-1,26505	1,801	49,4446	19
-1,266	1,802	-1,26603	1,802	49,4548	19
-1,267	1,803	-1,26702	1,803	49,465	19
-1,268	1,804	-1,26801	1,804	49,4752	19
-1,269	1,805	-1,269	1,805	49,4855	19
-1,269	1,806	-1,26938	1,806	49,4921	19
-1,27	1,807	-1,27039	1,807	49,5024	19
-1,271	1,808	-1,27139	1,808	49,5127	19
-1,272	1,809	-1,2724	1,809	49,523	19
-1,273	1,81	-1,2734	1,81	49,5333	19
-1,274	1,811	-1,27441	1,811	49,5436	19
-1,275	1,812	-1,27542	1,812	49,554	19
-1,276	1,813	-1,27643	1,813	49,5644	19
-1,277	1,814	-1,27744	1,814	49,5748	19
-1,278	1,815	-1,27845	1,815	49,5852	19
-1,279	1,816	-1,27946	1,816	49,5957	19
-1,28	1,817	-1,28048	1,817	49,6061	19
-1,281	1,818	-1,28149	1,818	49,6166	19
-1,282	1,819	-1,28251	1,819	49,6271	19

3 Conclusion

The chart of symmetric periodic solutions and of escape solutions of the problem and particularly of those of a great multiplicity provides a sufficiently complete and clear picture of the set of all solutions of the problem.

References:

[1] V. V. Markellos, W. Black, and P. E. Moran, Celestial Mechanics, Vol. 9, 1974, p.507.

[2] E. P.Valaris and M. A. Leftaki, Escape Solutions of Two-Degree of Freedom Dynamical System of the Coupled Non-Linear Double Oscillator with Third Order Potential, Wseas Transactions on Mathematics, Issue 4, Vol. 5, 2006, p.435.

[3] V. V. Markellos, Astrophysics and Space Science, Vol. 43, 1976, p. 449.

[4] E. P.Valaris and M. A. Leftaki, *1st International Conference "From Scientific Computing to Computational Engineering"*, Athens, 2004.