A variational principle for channel and pipe flows

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Abstract: - An approach to modeling channel and pipe flows of incompressible viscous fluid based on a unified variational principle valid for both laminar and turbulent regimes is proposed. For low Reynolds numbers this variational principle reduces to the principle of minimum dissipation. For high Reynolds numbers it enables one to calculate the velocity profiles and the corresponding friction factors with reasonably good accuracy.

Key-Words: - Channel, Pipe, Flow, Reynolds number, Laminar, Turbulent

1 Introduction

One of long-standing issues in fluid dynamics is how to propose a unified theory of channel and pipe flows which can predict the transition from laminar to turbulent regime and simultaneously calculate the velocity profile and the friction factors for all Reynolds numbers [1]. Up to now, most of researchers in the field share the believe that this could be done by solving the Navier-Stokes equation [2,3].

The present paper proposes an approach deviating largely from this main stream. It focuses on the turbulent modeling [4] rather than solving the Navier-Stokes equation. The starting point is the variational principle of minimum dissipation which is indeed the direct consequence of the Navier-Stokes equation and which can be used to determine the velocity profiles of laminar flows. However, as the Reynolds number becomes large, new vortices may occur and we assume that the Reynolds stress depends on the flow generated by these new vortices. We attempt at formulating a variational principle involving the statistically average velocities of turbulent flow and the flow generated by new vortices by taking into account the interaction between large-scale and small-scale vortices through the energy cascade [5]. The empirical law of wall [6,7] is used to find the nonlinear term responsible for the interaction between vortices. We will show that the proposed variational principle reduces to the principle of minimum dissipation for small Reynolds number. For large Reynolds numbers it enables one to predict the velocity profile and the friction factors for turbulent flows with reasonable accuracy as compared with experimental data.

2 Shear flow

Consider an incompressible viscous fluid bounded by two parallel plates of infinite extent. The lower plate occupying the plane $y=0$ is at rest. To the upper plate occupying the plane $y=2h$ the constant force τ per unit area is applied. The shear flow (or Couette flow) driven by the motion of this plate exerts the resistance to it through the viscous shear stress. If the resistance is equal to the applied force, the stationary flow will be settled. The problem is to determine the velocity of the stationary flow as function of τ.

It turns out that the solution of this problem exhibits extremely different behaviors at small and large τ. In the laminar case (at small τ) the solution can be found by the following variational principle: among all admissible velocity fields $u(y)$ satisfying $u(0)=0$ the true velocity field $\hat{u}(y)$ minimizes the dissipation functional

$$D=\int_0^{2h} \tfrac{1}{2}\eta u'^2 dy - \tau u(2h), \qquad (1)$$

with η being the viscosity and the prime denoting the derivative. The first term in (1) describes the dissipation (per unit area), while the second term corresponds to the power of the external force and can be regarded as the energy source.

It is easy to show that the minimizer of (1) satisfies the equation $\eta u'=\tau$ which leads to the linear profile of velocity $u(y)=\frac{\tau}{\eta}y$. Thus, the average velocity $\underline{u}$ is equal to the velocity in the middle of the flow at $y=h$, $\underline{u}=u_m=\tau h/\eta$. Consequently, the friction coefficient reads

$$c_f=\frac{2\tau}{\rho u_m^2}=\frac{2}{\mathrm{Re}},$$

with ρ the mass density, and $\mathrm{Re}=\rho u_m h/\eta$ the Reynolds number [1]. It is more convenient to use another definition of Reynolds number which is directly related to τ (the so-called friction Reynolds number)

$R = \rho u_\tau h / \eta, \quad u_\tau = \sqrt{\tau / \rho}$.

Because $\mathrm{Re} = R^2$ for laminar flows, we have in this case $c_f = 2 / R^2$.

As the Reynolds number increases and exceeds some critical value, new vortices may occur. The energy required for the nucleation and motion of these new vortices is transferred from the energy source through the Richardson cascade [4,5]. Because of this energy transfer the statistically average velocity of the turbulent flow $u(y)$ is reduced considerably. For turbulent regime many uncertainties arise except the following balance equation of mean momentum for $u(y)$ which seems to be firmly established [4,8]

$$\frac{d}{dy}(\eta u' + \tau F) = 0,$$

where τF is the so-called Reynolds stress. However, F is unknown, and the problem of how to close this equation remains (see, for example, the mixing length model or the $k-\varepsilon$ model in [4]). In this paper we shall adopt the following two main hypotheses:

1. *Function F depends only on $\varphi(y)$ which is the statistically average velocity of the flow generated by new vortices.*
2. *The governing equations for $u(y)$ and $\varphi(y)$ can be obtained from a variational principle.*

Our aim is to show that the following variational principle leads to a satisfactory model for both laminar and turbulent flows: among all admissible velocity fields $u(y)$ and $\varphi(y)$ satisfying the boundary conditions

$$u(0) = 0, \quad \varphi(0) = \varphi(2h) = 0, \tag{2}$$

the shear flow is described by those for which the dissipation functional

$$D = \int_0^{2h} (\tfrac{1}{2}\eta u'^2 + \tau F(\varphi)|u'| - \tfrac{1}{2}\eta\varphi'^2) dy - \tau u(2h) \tag{3}$$

has an extremum. The first term in (3) is the dissipation due to viscosity, the second term describes the energy transfer from large-scale to small-scale vortices, with $F(\varphi)$ a nonlinear function of φ, the third term is responsible for the reduction in dissipation due to the energy storage by small scale vortices, and finally, the last term corresponds to the power of the external force (or the energy source). Here $|u'|$ and $|\varphi'|$ are the scalar densities of the corresponding vortices. We also assume that the wall is ideally smooth so that new vortices cannot be nucleated there. This assumption is reflected in the second boundary conditions (2).

It is convenient to introduce further the following dimensionless quantities

$$\varsigma = \frac{Ry}{h}, \quad \tilde{u} = \frac{u}{u_\tau}, \quad \tilde{\varphi} = \frac{\varphi}{u_\tau}, \quad \tilde{F}(\tilde{\varphi}) = F(\tilde{\varphi} u_\tau),$$

with R and u_τ being previously described. We shall deal further only with these dimensionless quantities, therefore the tildes can be dropped for short. In this problem the velocity is monotone increasing, therefore $|u'| = u'$. Thus, the dimensionless dissipation $D / \tau u_\tau$ becomes

$$\int_0^{2R} (\tfrac{1}{2}u'^2 + F(\varphi)u' - \tfrac{1}{2}\varphi'^2) d\varsigma - u(2R) \tag{4}$$

Varying functional (4) we derive from it the Euler equations

$$u' + F(\varphi) = 1, \quad F'(\varphi)u' + \varphi'' = 0 \tag{5}$$

The first equation follows directly from the balance of mean momentum and the boundary condition at $\varsigma = 2R$. Expressing u' through $F(\varphi)$ in the first equation and substituting it into the second equation we obtain the governing equation for φ

$$\varphi'' + F'(\varphi)(1 - F(\varphi)) = 0. \tag{6}$$

Equation (6) has the form of the equation of motion of a particle with mass 1 in the potential well

$$V(\varphi) = F(\varphi) - \tfrac{1}{2}F^2(\varphi). \tag{7}$$

The immediate consequence of this is the conservation law

$$\tfrac{1}{2}\varphi'^2 + V(\varphi) = V(\varphi_m), \tag{8}$$

where φ_m is the maximal velocity of the vortex flow which is achieved at $\varsigma = R$. We assume that both $F(\varphi)$ and $V(\varphi)$ are even function of φ.

The determination of $F(\varphi)$ is based on the law of wall [6,7]. This law states that for Reynolds numbers approaching infinity the Reynolds stress $F(\varphi_\infty(\xi))$ becomes a universal function $f(\xi)$, with $\xi \in (0, \infty)$ the so-called ``wall coordinate". From various experimental data (see [4,8]) we know that $f(\xi) \sim 1 - \frac{1}{\kappa\xi}$ as $\xi \to \infty$, with $\kappa = 0.41$ the Karman constant, and $f(\xi) \sim a\xi^3$ for small ξ. We use therefore the following semi-empirical formula

$$f(\xi) = 1 - \frac{1}{1 + (a+b)\xi^3} - \frac{b\xi^3}{1 + \kappa b \xi^4} \tag{9}$$

for this universal function, with $a = 6.\ 10^{-4}$, $b = 1.085\ 10^{-6}$ (compare with [8]). Thus,

$$\tfrac{1}{2}\varphi_\infty'^2 + V(\varphi_\infty(\xi)) = \tfrac{1}{2}$$

and, consequently

$$\varphi_\infty'(\xi) = \sqrt{1 - 2V(\varphi_\infty(\xi))} = 1 - f(\xi). \tag{10}$$

Equations $(5)_1$ and (10) show that for very large Reynolds numbers the vorticity densities of turbulent and vortex flows are equal, what seems to be quite natural. With function $f(\xi)$ from (9) we find

$$\varphi_\infty(\xi) = \tfrac{1}{\Lambda}[\tfrac{1}{3}\ln\frac{\Lambda\xi+1}{\sqrt{(\Lambda\xi)^2-\Lambda\xi+1}} + \tfrac{1}{\sqrt{3}}(\arctan\frac{2\Lambda\xi-1}{\sqrt{3}} + \tfrac{\pi}{6})] + \tfrac{1}{4\kappa}\ln(1+\kappa b\xi^4) \qquad (11)$$

where $\Lambda = (a+b)^{1/3}$. The plot of $V(\varphi)$ using ξ as parameter is shown in Fig.1. This function applies to *all Reynolds numbers*. It is interesting to mention that function $V(\varphi)$ behaves like $a\varphi^3$ for small φ and $1-e^{-2\kappa\varphi}/\kappa^2$ for large φ.

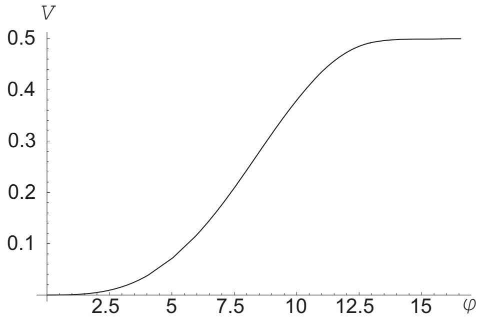

Fig.1: Function $V(\varphi)$

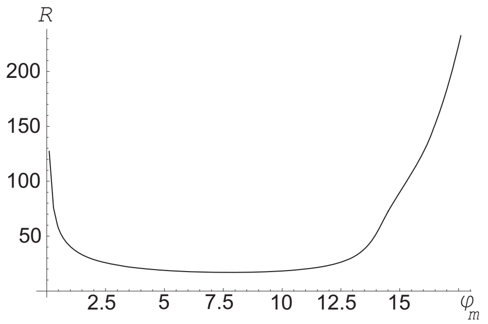

Fig.2: Reynolds number as function of φ_m

Knowing $V(\varphi)$ one can integrate equation (8) to determine φ and then u. In particular,

$$R = \int_0^{\varphi_m} \frac{d\varphi}{\sqrt{2[V(\varphi_m) - V(\varphi)]}} . \qquad (12)$$

It is easy to show that R tends to infinity as ξ tends to zero like

$$R \sim \sqrt{\frac{\pi}{2a\xi_m}}\frac{\Gamma(4/3)}{\Gamma(5/6)},$$

and tends to infinity as $\xi_m \to \infty$ tends to infinity like a linear function. It has one minimum R_c=16.9714 which is achieved at ξ_m=8.4397 (or φ_m=7.805). For $R < R_c$ the extremal φ must vanish. Thus, the value R_c=16.9714 can be regarded as the critical Reynolds number, at which the transition from laminar to turbulent shear flow takes place (this corresponds to Re_c=288.028). The plot of R as function of φ_m is shown in Fig.2. One can see that the transition from laminar to turbulent regime is ``hard" in the sense that a finite amplitude of velocity of vortex motion is required for it. For example, if the disturbances of velocity of vortex motion is smaller than 0.163, then the laminar regime can be maintained up to the Reynolds number R=100. This agrees qualitatively with the stability analysis of the Navier-Stokes equation [9]. Mention that the continuum description of vortex motion through φ may not work appropriately for intermittent turbulent flows, so the quantitative agreement can hardly be expected.

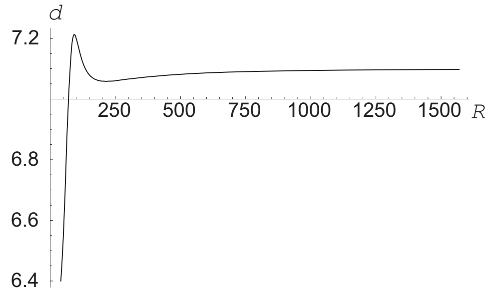

Fig.3: Difference $d = u_m - \ln R/\kappa$ as function of Reynolds number

Finally we find the distribution of velocity $u(\varsigma)$ of turbulent flow from $\varphi(\varsigma)$ by integrating the equation $(5)_1$. The velocity in the middle of the flow is the function of the Reynolds number R which behaves asymptotically like $\ln R/\kappa$ for large R. The plot of the difference $d = u_m - \ln R/\kappa$ as function of R is shown in Fig.3. As R tends to infinity, this difference tends to the value 7.1 which coincides with the empirical value given in [8].

3 Channel flow

Consider next the 2-D flow of the incompressible viscous fluid in a channel driven by the constant pressure gradient (Poiseuille flow). Both plates are now at rest. In the laminar case (low pressure gradient) the velocity $u(y)$ is the minimizer of the dissipation functional

$$D = \int_{-h}^{h} \tfrac{1}{2}\eta u'^2 dy + p'\int_{-h}^{h} u(y)dy \qquad (13)$$

under the constraint

$u(\pm h) = 0$.

Here p' denote the pressure gradient with respect to x which is constant over the cross-section. It is easy to show that the velocity profile is parabolic: $u(y) = \frac{p'}{2\eta}(h^2 - y^2)$, so the average velocity is equal to $\bar{u} = -p'h^2/3\eta$. This yields the following friction coefficient

$$c_f = \frac{-p'h}{\tfrac{1}{2}\rho\bar{u}^2} = \frac{18}{R^2},$$

where $R = \rho u_p h/\eta$, $u_p = \sqrt{-p'h/\rho}$.

As the pressure gradient becomes large, a flow induced by new vortices occurs. We let $u(y)$ denote as before the statistically average velocity of turbulent flow and $\varphi(y)$ the statistically average velocity of flow generated by new vortices. Adopting the same hypotheses as in the previous case, we formulate the following variational principle: among all admissible velocity fields $u(y)$ and $\varphi(y)$ satisfying the boundary conditions

$$u(\pm h)=0, \qquad \varphi(\pm h)=0, \tag{14}$$

the channel flow is described by those for which the dissipation functional

$$D=\int_{-h}^{h}(\tfrac{1}{2}\eta u'^2-p'hF(\varphi)|u'|+p'h\frac{\alpha}{R^{2/5}}|u'|-\tfrac{1}{2}\eta\varphi'^2)dy + p'\int_{-h}^{h}u(y)dy \tag{15}$$

has an extremum. In comparison with the shear flow the only new term added in this dissipation functional is the third term which is responsible for the energy storage of large scale vortices, where α is now the universal parameter which is chosen to be $\alpha=1.442$.

By changing the variables and unknown functions

$$\varsigma=\frac{Ry}{h}, \quad u\mapsto\frac{u}{u_p}, \quad \varphi\mapsto\frac{\varphi}{u_p}, \quad D\mapsto-\frac{D}{p'hu_p},$$

we transform the dissipation functional to

$$D=\int_{-R}^{R}(\tfrac{1}{2}u'^2+F(\varphi)|u'|-\frac{\alpha}{R^{2/5}}|u'|-\tfrac{1}{2}\varphi'^2)d\varsigma - \int_{-R}^{R}\frac{1}{R}u(\varsigma)d\varsigma. \tag{16}$$

Observe first that the variational problem (16) always has the extremal $\varphi=0$ leading to the laminar velocity profile. However, for sufficiently large R there is another extremal describing the turbulent flow. For the developed turbulent flow (large R) we assume that

$$u(\varsigma)=\begin{cases}u_1(\varsigma)\\ u_m\\ u_1(-\varsigma)\end{cases}, \quad \varphi(\varsigma)=\begin{cases}\varphi_1(\varsigma)\\ \varphi_m(\varsigma)\\ -\varphi_1(\varsigma)\end{cases},$$

for $\varsigma\in(-R,-l)$, $\varsigma\in(-l,l)$, and $\varsigma\in(l,R)$, respectively, with u_m=const and l being the unknown length which must be subject to variation. This assumption means that the flow in the middle of the channel can be regarded as the flow of ideal fluid. Then the half-dissipation becomes

$$\frac{D}{2}=\int_{-R}^{-l}(\tfrac{1}{2}u_1'^2+F(\varphi_1)u_1'-\frac{\alpha}{R^{2/5}}u_1'-\tfrac{1}{2}\varphi_1'^2)d\varsigma - \int_{-l}^{0}\tfrac{1}{2}\varphi_m'^2d\varsigma-\int_{-R}^{-l}\frac{1}{R}u_1(\varsigma)d\varsigma-\frac{u_m l}{R}. \tag{17}$$

The standard calculus of variation leads to the differential equations

$$-\frac{d}{d\varsigma}(u'+F(\varphi))-\frac{l}{R}=0, \qquad F'(\varphi)u'+\varphi''=0, \tag{18}$$

for $\varsigma\in(-R,-l)$ and

$$\varphi''=0, \tag{19}$$

for $\varsigma\in(-l,0)$. The boundary conditions read

$$\begin{aligned}&u(-R)=\varphi(-R)=0, \qquad \varphi(0)=0,\\ &u'(-l)=0, \qquad \varphi'(-l+0)=\varphi'(-l-0),\\ &F(\varphi(-l))-\frac{\alpha}{R^{2/5}}=\frac{l}{R}.\end{aligned} \tag{20}$$

Equation $(18)_1$ and the last boundary condition of (20) imply that

$$u'+F(\varphi)=-\frac{\varsigma}{R}+\frac{\alpha}{R^{2/5}}. \tag{21}$$

Expressing u' through $F(\varphi)$ and substituting into the second equation of (18) we obtain the following governing equation for φ in $(-R,-l)$

$$\varphi''-F'(\varphi)(\frac{\varsigma}{R}-\frac{\alpha}{R^{2/5}}+F(\varphi))=0. \tag{22}$$

This equation is subject to the boundary conditions

$$\varphi(-R)=0, \qquad F(\varphi(-l))=\frac{l}{R}+\frac{\alpha}{R^{2/5}}, \qquad \varphi'(-l)=-\frac{\varphi(-l)}{l}. \tag{23}$$

Since $F(\varphi)$ does not have the explicit analytical form, it is convenient to choose the wall coordinate $\xi(\varsigma)$ as unknown function in accordance to

$$\varphi(\varsigma)=\varphi_\infty(\xi(\varsigma)), \qquad F(\varphi)=f(\xi), \tag{24}$$

$$\varphi'=(1-f(\xi))\xi', \qquad \varphi''=(1-f(\xi))\xi''-f'(\xi)\xi'^2.$$

In terms of the wall coordinate the governing equation becomes

$$(1-f(\xi))\xi''-f'(\xi)\xi'^2-\frac{f'(\xi)}{1-f(\xi)}(\frac{\varsigma}{R}-\frac{\alpha}{R^{2/5}} + f(\xi))=0. \tag{25}$$

This equation is subject to the boundary conditions

$$\xi(-R)=0, \qquad \xi(-l)=f^{-1}(\frac{l}{R}+\frac{\alpha}{R^{2/5}}), \qquad \xi'(-l)=-\frac{\varphi_\infty(f^{-1}(l/R+\alpha/R^{2/5}))}{l(1-l/R-\alpha/R^{2/5})} \tag{26}$$

where f^{-1} is the inverse function of f. Equation (25) can also be directly derived from the variational principle (15).

The shooting method can be applied to find the solution of the two-point boundary-value problem (25), (26). Take for example R=1000. By using the shooting method one can show that l=860.042 leads to the solution satisfying the boundary conditions (26). The plot of phase curve (ξ,ξ') in the phase plane is shown in Fig.4. The plot of the velocity gradient as function of ς from $-R$ to $-l$ is shown in Fig.5. In the turbulent core region $(-l,0)$ the velocity gradient u' is equal to zero. This velocity gradient profile agrees with the direct numerical simulation of the Navier-Stokes equations for channel flow [10] except at the viscous sub-layer where $u' \sim 1+\alpha / R^{2/5}$. The difference becomes vanishingly small as $R \to \infty$.

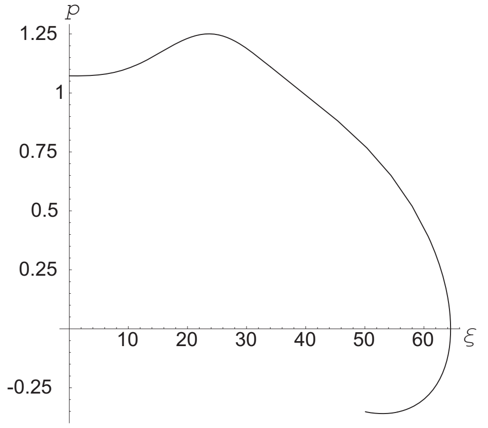

Fig.4: The phase curve in the phase plane for R =1000

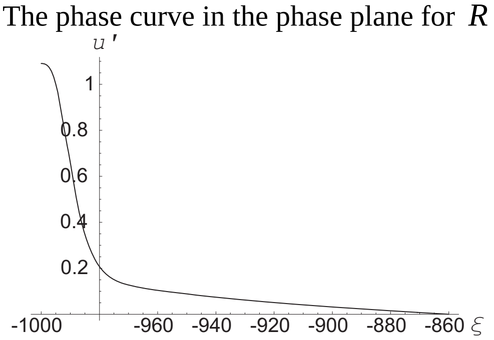

Fig.5: The velocity gradient as function of ς (R =1000)

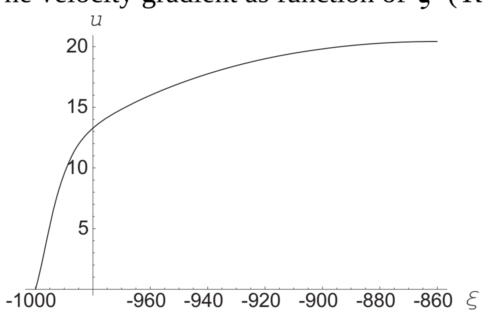

Fig.6: The velocity profile for R =1000

Knowing $u'(\varsigma)$ we can find the velocity profile $u(\varsigma)$ by integration. The result is shown in Fig.6 for R =1000. In the turbulent core region the velocity achieves its maximal value u_m which is equal to 20.4242. Then, the average velocity can also be found in accordance with

$$\bar{u} = \tfrac{1}{R}\int_{-R}^{0} u(\varsigma)d\varsigma = \tfrac{1}{R}\left(\int_{-R}^{-l} u(\varsigma)d\varsigma + u_m l\right).$$

For R =1000 the numerical calculation gives the value $\bar{u}$ =19.9336. Thus, the difference $\bar{u} - \ln R/\kappa$ =3.09 is close to the empirical value 3.3 (see [8]).

For the Reynolds number R =10000 the numerical calculations yield u_m =24.964 and $\bar{u}$ =24.7906. Thus, $\bar{u} = \ln R/\kappa + d$, where d =2.326.

4 Pipe flow

For the laminar pipe flow driven by the constant pressure gradient (3-D Poiseuille flow) the velocity $u(r)$ is the minimizer of the dissipation functional

$$D = \int_0^a \tfrac{1}{2}\eta u'^2 2\pi r dr + p'\int_0^a u(r)2\pi r dr$$

under the constraint

$$u(a) = 0.$$

Here the polar coordinate r is used, with a the radius of the circular cross-section, and p' is the pressure gradient which is constant over the cross-section. It is easy to show that the velocity profile is parabolic: $u(r) = -\frac{p'}{4\eta}(a^2 - r^2)$, which leads to the following resistance law

$$c_f = -\frac{p'a}{\frac{1}{2}\rho\bar{u}^2} = \frac{64}{R^2}$$

where $R = \rho u_p a/\eta$, $u_p = \sqrt{-p'a/2\rho}$.

As the pressure gradient becomes large, a flow induced by new vortices occurs. We let $u(r)$ denote as before the statistically average velocity of turbulent flow and $\varphi(r)$ the statistically average velocity of flow generated by new vortices. We formulate the following variational principle: among all admissible velocity fields $u(r)$ and $\varphi(r)$ satisfying the boundary conditions

$$u(a) = 0, \quad \varphi(a) = 0, \tag{27}$$

the pipe flow is described by those for which the dissipation functional

$$D = \int_0^a \left(\tfrac{1}{2}\eta u'^2 - \tfrac{1}{2}p'aF(\varphi)|u'| + \tfrac{1}{2}p'a\frac{\alpha}{R^{2/5}}|u'| - \tfrac{1}{2}\eta\varphi'^2\right)2\pi r dr + p'\int_0^a u(r)2\pi r dr \tag{28}$$

has an extremum, where, as before, α =1.442.

By changing the variables and unknown functions

$$\varsigma = \frac{Rr}{a}, \quad u \mapsto \frac{u}{u_p}, \quad \varphi \mapsto \frac{\varphi}{u_p}, \quad D \mapsto -\frac{DR}{\pi p' a^2 u_p}$$

we transform the dissipation functional to

$$D = \int_0^R \left(\tfrac{1}{2}u'^2 + F(\varphi)|u'| - \frac{\alpha}{R^{2/5}}|u'| - \tfrac{1}{2}\varphi'^2\right)\varsigma d\varsigma - \int_0^R \frac{2}{R}u(\varsigma)\varsigma d\varsigma. \tag{29}$$

The variational problem (29) always has the extremal $\varphi = 0$ leading to the laminar velocity profile. However, for sufficiently large R there is another extremal describing the turbulent flow. For the developed turbulent flow (large R) we assume that

$$u(\varsigma) = \begin{cases} u_m \\ u_1(\varsigma) \end{cases}, \quad \varphi(\varsigma) = \begin{cases} \varphi_m(\varsigma) \\ \varphi_1(\varsigma) \end{cases}$$

for $\varsigma \in (0,l)$ and $\varsigma \in (l,R)$, respectively, with u_m=const and l being the unknown length which must be subject to variation. Thus, the flow in the middle of the pipe can be regarded as the flow of ideal fluid. Then

$$D = -\int_0^l \tfrac{1}{2}\varphi_m'^2 \varsigma d\varsigma + \int_l^R (\tfrac{1}{2}u_1'^2 - F(\varphi_1)u_1' + \frac{\alpha}{R^{2/5}}u_1' - \tfrac{1}{2}\varphi_1'^2)\varsigma d\varsigma - \frac{u_m l^2}{R} - \int_l^R \frac{2}{R}u_1(\varsigma)\varsigma d\varsigma .$$

By the similar method as in the previous sections we derive, in terms of the wall coordinate, the governing equation

$$(1-f(\xi))\xi''\varsigma - f'(\xi)\xi'^2\varsigma + (1-f(\xi))\xi' - \frac{f'(\xi)}{1-f(\xi)}(f(\xi) - \frac{\varsigma}{R} - \frac{\alpha}{R^{2/5}})\varsigma = 0, \qquad (30)$$

and the boundary conditions

$$\xi(l) = f^{-1}(\frac{l}{R} + \frac{\alpha}{R^{2/5}}), \quad \xi'(l) = 0, \quad \xi(R) = 0 . \qquad (31)$$

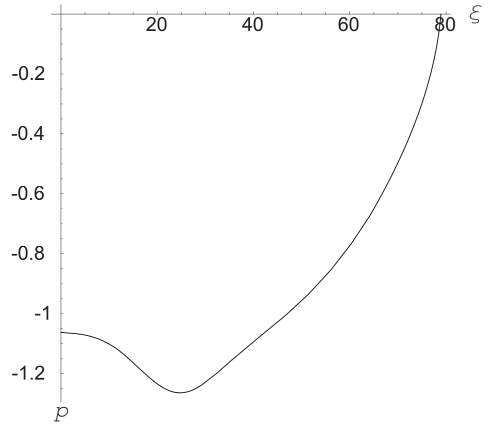

Fig.7: The phase curve in the phase plane for R=1000

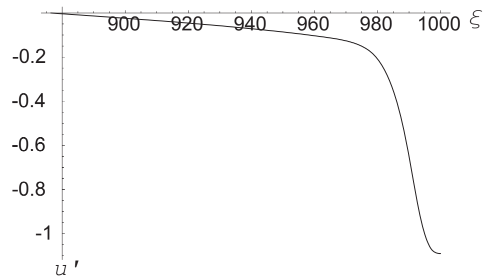

Fig.8: The velocity gradient (ς from l to R)

Consider for example R=1000. By using the shooting method one can show that l=876.416 leads to the solution of (30) satisfying the boundary conditions (31). The plot of phase curve $(\xi(\varsigma),\xi'(\varsigma))$ in the phase plane is shown in Fig.7. The plot of the velocity gradient $u' = f(\xi) - \varsigma/R - \alpha/R^{2/5}$ as function of ς from l to R is shown in Fig.8. Knowing $u'(\varsigma)$ we can find the velocity profile $u(\varsigma)$ by integration. The result is shown in Fig.9 for R=1000. In the turbulent core region the velocity achieves its maximal value u_m which is equal to 19.8935. Then, the average velocity can also be found. For R=1000 the numerical calculation gives the value $\bar{u}$=19.0489. Thus, the difference $\bar{u} - \ln R/\kappa$=2.2 is close to the empirical value 1.96 (see [8]).

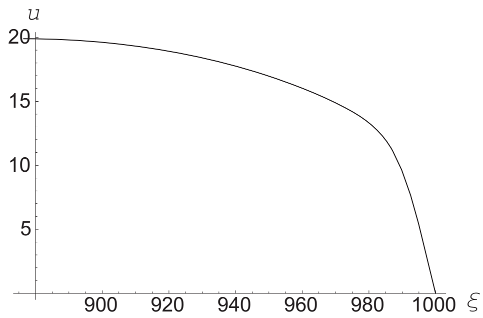

Fig.9: The velocity profile $u(\varsigma)$ for R=1000

For the Reynolds number R=10000 the numerical calculations yield u_m=24.7894 and $\bar{u}$=24.4576. Thus, $\bar{u} = \ln R/\kappa + d$, where d=1.9934 for R=10000. This is in excellent agreement with the empirical value 1.96.

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