The Principle of Turbulent Frame Indifference and new closure relations in LES

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Abstract: - In this paper the principle of Turbulent Frame Indifference is revised. The present-day LES models and the drawbacks of the dynamic calculation of the closure coefficient for the generalized SGS turbulent stress tensor are analyzed. A new closure relation for the generalized SGS turbulent stress tensor is proposed. The proposed closure relation for the generalized SGS turbulent stress tensor: complies with the principle of turbulent frame indifference; takes into account both the anisotropy of the turbulence velocity scales and turbulence length scales; removes any balance assumption between the production and dissipation of SGS turbulent kinetic energy. In the proposed model the generalized SGS turbulent stress tensor is related exclusively to the generalized SGS turbulent kinetic energy (which is calculated by means of its balance equation) and the modified Leonard tensor; the SGS viscous dissipation ε of the generalized SGS turbulent kinetic energy is calculated by solving the ε balance equation. The modelled balance equation of ε respects the properties of form-invariance and frame-dependence of the exact balance equation.

Key-Words: - LES, Turbulent frame indifference, closure relations

1 The principle of Turbulent Frame Indifference

To date many efforts have been made to establish new closure relations for the unknown turbulent quantities in the resolved equations that govern turbulent, three-dimensional unsteady flows (LES).

In the framework of ordinary continuum thermodynamics these relations (or turbulence models) could be interpreted as constitutive equations which are necessary to close the equations of motion and the internal energy equation.

The constitutive equations represent, in an idealized form, the behaviour of the materials and, consequently, they must fulfil the principle of Material Frame Indifference [1].

This basic working principle of continuum mechanics requires the constitutive equations to be the same for observers in inertial systems and in non-inertial ones. The above mentioned principle imposes that the functional relations (between unknown tensors and kinematic quantities) must fulfil two distinct requirements [2]:

a) form-invariance under the most general class of transformations of the reference frame;

b) frame indifference, in particular independence of the translational and angular velocity of the frame.

The principle of Turbulent Frame Indifference is the equivalent in turbulence of the principle of Material Frame Indifference and implies form-invariance and frame indifference on the turbulent closure relations [3].

Sadiki and Hutter [2] emphasized that form-invariance in a) and frame-independence in b) are two distinct matters.

A turbulent closure relation is form-invariant if it does not modify its formal expression under transformations of the frame and is constructed only with objective tensors. The objective tensors of rank n (n=0,1,2) are said to be objective because they transform like geometric objects [2] [4].

Considering an inertial frame, in which a material point has coordinate x_i at time t, and a non-inertial frame in which the same point has coordinate x_i^* at time t^*, the most general law which governs the transformations of the coordinates and the time expressed in the two frames is that given by the Euclidean transformations

$$x_i = Q_{ij}(t) x_j^* + b_i(t), \qquad t = t^* + a \tag{1}$$

where $Q_{ij}(t)$ are the components of a time-dependent proper orthogonal tensor, $b_i(t)$ is the time-dependent distance between the origins of the two frames and a is any constant. Tensors of rank n ($n = 0,1,2$) are said to be objective, if the components transform according to

$S = S^*$ objective scalar

$V_i = Q_{ij} V_j^*$ objective vector (2)

$A_{ij} = Q_{im} Q_{jn} A^*_{mn}$ objective tensor

A turbulent closure relation is frame indifferent if it is expressed in terms of tensors that are independent of the angular and translational velocity of the frame. Weiss and Hutter [4] emphasized the difference between the form-invariance and the frame indifference by underlining the existence of tensors that are objective but dependent on the angular velocity of the frame.

For example the antisymmetric part of the velocity gradient is not an objective quantity: let W_{ij} and W_{ij}^* be, respectively, the representations of this quantity in an inertial and non-inertial frame, the law of transformation of these representations is given by

$$W_{ij} = Q_{im} Q_{jn} W^*_{mn} + \dot{Q}_{ik} Q_{jk} \quad (3)$$

It is possible to associate an objective tensor, called absolute velocity tensor W_{ij}^{Ω} to the quantity W_{ij}. The law of transformation between the representations of this tensor in the different frames of reference is given by

$$W_{ij}^{\Omega} = Q_{im} Q_{jn} W^{\Omega*}_{mn}, \text{ where } W^{\Omega*}_{mn} = W^*_{mn} + Q_{km} \dot{Q}_{kn} \quad (4)$$

The absolute vorticity tensor W_{ij}^{Ω} is an objective tensor (since its representations in the different frames transform according to Equation 2) but is frame-dependent since its representations depend on the frame by means of the term $Q_{km}\dot{Q}_{kn}$, associated with the angular velocity of the non-inertial frame [3], [4]. Consequently, it is always possible to deduce an objective but frame-dependent tensor from the representation in an inertial frame of a non-objective quantity.

A turbulent closure relation, which is expressed in terms of objective tensors that are dependent on the angular velocity of the frame, does not fulfil the principle of Turbulent Frame Indifference, because it is form invariant but frame dependent.

Must all the turbulent closure relations fulfil the Principle of Turbulent Frame Indifference?

The turbulent phenomena are not associated to the properties of the materials: consequently, turbulent closure relations do not represent the material behaviour.

Turbulent closure relations must always be form invariant, but must not necessarily be frame indifferent [3], [4].

In other words not all the turbulent closure relations must fulfil the principle of Turbulent Frame Indifference.

In the turbulent closure relations, the modelled expressions of an unknown objective tensor must be formulated in terms of objective tensors (allowing the closure relations to fulfil the requirement of form invariance) and must retain the same dependence (on the angular velocity of the frame) of the unknown tensor.

As demonstrated by Gallerano et al. [5], the generalised SGS turbulent stress tensor τ_{ij} is an objective tensor and is independent of the angular and translational velocity of the frame. Consequently, the closure relation for this tensor must fulfil the principle of Turbulent Frame Indifference. In other words, the closure relations for the generalised SGS turbulent stress tensor τ_{ij}:

- must be form invariant (or rather, must be expressed in terms of objective tensors);
- must be frame indifferent (or rather, must be expressed in terms of tensors that are independent of the angular and translational velocity of the frame).

2 Closure relations based on the Smagorinsky model

Among the most common LES models present in literature are the dynamic mixed models [6] [7] [8] [9] [10] based on the Smagorinsky closure relation, in which the generalized SGS turbulent stress tensor is related to the resolved strain-rate tensor, $\overline{S}_{ij}$, by means of a scalar eddy viscosity, ν_T:

$$\tau_{ij} = L^m_{ij} - 2\nu_T \overline{S}_{ij} \quad (5)$$

where the overbar $\overline{(.)}$ indicates the filter operation and L^m_{ij} is the modified Leonard tensor [11]. It is assumed, in these models, that the eddy viscosity is a scalar proportional to the cubic root of the generalized SGS turbulent kinetic energy dissipation and that such dissipation is locally and instantaneously balanced by the production of the generalized SGS turbulent kinetic energy (i.e., by the rate of kinetic energy per unit of mass transferred from the large scales, larger than the filter size, to the unresolved ones). Consequently, the eddy viscosity takes the form:

$$\nu_T = \left(C_S \overline{\Delta}\right)^2 \left(2\overline{S}_{mn}\overline{S}_{mn}\right)^{1/2} \quad (6)$$

where $\overline{\Delta}$ is the filter width and C_S is the closure coefficient. It is evident that the dynamic mixed models based on the Smagorinsky closure relation are fraught with four relevant drawbacks. The first drawback is represented by the scalar definition of the eddy viscosity; the second one concerns the local balance assumption of the generalized SGS turbulent kinetic energy production and dissipation, the third drawback is related to the dynamic calculation of the coefficient used to model the eddy viscosity, whilst the fourth drawback is related to the problems arising from the numerical scheme adopted for the simulations of three-dimensional unsteady flows (LES).

The scalar definition (first inconsistency) of the eddy viscosity is equivalent to assuming that the principal axes of the generalized SGS turbulent stress tensor, or the unresolved part of it (represented by the cross and Reynolds terms), are aligned with the principal axes of the resolved strain-rate tensor. Moreover, the eddy viscosity is proportional to the product of two terms, of which the dimensions are, respectively, those of a length and a velocity [12]. These terms, which represent, respectively, the turbulence length scales and turbulence velocity scales, are, more generally, second-order tensors of which the product is a fourth-order tensor which represents the eddy viscosity [13]. The scalar definition of the eddy viscosity, used in the above-mentioned dynamic mixed models based on the Smagorinsky closure relation, presupposes the existence of a single turbulence velocity scale and a single turbulence length scale. In this manner, the turbulence anisotropy induced by the continuous transfer of energy from the mean flow towards the turbulent fluctuations, which is generally extremely anisotropic, is not considered.

The second inconsistency of the dynamic mixed models based on the Smagorinsky closure relation is related to the assumption of a local and instantaneous balance between production and dissipation of the generalized SGS turbulent kinetic energy, formulated in the above-mentioned models to obtain the turbulent viscosity expression. This assumption is confirmed statistically and never instantaneously, and only locally at the scales associated with wavenumbers within the inertial subrange, and the latter exists only for isotropic turbulence and at high Reynolds numbers.

The third inconsistency of these dynamic models concerns the calculation of the above mentioned closure coefficient C_S. When simulating confined flows at high Reynolds number, the results of the dynamic procedure are of doubtful reliability in the region close to the wall including both the viscous sublayer and the buffer layer [14]. In this region, the filter width used in the dynamic procedure is larger than most eddies that govern the momentum and energy transfer. Consequently, the dynamic procedure used under these conditions for the calculation of the coefficient C_S is not able to fully account for the local subgrid dissipative processes that affect the entire domain.

The fourth inconsistency of the dynamic mixed models based on the Smagorinsky closure relation is connected to the problems arising from the numerical scheme adopted for the simulations of three-dimensional unsteady flows (LES).

The three-dimensional unsteady flows simulations require numerical schemes with a high order of accuracy: a low order of accuracy of centered finite difference schemes introduces (in the three-dimensional unsteady flows simulations) an anti-dissipative factor, which reduces the ability of the generalised SGS turbulent stress tensor to represent the kinetic energy transfer from the resolved scales to the unresolved ones, with an increase of the resolved kinetic energy.

The numerical scheme, besides being accurate, must fulfil the conservation requirement.

As suggested in [15] and [16], conservation properties of the mass, the momentum and the kinetic energy equations, for incompressible flows, are regarded as analytical requirements for a proper set of discrete equations. Consider the following governing equation for the scalar quantity ϕ:

$$\frac{\partial \phi}{\partial t} + {}^{1}Q(\phi) + {}^{2}Q(\phi) + {}^{3}Q(\phi) + \ldots = 0 \qquad (7)$$

the term ${}^{k}Q(\phi)$ is conservative (conserves ϕ) if it can be written in divergence form [8]

$${}^{k}Q(\phi) = \frac{\partial\left({}^{k}F_j\right)}{\partial x_j} \qquad (8)$$

Note that mass is conserved *a priori* since the continuity equation appears in divergence form. For the same reason the convective term of the momentum equation is conservative *a priori* (conserves momentum) if it is written in divergence form:

$$\left(Div.\right)_i = \frac{\partial u_j u_i}{\partial x_j} \qquad (9)$$

This definition of the conservation *a priori* indicates the property of conserving momentum (in periodic field) independently of the modalities by which the continuity equation is satisfied.

The governing equation for the kinetic energy, $K=u_i u_i / 2$, can be developed by taking the vector dot product of the velocity and the momentum equation,

$$u_i\left(\frac{\partial u_i}{\partial t} + \frac{\partial u_j u_i}{\partial x_j} + \frac{\partial p}{\partial x_i} - \frac{\partial T_{ij}}{\partial x_j}\right) = 0 \qquad (10)$$

where p is the pressure divided by the constant density, and T_{ij} is the viscous stress. In the above equation the convective term can be rewritten in the following form, corresponding to that in the momentum equation,

$$u_i \frac{\partial u_j u_i}{\partial x_j} = \frac{\partial u_j u_i u_i / 2}{\partial x_j} + \frac{1}{2} u_i u_i \frac{\partial u_j}{\partial x_j} \qquad (11)$$

This term is composed by two parts: the first is in conservative form and the second involves the continuity equation. The convective term (expressed in divergence form in the momentum equation) conserves *a priori* momentum but does not conserve *a priori* kinetic energy: in fact Equation 11 shows how the continuity equation is involved in the kinetic energy conservation property of the convective terms that are expressed in divergence form. In other words kinetic energy is conserved (by divergence form of convective terms) only when the continuity equation is perfectly satisfied.

The passage from the previous analytical considerations to the effect that they produce on numerical simulations imposes a reflection on the following statement:

the continuity equation cannot be perfectly satisfied by numerical simulation.

In the simulations of three-dimensional unsteady flows (LES) (that are realized by a high-order finite difference scheme with the divergence form of the convective terms) the resolved kinetic energy is not perfectly conserved because the continuity equation is not perfectly satisfied. Consequently, the resolved kinetic energy is destined to rise in long time simulations.

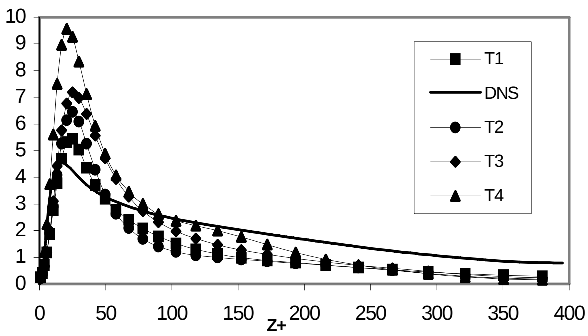

Fig. 1. Profiles of resolved kinetic energy Reynolds averaged over successive intervals of time (T1, T2, T3, T4). Simulation performed by using the turbulence model of Zang et al. [14]. Channel flow, Re*=395.

In the dynamic mixed models based on the Smagorinsky closure relation, the calculation of the closure coefficient C_S (by dynamic procedure) is not able to compensate the effects produced by the convective terms (expressed in divergence form in the resolved momentum equation) that are not able to perfectly conserve resolved kinetic energy.

In order to verify the above mentioned inconsistency of the dynamic mixed models based on the Smagorinsky closure relation, simulations of a turbulent channel flow at Re*=395 (Re* is the friction-velocity-based Reynolds number) have been performed, by a fourth-order staggered finite difference scheme proposed by Morinishi et al [15]. The generalised SGS turbulent stress tensor has been calculated by means of the mixed dynamic model of Zang et al. [6]. The resolved kinetic energy has been averaged over time intervals greater than the integral turbulent time scale. In Figure 1 the over time averaged resolved kinetic energy profiles are shown.

From the figure it is possible to deduce that the resolved kinetic energy increases.

In this paper it is demonstrated that, in order overcome the inconsistencies of the dynamic mixed Smagorinsky-type models and in order to contain the increase of resolved kinetic energy, the turbulent closure relation for the generalised SGS turbulent stress tensor must be expressed directly as a function of the generalised SGS turbulent kinetic energy E and of the SGS viscous dissipation ε.

The generalised SGS turbulent kinetic energy and the SGS viscous dissipation are unknown quantities that are calculated by solving the relative balance equations. In these equations there are unknown terms that are calculated by dynamic procedures. In this paper it is shown that the dynamic procedures for the production and dissipation terms of the SGS viscous dissipation balance equation are able to compensate dynamically the increase of resolved kinetic energy, and then allow the large eddy simulation of three-dimensional unsteady flows, also for long time simulations.

3 The balance equations of the generalised SGS turbulent kinetic energy E and of the SGS viscous dissipation ε

The balance equation of the generalized SGS turbulent kinetic energy, *E*, expressed in terms of the generalized central moments takes the form [5].

$$\begin{aligned} \frac{DE}{Dt} = & -\frac{1}{2}\frac{\partial \tau(u_k, u_k, u_m)}{\partial x_n} - \tau_{mk}\frac{\partial \overline{u_k}}{\partial x_m} - \frac{\partial \tau(p, u_m)}{\partial x_m} + \\ & + \nu \frac{\partial^2 E}{\partial x_m \partial x_m} + \tau(F_{Ok}, u_k) - \nu\tau\left(\frac{\partial u_k}{\partial x_m}, \frac{\partial u_k}{\partial x_m}\right) \end{aligned} \qquad (12)$$

where F_{Ok} is the force density and

$$\tau(f,g) = \overline{fg} - \overline{f}\,\overline{g}$$

$$\tau(f,g,h) =$$

$$\overline{fgh} - \overline{f}\,\overline{g}\,\overline{h} - \overline{f}\tau(g,h) - \overline{g}\tau(f,h) - \overline{h}\tau(f,g)$$

are, respectively, the generalized second and third order central moment related to the generic quantities f, g and h [11].

As demonstrated in [5], the generalized SGS turbulent kinetic energy balance equation is form-invariant and frame-indifferent, in so much that each of the terms that appear in it are representations, in inertial and non-inertial frames, of objective tensors that are independent of the angular and translational velocity of the frame.

The last term on the right-hand side of Equation 12 is defined as the viscous dissipation ε of SGS turbulent kinetic energy.

$$\varepsilon = \nu\tau\left(\frac{\partial u_i}{\partial x_j},\frac{\partial u_i}{\partial x_j}\right) \qquad (13)$$

In the proposed LES model a further balance equation is introduced for the viscous dissipation ε. The balance equation of ε, expressed in terms of the generalized central moments, takes the form

$$\begin{aligned}
&\frac{\partial \varepsilon}{\partial t}+\frac{\partial \overline{u_k \varepsilon}}{\partial x_k}-\nu\frac{\partial^2 \varepsilon}{\partial x_k \partial x_k}+\nu\frac{\partial}{\partial x_k}\tau\left(u_k,\frac{\partial u_i}{\partial x_j},\frac{\partial u_i}{\partial x_j}\right)\\
&+2\nu\frac{\partial}{\partial x_k}\left(\frac{\partial \overline{u_i}}{\partial x_j}\tau\left(u_k,\frac{\partial u_i}{\partial x_j}\right)\right)+2\nu\frac{\partial}{\partial x_k}\tau\left(\frac{\partial u_k}{\partial x_i},\frac{\partial p}{\partial x_i}\right)\\
&-2\nu\frac{\partial}{\partial x_k}\left(\frac{\partial \overline{u_i}}{\partial x_j}\frac{\partial \tau_{ik}}{\partial x_j}\right)+2\nu\tau\left(\frac{\partial u_i}{\partial x_k},\frac{\partial u_k}{\partial x_j},\frac{\partial u_i}{\partial x_j}\right)\\
&+2\nu\frac{\partial \overline{u_k}}{\partial x_j}\tau\left(\frac{\partial u_i}{\partial x_k},\frac{\partial u_i}{\partial x_j}\right)+2\nu\frac{\partial \overline{u_i}}{\partial x_k}\tau\left(\frac{\partial u_k}{\partial x_j},\frac{\partial u_i}{\partial x_j}\right)\\
&+2\nu\frac{\partial \overline{u_i}}{\partial x_j}\tau\left(\frac{\partial u_i}{\partial x_k},\frac{\partial u_k}{\partial x_j}\right)+2\nu\frac{\partial \tau_{ik}}{\partial x_j}\frac{\partial^2 \overline{u_i}}{\partial x_j \partial x_k}\\
&+2\nu^2\tau\left(\frac{\partial^2 u_i}{\partial x_j \partial x_k},\frac{\partial^2 u_i}{\partial x_j \partial x_k}\right)-2\nu\tau\left(\frac{\partial u_i}{\partial x_j},\frac{\partial F_{Oi}}{\partial x_j}\right)=0
\end{aligned} \qquad (14)$$

This equation is obtained from the Navier-Stokes equation and the filtered Navier-Stokes equation. Above all, it is demonstrated that Equation 14 is form-invariant and frame-dependent. In a non-inertial frame, Equation 14 takes the form:

$$\begin{aligned}
&\frac{\partial \varepsilon^*}{\partial t^*}+\frac{\partial \overline{u}^*_k \varepsilon^*}{\partial t^*}-\nu^*\frac{\partial^2 \varepsilon^*}{\partial x^*_r \partial x^*_r}+\nu^*\frac{\partial}{\partial x^*_r}\tau\left(u^*_r,\frac{\partial u^*_i}{\partial x^*_j},\frac{\partial u^*_i}{\partial x^*_j}\right)\\
&+2\nu^*\frac{\partial}{\partial x^*_m}\left[\frac{\partial \overline{u}^*_n}{\partial x^*_s}\tau\left(u^*_m,\frac{\partial u^*_n}{\partial x^*_s}\right)\right]+\\
&+\boxed{2\nu^*\dot{Q}_{in}Q_{it}\frac{\partial}{\partial x^*_m}\tau\left(u^*_m,\frac{\partial u^*_t}{\partial x^*_n}\right)}\\
&+2\nu^*\frac{\partial}{\partial x^*_s}\tau\left(\frac{\partial \overline{u}^*_s}{\partial x^*_i},\frac{\partial p^*}{\partial x^*_i}\right)\\
&-2\nu^*\frac{\partial}{\partial x^*_k}\left(\frac{\partial \overline{u}^*_i}{\partial x^*_j},\frac{\partial \tau_{ik}}{\partial x_j}\right)-\boxed{2\nu^*\dot{Q}_{in}Q_{it}\frac{\partial}{\partial x^*_k}\left(\frac{\partial \tau^*_{tk}}{\partial x^*_n}\right)}\\
&+2\nu^*\tau\left(\frac{\partial u^*_i}{\partial x^*_k},\frac{\partial u^*_k}{\partial x^*_j},\frac{\partial u^*_i}{\partial x^*_j}\right)+2\nu^*\frac{\partial \overline{u}^*_m}{\partial x^*_n}\tau\left(\frac{\partial \overline{u}^*_i}{\partial x^*_m},\frac{\partial \overline{u}^*_i}{\partial x^*_n}\right)\\
&+2\nu^*\frac{\partial \overline{u}^*_m}{\partial x^*_n}\tau\left(\frac{\partial u^*_n}{\partial x^*_j},\frac{\partial u^*_m}{\partial x^*_j}\right)+2\nu^*\frac{\partial \overline{u}^*_m}{\partial x^*_n}\tau\left(\frac{\partial u^*_m}{\partial x^*_k},\frac{\partial u^*_k}{\partial x^*_n}\right)\\
&+\boxed{2\nu^*\dot{Q}_{in}Q_{it}\tau\left(\frac{\partial u^*_t}{\partial x^*_k},\frac{\partial u^*_k}{\partial x^*_n}\right)}+2\nu^*\frac{\partial \tau^*_{ns}}{\partial x^*_m}\frac{\partial^2 \overline{u}^*_n}{\partial x^*_m \partial x^*_s}\\
&+2\nu^{*2}\tau\left(\frac{\partial^2 u^*_i}{\partial x^*_j \partial x^*_k},\frac{\partial^2 u^*_i}{\partial x^*_j \partial x^*_k}\right)-2\nu^*\tau\left(\frac{\partial u^*_l}{\partial x^*_m},\frac{\partial F^*_{Ol}}{\partial x^*_m}\right)=0
\end{aligned} \qquad (15)$$

in which the tensor $\dot{Q}_{in}Q_{it}$ that appears in the representation of the 5th, 7th and 11th terms (within the respective boxes) is associated with the angular velocity of the non-inertial frame with respect to the inertial one. The 5th, 7th and 11th terms of equations 14 and 15 are the representations (in the inertial and non-inertial frame) of zero-order objective tensors that are frame-dependent: the objectivity of these tensors is guaranteed by the fact that the respective representations in the different frames are related to each other according to the laws of transformation expressed by Equation 2; the frame-dependence is given by the presence of the above mentioned term $\dot{Q}_{in}Q_{it}$ associated with the angular velocity of the frame. All the other terms that appear in Equations 14 and 15 are representations, in the different frames, of objective tensors that are independent of the angular velocity of the frame. From this consideration and for the assumption that an equation is form-invariant if it is expressed only in terms of objective tensors, it results that the balance equation of ε is form-invariant. This equation is also frame-dependent, through the appearance of $\dot{Q}_{in}Q_{it}$ in the transformations of the representations of the 5th, 7th and 11th terms in the different frames.

4 New closure relations

According to Bardina's scale similarity assumption, a closure relation is proposed for the generalized SGS turbulent stress tensor, in which there appears no coefficient to be calibrated or to be calculated dynamically:

$$\tau_{ij}=\left(\frac{2E}{L^m_{kk}}\right)L^m_{ij} \qquad (16)$$

The closure relation 16 is obtained without any assumption of local balance between the production and dissipation of generalized SGS turbulent kinetic energy and may thus be considered applicable to LES with the filter width falling into the range of wave numbers greater than the wave number corresponding to the maximum turbulent kinetic energy. The closure relation 16 for the generalized SGS turbulent stress tensor: a) respects the principle of turbulent frame indifference; b) takes into account both the anisotropy of the turbulence velocity scales and turbulence length scales; c) assumes scale similarity; d) guarantees an adequate energy drain from the grid scales to the subgrid scales and guarantees backscatter; e) overcomes the inconsistencies linked to the dynamic calculation of the closure coefficient used in the modelling of the generalized SGS turbulent stress tensor; f) is able to eliminate the effects produced by the non-conservation *a priori* of the resolved kinetic energy.

Even though the closure relations in turbulence must not necessarily respect the requirement of frame-indifference, since the generalized SGS turbulent stress tensor is an objective tensor and frame-indifferent [5], the closure relation for this tensor must also be form-invariant and frame-indifferent. In other words, the principle of turbulent frame indifference must not necessarily be applied to all the closure relations, but must be applied to all the functional relations that express the generalized SGS turbulent stress tensor (objective and frame-indifferent) in terms of the resolved kinematic quantities. The closure relation 16 for the generalized SGS turbulent stress tensor respects the principle of turbulent frame indifference.

In this paper, the modelled expressions used for the unknown terms of the exact generalized SGS turbulent kinetic energy balance equation are such as to guarantee the respect of the principle that the modelled balance equation of E must be form-invariant and frame-indifferent, like the exact balance equation of E. The modelled form of Equation 12 is

$$\frac{DE}{Dt} = -\frac{\partial}{\partial x_k}\left(D\sqrt{E\Delta}\,\frac{\partial E}{\partial x_k}\right) - \tau_{mk}\frac{\partial \overline{u_k}}{\partial x_m} + \nu\frac{\partial^2 E}{\partial x_m \partial x_m} - \varepsilon \qquad (17)$$

where the scalar coefficient D is dynamically calculated by means of a Germano identity applied to the 1st and 3rd terms on the right-hand side of Equation 12 [5].

As demonstrated in section 3, the balance equation of the SGS viscous dissipation, ε, is form-invariant because it is expressed only in terms of objective tensors. This balance equation is also frame-dependent, through the appearance of $\dot{Q}_{in}Q_{it}$ in the transformations of the representations of the 5th, 7th and 11th terms of the balance equation in the different frames. Consequently, the closure relations for the 5th, 7th and 11th terms of Equation 14 must be formulated in full respect of the requirement that the modelled equation must be form-invariant and must maintain the same frame-dependence as the exact equation. In this paper the following modelled balance equation of the viscous dissipation is proposed.

$$\begin{aligned}
&\frac{\partial \varepsilon}{\partial t} + \frac{\partial \overline{u_k}\varepsilon}{\partial x_k} - \nu\frac{\partial^2 \varepsilon}{\partial x_k \partial x_k} + \frac{\partial}{\partial x_k} C_{F_\varepsilon}\frac{E^2}{\varepsilon}\frac{L^m_{kl}}{L^m_{jj}}\frac{\partial \varepsilon}{\partial x_l} \\
&+ 2\nu\frac{\partial}{\partial x_k}\left(\frac{\partial \overline{u_i}}{\partial x_j}\left(\frac{C_{F^l_\varepsilon}\frac{E}{\varepsilon}\frac{\partial \varepsilon}{\partial x_q}\delta_q}{\tau\left(\overline{u_n},\frac{\partial \overline{u_n}}{\partial x_n}\right)}\tau\left(\overline{u_k},\frac{\partial \overline{u_i}}{\partial x_j}\right)\right)\right) \\
&- 2\nu\frac{\partial}{\partial x_k}\left(\frac{\partial \overline{u_i}}{\partial x_j}\frac{\partial}{\partial x_j}\left(\frac{2E}{L^m_{qq}}L^m_{ik}\right)\right) - C_{P_\varepsilon}\frac{\varepsilon\left(-L^m_{ij}S_{ij}\right)}{L_{kk}} \\
&+ 2\nu\frac{\partial \overline{u_i}}{\partial x_j}\frac{\frac{\varepsilon}{\nu}\tau\left(\frac{\partial \overline{u_i}}{\partial x_k},\frac{\partial \overline{u_i}}{\partial x_j}\right)}{\tau\left(\frac{\partial \overline{u_q}}{\partial x_n},\frac{\partial \overline{u_q}}{\partial x_n}\right)} + 2\nu\frac{\partial u_i}{\partial x_k}\frac{\frac{\varepsilon}{\nu}\tau\left(\frac{\partial \overline{u_k}}{\partial x_j},\frac{\partial \overline{u_i}}{\partial x_j}\right)}{\tau\left(\frac{\partial \overline{u_q}}{\partial x_n},\frac{\partial \overline{u_q}}{\partial x_n}\right)} \\
&+ 2\nu\frac{\partial \overline{u_i}}{\partial x_j}\frac{\frac{\varepsilon}{\nu}\tau\left(\frac{\partial \overline{u_i}}{\partial x_k},\frac{\partial \overline{u_k}}{\partial x_j}\right)}{\tau\left(\frac{\partial \overline{u_q}}{\partial x_s},\frac{\partial \overline{u_s}}{\partial x_q}\right)} + 2\nu\frac{\partial}{\partial x_j}\left(\frac{2E}{L^m_{nn}}L^m_{ik}\right)\frac{\partial^2 \overline{u_i}}{\partial x_j \partial x_k} \\
&+ C_{D_\varepsilon}\frac{\varepsilon^2}{E} = 0 \qquad (18)
\end{aligned}$$

where $\delta_i=(1,1,1)$ and in which the closure coefficients are calculated dynamically by means of the Germano identities. This equation is form-invariant and has the same dependence on the frame as the exact equation. The 5th, 7th and 11th terms of Equation 18 are the modelled expression of the corresponding unknown terms of the Equation 14.

These modelled expressions are formulated by using the hypothesis of scale-similarity and respecting the same frame-dependence as the respective unknown terms in 15.

For example, in the 5th term of the balance equation of ε, $2\nu\frac{\partial}{\partial x_k}\left(\frac{\partial \overline{u_i}}{\partial x_j}\tau\left(u_k,\frac{\partial u_i}{\partial x_j}\right)\right)$, is modelled only the unknown part, $\tau\left(u_k,\frac{\partial u_i}{\partial x_j}\right)$.

The hypothesis of scale similarity gives

$$\tau\left(u_k, \frac{\partial u_i}{\partial x_j}\right) = \frac{\tau\left(u_r, \dfrac{\partial u_r}{\partial x_r}\right)}{\tau\left(\bar{u}_n, \dfrac{\partial u_n}{\partial x_n}\right)} \tau\left(\bar{u}_k, \frac{\partial \overline{u_i}}{\delta x_j}\right) \qquad (19)$$

By introducing the following closure relation for the unknown term of Equation 19,

$$\tau\left(u_r \frac{\partial u_r}{\partial x_r}\right) = C_{F_\varepsilon^l} \frac{E}{\varepsilon} \frac{\partial \varepsilon}{\partial x_r} \delta_r \qquad (20)$$

the final modelled form of the 5th term is

$$2\nu \frac{\partial}{\partial x_k}\left(\frac{\partial \overline{u_i}}{\partial x_j} \left(\frac{C_{F_\varepsilon^l} \dfrac{E}{\varepsilon} \dfrac{\partial \varepsilon}{\partial x_r} \delta_r}{\tau\left(\overline{u_n}, \dfrac{\partial \overline{u_n}}{\partial x_n}\right)} \tau\left(\overline{u_k}, \frac{\partial \overline{u_i}}{\partial x_j}\right) \right) \right) \qquad (21)$$

where the closure coefficient $C_{F_\varepsilon^l}$ is calculated dynamically by means of a Germano identity.

Under a Euclidean transformation of the frame, the exact 5th term of the balance equation of ε transforms as.

$$2\nu \frac{\partial}{\partial x_k}\left(\frac{\partial \overline{u_i}}{\partial x_j} \tau\left(\overline{u_k}, \frac{\partial \overline{u_i}}{\partial x_j}\right) \right) =$$

$$2\nu \frac{\partial}{\partial x_k}\left[\frac{\partial \overline{u_n}}{\partial x_s} \left(u_m^*, \frac{\partial u_n^*}{\partial x_s} \right) \right] +$$

$$\boxed{2\nu^* \dot{Q}_{in} Q_{it} \frac{\partial}{\partial x_m^*} \tau\left(u_m^*, \frac{\partial u_n^*}{\partial x_s}\right)} \qquad (22)$$

Under the same Euclidean transformation of the frame the modelled form of the 5th term of the transport equation of ε transforms as

$$2\nu \frac{\partial}{\partial x_k}\left(\frac{\partial \overline{u_i}}{\partial x_j} \left(\frac{C_{F_\varepsilon^l} \dfrac{E}{\varepsilon} \dfrac{\partial \varepsilon}{\partial x_r} \delta_r}{\tau\left(\overline{u_n}, \dfrac{\partial \overline{u_n}}{\partial x_n}\right)} \tau\left(\overline{u_k}, \frac{\partial \overline{u_i}}{\partial x_j}\right) \right) \right) =$$

$$2\nu^* \frac{\partial}{\partial x_m^*}\left(\frac{\partial \overline{u_n^*}}{\partial x_s^*} \left(\frac{C_{F_\varepsilon^l}^* \dfrac{E^*}{\varepsilon^*} \dfrac{\partial \varepsilon^*}{\partial x_r^*} \delta_r^*}{\tau\left(\overline{u_i^*}, \dfrac{\partial \overline{u_i^*}}{\partial x_i^*}\right)} \tau\left(\overline{u_m^*}, \frac{\partial \overline{u_n^*}}{\partial x_s^*}\right) \right) \right)$$

$$+ 2\nu^* \dot{Q}_{in} Q_{it} \frac{\partial}{\partial x_m^*} \tau\left(\frac{C_{F_\varepsilon^l}^* \dfrac{E^*}{\varepsilon^*} \dfrac{\partial \varepsilon^*}{\partial x_i^*} \delta_i^*}{\tau\left(\overline{u_i^*}, \dfrac{\partial \overline{u_n^*}}{\partial x_n^*}\right)} \tau\left(\overline{u_m^*}, \frac{\partial \overline{u_t^*}}{\partial x_n^*}\right) \right) \qquad (23)$$

The modelled term results as being dependent on the frame of reference in the same manner as the exact term. A similar result is obtained by repeating the same procedure for the others terms of Equation 18. The demonstration is omitted for the sake of brevity.

5 Results and Discussion

Turbulent channel flows (between two flat parallel plates placed at a distance of 2L) are simulated with the proposed Large Eddy Simulation model at different friction-velocity-based Reynolds numbers (Re*), ranging from 395 to 2340. In order to validate the proposed closure relation for the generalized SGS turbulent stress tensor, the numerical results obtained with the proposed model are compared with DNS results [19] and with experimental data [20]

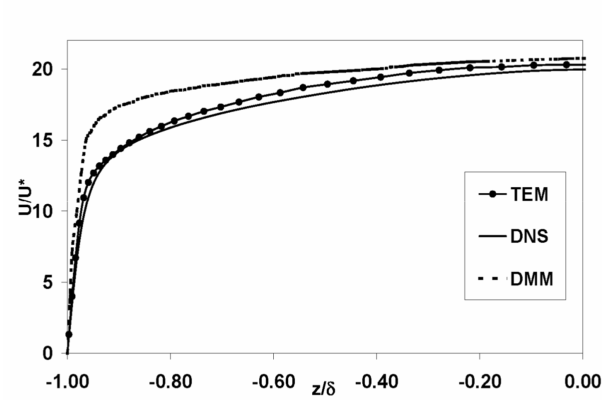

Fig. 2. Time-averaged streamwise velocities. Comparison between DNS and LES results obtained with DMM and the proposed model (TEM). Channel flow, Re* = 395.

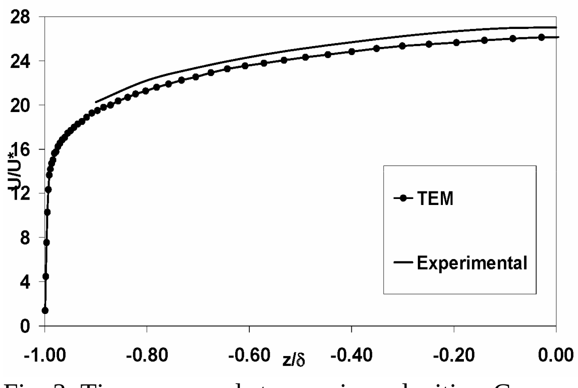

Fig. 3. Time-averaged streamwise velocities. Comparison between experimental measurements and LES results obtained with the proposed model (TEM). Channel flow, Re* = 2340.

In Figure 2 is plotted the profile of the time-averaged streamwise velocity component obtained with the proposed model compared with the profile obtained with DNS [19] and the Dynamic Mixed Model, DMM [14], for channel flow at Re* = 395.

The figure shows that the profile obtained with the proposed model agrees more with the DNS velocity profile than with the profile obtained with the DMM, both in the boundary layer and in the region inside the channel. Figure 3 shows the profile of the time-averaged streamwise velocity component for a channel flow at Re*=2340 obtained with the proposed model, compared with the profile of the analogous velocity component measured experimentally [20]. The agreement between the two velocity profiles is very good.

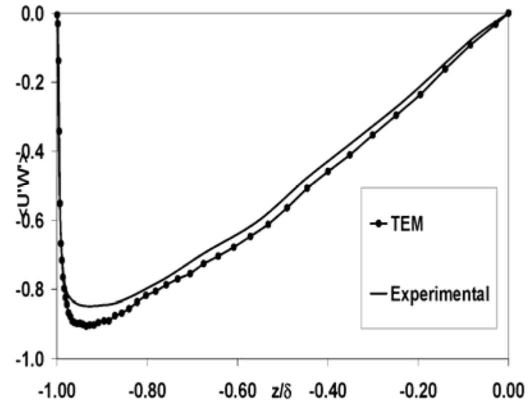

Fig. 4. Reynolds stress $\langle u_1'u_3' \rangle$. Comparison between experimental measurements and LES results obtained with the proposed model (TEM). Channel flow, Re* = 2340.

Figure 4 compares the profile of the component $\langle u_1'u_3' \rangle$ of the Reynolds stress tensor (where indexes (1) and (3) denote, respectively, the streamwise and wall-normal directions), calculated with the proposed model, with the profile of the similar component of the Reynolds stress tensor obtained from experimental measurements [20], for a channel flow at Re* = 2340. Figure 4 shows that at Re* = 2340 the proposed model provides a profile of the component $\langle u_1'u_3' \rangle$ in agreement with that of the corresponding component of the Reynolds stress tensor obtained from the experimental measurements.

Figure 5 shows the instantaneous profiles of the terms of the balance equations of *E* averaged over homogeneous planes, for channel flow at Re*=2340. Figures 5 demonstrates that the balance between production and dissipation of the generalized SGS turbulent kinetic energy is confirmed only in a limited region between the buffer layer and the log layer (20<z+<40) whilst it is not confirmed in other regions of the domain. The viscous dissipation of *E* is balanced in the viscous sublayer (z+<5) by the viscous diffusion term whilst the production of *E* is practically negligible. Moving away from the wall, in the first part of the buffer layer, the production term of *E* increases until it reaches its maximum value (z+ ≈ 10) and the terms of turbulent transport and viscous diffusion of *E* are comparable with the production term of *E*. In the region between the buffer layer and the log layer (20<z+<40) the convective and turbulent transport terms and the viscous diffusion term are negligible compared with the production and dissipation terms. Only in this limited region is there a balance between the production and the dissipation of *E*. Towards the center of the channel (z+>30) the viscous dissipation tends towards a minimum but not negligible value. In this region the production term of *E* is balanced not only by the dissipation but also by the turbulent transport of *E*.

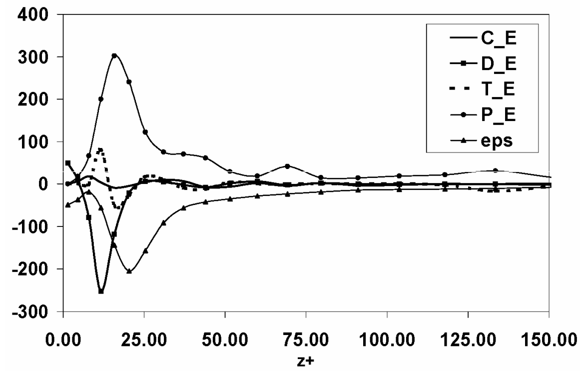

Fig. 5. Instantaneous generalized SGS turbulent kinetic energy balance terms averaged over homogeneous planes. Production: P_E; Turbulent transport: T_E; Convection: C_E; Viscous diffusion: D_E; Viscous dissipation: eps. Channel flow, Re*=2340.

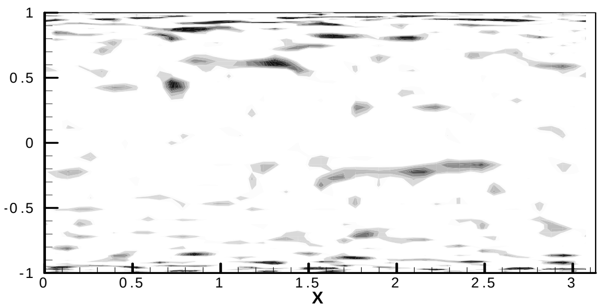

Figure 6: Vortex identification with λ_2 method [21], x-z plane

In Figure 6 the near wall vortex structures (inside the turbulent boundary layer) are clearly identified by the λ_2 method [21]: the dimensions of the spatial discretisation steps allow the optimal simulation of the above mentioned vortex structures that govern the transport, the production and the dissipation of the turbulent kinetic energy.

6 Conclusions

In this paper the principle of Turbulent Frame Indifference is revised. The present-day LES models and the drawbacks of the dynamic calculation of the closure coefficient for the generalized SGS turbulent stress tensor are analyzed. A new closure relation for the generalized SGS turbulent stress tensor is proposed. The proposed closure relation for the generalized SGS turbulent stress tensor: complies with the principle of turbulent frame indifference; takes into account both the anisotropy of the turbulence velocity scales and turbulence length scales; removes any balance assumption between the production and dissipation of SGS turbulent kinetic energy. In the proposed model the generalized SGS turbulent stress tensor is related exclusively to the generalized SGS turbulent kinetic energy (which is calculated by means of its balance equation) and the modified Leonard tensor; the viscous ε of the generalized SGS turbulent kinetic energy is calculated by solving the ε balance equation. The modelled balance equation of ε respects the properties of form-invariance and frame-dependence of the exact balance equation. The proposed model has been tested for a turbulent channel flow at Reynolds numbers (based on friction velocity and channel half-width) ranging from 395 to 2340. The proposed model improves the agreement between the results obtained with LES and those obtained with DNS.

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