A parameter robust numerical method for a singularly perturbed Volterra equation in security technologies

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Abstract: In this paper, we study convergence properties of a finite difference scheme on a Shishkin mesh applied to a singularly perturbed Volterra integro-differential equation in security technologies. We derive a priori error estimate that is robust with respect to perturbation parameter ε, and prove that the finite difference scheme is almost second order accurate. Numerical results support the theoretical results.

Key-Words: Singularly perturbed; Volterra integro-differential equation; finite difference; Shishkin mesh; uniform convergence

1 Introduction

Singular perturbation problems arise in several branches of engineering and applied mathematics which include fluid dynamics, security technologies, chemical reactor theory, gas porous electrodes theory, etc. To solve these types of problems various methods are proposed in the literature, more details can be found in the books of Farrell et al. [1] and Roos et al. [2].

In this paper we consider the following singularly perturbed Volterra integro-differential equation in security technologies:

$$\varepsilon u'(t) + a(t)u(t) + \int_0^t K(t,s)u(s)ds = f(t), \quad t \in I := [0,1], \tag{1}$$

$$u(0) = A, \tag{2}$$

where $0 < \varepsilon << 1$ is the perturbation parameter, $\alpha_* \geq a(t) \geq \alpha > 0$, $f(t)$ $(t \in I)$ and $K(t,s)$ ($(t,s) \in I \times I$) are sufficiently smooth functions and A is a given constant. On putting $\varepsilon = 0$ in equation (1), we obtain the reduced equation

$$a(t)u_0(t) + \int_0^t K(t,s)u_0(s)ds = f(t),$$

which is a Volterra integral equation of the second kind. The solution u of (1),(2) has a boundary layer at $t = 0$ (see, e.g., [3,4,5]).

The asymptotic structure of the solution to equation (1),(2) was examined by Angell and Olmstead [6,7]. The numerical discretization of singularly perturbed Volterra integro-differential equations and Volterra integral equations by tension spline collocation methods in certain tension spline spaces are considered in Ref. [8]. An exponentially fitted difference scheme on a uniform mesh is discussed in Ref. [9].

This present study is devoted to a finite difference method for the Volterra integro-differential equation (1),(2) on a Shishkin mesh. We first present bounds for u and its derivatives. These bounds enable us to construct a special piecewise uniform mesh on which we can prove that the finite difference scheme is almost second-order accurate, uniformly in ε. Our analysis is based on discrete comparison principle, truncation error analysis and appropriate barrier functions.

An outline of the paper is as follows: in section 2 we state some properties of the exact solution. Based on these results we introduce a Shishkin mesh and a finite difference scheme in section 3. In section 4 we analyse the convergence properties of the scheme. Finally, numerical results are presented in section 5.

Notation 1. Throughout the paper, C will denote a generic positive constant (possibly subscripted) that is independent of ε and of the mesh. Note that C is not necessarily the same at each occurence.

2 Properties of the exact solution

To construct layer-adapted meshes correctly, it is crucial to have a precise knowledge of the asymptotic behavior of the exact solution.

Lemma 1. The solution $u(x)$ of the problem (1),(2) satisfies the following bound

$$\left|u^{(k)}(t)\right| \leq C(1 + \varepsilon^{-k} e^{-\alpha t/\varepsilon}), \ t \in I \ , k = 0,1,2,3.$$

Proof. See [9] for a proof with $k=0$ and $k=1$; the argument works also for $k=2,3$.

3 The Shishkin mesh and a finite difference scheme

In this section we describe the piecewise-uniform Shishkin mesh and a finite difference scheme based on trapezoidal integration.

The construction of the Shishkin mesh is based on the bounds of the exact solution and its derivatives. Let λ denote a mesh transition parameter defined by

$$\lambda = \min\{\frac{1}{2}, \frac{2\varepsilon}{\alpha}\ln N\}. \quad (3)$$

Assumption 1. We now make the mild assumption that $\lambda = \frac{2\varepsilon}{\alpha}\ln N$, as otherwise N^{-1} is exponentially small compared with ε. We shall also assume throughout the paper that $\varepsilon \le CN^{-1}$ as is generally the case in practice.

We divide each of the subintervals $[0,\lambda]$ and $[\lambda,1]$ into $N/2$ equidistant subintervals. Then our mesh is

$$t_i = \begin{cases} 2i\lambda/N & i=0,1,\cdots,N/2, \\ 1-2(1-\lambda)(N-i)/N & i=N/2+1,\cdots,N. \end{cases} \quad (4)$$

We denote by h and H the mesh widths inside and outside the boundary layer, i.e.,

$$h = 4\varepsilon\alpha^{-1}N^{-1}\ln N,\ H = 2(1-\lambda)N^{-1}. \quad (5)$$

The finite difference scheme that we present consists of midpoint difference operator and trapezoidal integration in approximating the Volterra integral. Based on the Shishkin mesh, we propose the finite difference scheme for problem (1),(2):

$$\varepsilon\frac{u_i^N - u_{i-1}^N}{h_i} + a_{i-1/2}(u_i^N + u_{i-1}^N)/2 +$$

$$\frac{h_i}{4}[\frac{3}{2}K(t_{i-1/2},t_{i-1})u_{i-1}^N + \frac{1}{2}K(t_{i-1/2},t_i)u_i^N] \quad (6)$$

$$+\tilde{K}(t_0,\cdots,t_{i-1};u_0^N,\cdots u_{i-1}^N) = f_{i-1/2}, i=1,\cdots,N,$$

$$u_0^N = A, \quad (7)$$

where

$$\tilde{K}(t_0,\cdots,t_{i-1};u_0^N,\cdots u_{i-1}^N) =$$

$$\begin{cases} 0 & i=1, \\ \sum_{j=1}^{i-1}\frac{h_j}{2}[K(t_{i-1/2},t_j)u_j^N + K(t_{i-1/2},t_{j-1})u_{j-1}^N] & i>1 \end{cases} \quad (8)$$

and $a_{i-1/2} = a((t_{i-1}+t_i)/2)$; similarly for $f_{i-1/2}$ and $K(t_{i-1/2},t_{j-1})$.

4 Analysis of the scheme

The analysis is based on the discrete comparison principle and barrier function technique introduced in [10,11].

Lemma 2. Assume that

$$a(t) + \frac{H}{4}K(t,t) \ge 2\alpha_* > 0. \quad (9)$$

Then the operator l^N defined by

$$l^N y_i \equiv \varepsilon\frac{y_i - y_{i-1}}{h_i} + [\frac{1}{2}a_{i-1/2} + \frac{h_i}{8}K(t_{i-1/2},t_i)]y_i,$$

$$1 \le i \le N \quad (10)$$

satisfies a discrete comparison principle, i.e., if $\{v_i\}$ and $\{w_i\}$ are mesh functions that satisfy $v_0 \le w_0$ and $l^N v_i \le l^N w_i$ for $i=1,2,\cdots,N$, then $v_i \le w_i$ for all i.

Proof. It is easy to verify that the $(N+1)\times(N+1)$ matrix associated with l^N is an M-matrix, as in the proof of [11, Lemma 3.1].

An immediate consequence of this discrete comparison principle is the following stability result.

Lemma 3. Under the condition (9), the solution of the difference initial value problem

$$l^N y_i = F_i, i=1,2,\cdots,N,\ y_0 = B$$

satisfies the following estimate

$$|y_i| \le |B| + \alpha_*^{-1}|F_i|,\ i=1,2,\cdots,N,$$

where $F_i \ge 0$ is nondecreasing.

Proof. Applying Lemma 2 to the barrier function $W_i = |B| + \alpha_*^{-1}|F_i| \pm y_i$, we can easily get the desired result.

For the Shishkin mesh (4) we have the following result that will be used later.

Lemma 4. There exists a constant C such that

$$\int_{t_{k-1}}^{t_k}(1+\varepsilon^{-1}e^{-\alpha t/(2t)})dt \le CN^{-1}\ln N$$

for $k=1,2,\cdots,N$.

Proof. For $k=N/2+1,\cdots,N$, we have

$$\int_{t_{k-1}}^{t_k}(1+\varepsilon^{-1}e^{-\alpha t/(2\varepsilon)})dt = h_k - \frac{2}{\alpha}e^{-\alpha t/(2\varepsilon)}\Big|_{t_{k-1}}^{t_k}$$

$$\le CN^{-1} - \frac{2}{\alpha}(e^{-\alpha t_k/(2\varepsilon)} - e^{-\alpha t_{k-1}/(2\varepsilon)})$$

$$\le CN^{-1} + \frac{2}{\alpha}e^{-\alpha t_{N/2}/(2\varepsilon)} \le CN^{-1}.$$

For $k = 1,2,\cdots,N/2$, we have

$$\int_{t_{k-1}}^{t_k}(1+\varepsilon^{-1}e^{-\alpha t/(2\varepsilon)})dt = h_k - \frac{2}{\alpha}e^{-\alpha t/(2\varepsilon)}\Big|_{t_{k-1}}^{t_k}$$

$$\le h_k - \frac{2}{\alpha}e^{-\alpha t_k/(2\varepsilon)}(1-e^{\alpha h_k/(2\varepsilon)})$$

$$\le CN^{-1} + C\frac{h_k}{\varepsilon}e^{\alpha t_k/(2\varepsilon)} \le CN^{-1}\ln N.$$

Combine the two inequalities to complete the proof.

The next lemma gives us a useful formula for the truncation error of trapezoidal integration in approximating the Volterra integral.

Lemma 5. For $1 \le i \le N$, there exists a constant C such that

$$\tau_i = \Big|\frac{h_i}{4}[\frac{3}{2}K(t_{i-1/2},t_{i-1})u_{i-1} + \frac{1}{2}K(t_{i-1/2},t_i)u_i]$$
$$+\tilde{K}(t_0,\cdots,t_{i-1};u_0,\cdots,u_{i-1})$$
$$-\int_0^{t_{i-1/2}}K(t_{i-1/2},s)u(s)ds\Big| \le CN^{-2}\ln^2 N,$$

where $\tilde{K}(t_0,\cdots,t_{i-1};u_0,\cdots,u_{i-1})$ is given by (8).

Proof. The truncation error of trapezoidal integration in approximating the Volterra integral satisfies

$$\tau_i \le \Big|\frac{h_i}{4}[\frac{3}{2}K(t_{i-1/2},t_{i-1})u_{i-1} + \frac{1}{2}K(t_{i-1/2},t_i)u_i\,]$$
$$-\int_{t_{i-1}}^{t_{i-1/2}}K(t_{i-1/2},s)u(s)ds\Big| + \Big|\tilde{K}(t_0,\cdots,t_{i-1};$$
$$u_0,\cdots,u_{i-1}) - \int_0^{t_{i-1}}K(t_{i-1/2},s)u(s)ds\Big|$$
$$\le Ch_i\int_{t_{i-1}}^{t_{i-1/2}}|u''(t)|(t-t_{i-1})dt$$
$$+C\sum_{j=1}^{i-1}h_j\int_{t_{j-1}}^{t_j}|u''(t)|(t-t_{j-1})dt$$
$$\le Ch_i\int_{t_{i-1}}^{t_{i-1/2}}(1+\varepsilon^{-2}e^{-\alpha t/\varepsilon})(t-t_{i-1})dt$$
$$+C\sum_{j=1}^{i-1}h_j\int_{t_{j-1}}^{t_j}(1+\varepsilon^{-2}e^{-\alpha t/\varepsilon})(t-t_{j-1})dt$$
$$\le C\max_{1\le j\le i}\int_{t_{j-1}}^{t_j}(1+\varepsilon^{-2}e^{-\alpha t/\varepsilon})(t-t_{j-1})dt, \qquad (11)$$

where we have used Taylor expansion, Lemma 1 and the assumpation that the Kernel $K(t,s)$ and its derivatives are bounded.

To bounded (11) we shall use the following inequality

$$\int_a^b g(x)(x-a)^{(k-1)}dx \le \frac{1}{k}\{\int_a^b g(x)^{1/k}dx\}^k, \qquad (12)$$

which holds true for any positive monotonically decreasing function g on $[a,b]$ and for arbitrary $k \in N^+$; see [12]. By this inequality we have

$$\int_{t_{j-1}}^{t_j}(1+\varepsilon^{-2}e^{-\alpha t/\varepsilon})(t-t_{j-1})dt$$
$$\le \frac{1}{2}\{\int_{t_{j-1}}^{t_j}(1+\varepsilon^{-1}e^{-\alpha t/(2\varepsilon)})dt\}^2. \qquad (13)$$

From this inequality and Lemma 4 we get the desired result.

Now we can get our main result for this difference scheme.

Theorem 1. Under the condition (9), for the difference problem (6),(7) we have

$$|u_i - u_i^N| \le CN^{-2}\ln^2 N,\ 0 \le i \le N. \qquad (14)$$

Proof. For $1 \le i \le N$, from (6) and (10) we have

$$|l^N(u_i-u_i^N)| = \Big|l^N u_i - \{f_{i-1/2} - \frac{1}{2}a_{i-1/2}u_{i-1}^N - \frac{3h_i}{8}$$
$$\cdot K(t_{i-1/2},t_{i-1})u_{i-1}^N\} - \tilde{K}(t_0,\cdots,t_{i-1};u_0^N,\cdots,u_{i-1}^N)\Big|$$
$$\le \Big|\varepsilon(\frac{u_i-u_{i-1}}{h_i} - u'_{i-1/2})\Big| + \Big|\frac{h_i}{8}K(t_{i-1/2},t_i)u_i -$$
$$\int_0^{t_{i-1/2}}K(t_{i-1/2},s)u(s)ds + \frac{1}{2}a_{i-1/2}u_i - a_{i-1/2}u_{i-1/2}$$
$$+\frac{1}{2}a_{i-1/2}u_{i-1}^N + \frac{3h_i}{8}K(t_{i-1/2},t_{i-1})u_{i-1}^N$$
$$-\tilde{K}(t_0,\cdots,t_{i-1};u_0^N,\cdots,u_{i-1}^N)\Big|$$
$$\le \Big|\varepsilon(\frac{u_i-u_{i-1}}{h_i} - u'_{i-1/2})\Big| + \Big|\frac{1}{2}a_{i-1/2}u_i - a_{i-1/2}u_{i-1/2}$$
$$+\frac{1}{2}a_{i-1/2}u_{i-1}\Big| + \Big|\frac{1}{2}a_{i-1/2}(u_{i-1}-u_{i-1}^N)\Big|$$
$$+\Big|\frac{h_i}{4}[\frac{3}{2}K(t_{i-1/2},t_{i-1})u_{i-1} + \frac{1}{2}K(t_{i-1/2},t_i)u_i]$$
$$+\tilde{K}(t_0,\cdots,t_{i-1};u_0,\cdots,u_{i-1})$$
$$-\int_0^{t_{i-1/2}}K(t_{i-1/2},s)u(s)ds\Big|$$
$$+\Big|\frac{3h_i}{8}K(t_{i-1/2},t_i)(u_{i-1}-u_{i-1}^N)\Big|$$
$$+\Big|\tilde{K}(t_0,\cdots,t_{i-1};u_0-u_0^N,\cdots,u_{i-1}-u_{i-1}^N)\Big|$$
$$\le C\varepsilon\int_{t_{i-1}}^{t_i}|u'''(t)|(t-t_{i-1})dt$$
$$+C\int_{t_{i-1}}^{t_i}|u''(t)|(t-t_{i-1})dt + CN^{-2}\ln^2 N$$

$$+C\sum_{j=1}^{i-1}h_j\,|u_j-u_j^N|$$
$$\le C\int_{t_{i-1}}^{t_i}(1+\varepsilon^{-2}e^{-\alpha t/\varepsilon})(t-t_{i-1})dt+CN^{-2}\ln^2 N$$
$$+C\sum_{j=1}^{i-1}h_j\,|u_j-u_j^N|\le C\{\int_{t_{i-1}}^{t_i}(1+\varepsilon^{-1}e^{-\alpha t/(2\varepsilon)})dt\}^2$$
$$+CN^{-2}\ln^2 N+C\sum_{j=1}^{i-1}h_j\,|u_j-u_j^N|$$
$$\le CN^{-2}\ln^2 N+C\sum_{j=1}^{i-1}h_j\,|u_j-u_j^N|,$$

where we have also used Taylor expansion, (13), Lemmas 1,4,5 and the assumpation that the Kernel $K(t,s)$ and its derivatives are bounded.

Using the discrete comparison principle we have

$$|u_i-u_i^N|\le w_i,\quad 0\le i\le N,$$

where w_i is the solution of the problem

$$l^N w_i=\begin{cases} CN^{-2}\ln^2 N & i=1,\\ CN^{-2}\ln^2 N+C\sum_{j=1}^{i-1}h_j\,|u_j-u_j^N| & 1<i\le N,\end{cases}$$
$$w_0=0.$$

From here by virtue of Lemma 3 it follows that

$$|u_i-u_i^N|\le|w_i|$$
$$\le CN^{-2}\ln^2 N+C\sum_{j=1}^{i-1}h_j\,|u_j-u_j^N|$$

and consequently

$$|u_{i-1}-u_{i-1}^N|\le CN^{-2}\ln^2 N+C\sum_{j=1}^{i-2}h_j\,|u_j-u_j^N|.$$

Application of the recurrence inequality gives

$$|u_i-u_i^N|\le CN^{-2}\ln^2 N,\ \text{ for } i=0,1,\cdots,N.$$

The proof of the theorem is completed.

5 Numerical experiments

In this section we present two examples to illustrate the method described in this paper.

Example 1. Consider the problem

$$\varepsilon u'(t)+2u(t)+\int_0^t su(s)ds=f(t)\quad\text{for } t\in(0,1],$$
$$u(0)=1,$$

where $f(t)$ is chosen such that $u(t)=t+e^{-t/\varepsilon}$.

Example 2. Consider the problem

$$\varepsilon u'(t)+(t+1)u(t)+\int_0^t(t+s)u(s)ds=f(t)\quad\text{for}$$
$$t\in(0,1],\quad u(0)=1,$$

where $f(t)$ is chosen such that $u(t)=\sin t+e^{-t/\varepsilon}$.

For our tests we take $\varepsilon=10^{-8}$ which is a sufficiently small choice to bring out the singularly perturbed nature of the problems. We measure the accuracy in the discrete maximum norm $\|u-u^N\|_\infty$, the rates of convergence

$$r^N=\log_2\left(\frac{\|u-u^N\|_\infty}{\|u-u^{2N}\|_\infty}\right)$$

and the constants in the error estimate

$$C^N=\frac{\|u-u^N\|_\infty}{N^{-2}\ln^2 N}.$$

Table1 Numerical results for example 1

N	Error	Rate	Constant
16	2.9290e-2	1.554	0.9754
32	9.9764e-3	1.488	0.8505
64	3.5564e-3	1.568	0.8422
128	1.1999e-3	1.615	0.8350
256	3.9181e-4	1.662	0.8351
512	1.2378e-4	1.697	0.8338
1024	3.8186e-5	-	0.8334

Table2 Numerical results for example 2

N	Error	Rate	Constant
16	1.5938e-2	1.088	0.5308
32	7.4951e-3	1.015	0.6390
64	3.7089e-3	1.013	0.8783
128	1.8383e-3	1.009	1.2793
256	9.1359e-4	1.005	1.9472
512	4.5537e-4	1.002	3.0674
1024	2.2732e-4	-	4.9612

The Table 1 and 2 correspond to the above examples respectively. The numerical results are clear illustrations of the convergence estimate of Theorem. They indicate that the theoretical results are fairly sharp.

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