Eulerian equilibria of a rigid body in the three body problem

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Abstract: We consider the non-canonical Hamiltonian dynamics of a rigid body in the three body problem. By means of geometric-mechanics methods we will study the approximate dynamics that arises when we develop the potential in series of Legendre and truncate this in an arbitrary order. Working in the reduced problem, we will study the existence of equilibria that we will denominate of Euler in analogy with classic results on the topic. In this way, we generalize the classical results on equilibria of the three-body problem and many of those obtained by other authors using more classic techniques for the case of rigid bodies.

Key–Words: Three Body Problem, Rigid body, Poisson dynamics, Equilibria, Eulerian

1 Introduction

In the study of configurations of relative equilibria by differential geometry methods or by more classical ones; we will mention here the papers of Wang et al. [7], about the problem of a rigid body in a central Newtonian field; Maciejewski [2], about the problem of two rigid bodies in mutual Newtonian attraction.

For the problem of three rigid bodies we would like to mention that Vidiakin [6] and Duboshin [1] proved the existence of Euler and Lagrange configurations of equilibria when the bodies possess symmetries; Zhuravlev [8] made a review of the results up to 1990.

In Vera [3] and a series of recent papers of Vera and Vigueras ([4],[5]) we study the non-canonical Hamiltonian dynamics of $n+1$ bodies in Newtonian attraction, where n of them are rigid bodies with spherical distribution of mass or material points and the other one is a triaxial rigid body.

In this paper, we take $n = 2$ and as a first approach to the qualitative study of this system, we will describe the approximate dynamics that arises in a natural way when we take the Legendre development of the potential function and truncate this until an first order. We will see global conditions on the existence of relative equilibria and in analogy with classic results on the topic, we will study the existence of relative equilibria that we will denominate of *Euler* in the case in which S_1, S_2 are spherical or punctual bodies and S_0 is a triaxial rigid body. We will obtain necessary and sufficient conditions for their existence and we will give explicit expressions of this relative equilibria, useful for the later study of the stability of the same ones. The analysis is done in vectorial form avoiding the use of canonical variables and the tedious expressions associated with them.

We should notice that the studied system, has potential interest both in astrodynamics (dealing with spacecrafts) as well as in the understanding of the evolution of planetary systems recently found (and more to appear), where some of the planets may be modeled like a rigid body rather than a rigid body. In fact, the equilibria reported might well be compared with the ones taken for the ‘parking areas’ of the space missions (GENESIS, SOHO, DARWIN, etc) around the Eulerian points of the Sun-Earth and the Earth-Moon systems.

To finish this introduction, we will describe the structure of the article. The paper is organized in five sections, two appendixes and the bibliography. In the five sections we study the equations of motion, Casimir functions and integrals of the system, the relative equilibria and the existence of Eulerian equilibria in an approximate dynamics, in particular the study of the bifurcations of Eulerian relative equilibria in an approximate dynamics of order zero and one.

2 Equations of motion

Following the line of Vera and Vigueras [5] let S_0 be a rigid body of mass m_0 and S_1, S_2 two spherical rigid bodies of masses m_1 and m_2. We use the following notation. For $\mathbf{u}, \mathbf{v} \in \mathbb{R}^3$, $\mathbf{u} \cdot \mathbf{v}$ is the dot product, $\mid \mathbf{u} \mid$ is the Euclidean norm of the vector $\mathbf{u}$ and $\mathbf{u} \times \mathbf{v}$ is the cross product. $\mathbf{I}_{\mathbb{R}^3}$ is the identity matrix and $\mathbf{0}$ is the zero matrix of order three. We consider $\mathbb{I} = diag(I_i, I_j, I_k)$, $I_i \neq I_j \neq I_k$, the diagonal tensor of inertia of the rigid body with

$$I_1 = A, B, C \qquad I_2 = A, B, C \qquad I_3 = A, B, C$$

being A, B and C the principal inertia moments of S_0.

The vector $\mathbf{z} = (\mathbf{\Pi},\ \boldsymbol{\lambda},\ \mathbf{p}_{\boldsymbol{\lambda}},\ \boldsymbol{\mu},\ \mathbf{p}_\mu) \in \mathbb{R}^{15}$ is a generic element of the twice reduced problem obtained using the symmetries of the system. We consider $\mathbf{\Omega}$ the angular velocity of S_0, $\mathbf{\Pi} = \mathbb{I}\mathbf{\Omega}$ the total rotational angular momentum vector of the rigid body in the body frame $\mathfrak{J}$, which is attached to its rigid part and whose axes have the direction of the principal axes of inertia of S_0. The elements $\boldsymbol{\lambda}$, $\boldsymbol{\mu}$, $\mathbf{p}_{\boldsymbol{\lambda}}$ and $\mathbf{p}_\mu$ are respectively the barycentric coordinates and the linear momenta expressed in the body frame $\mathfrak{J}$.

The twice reduced Hamiltonian of the system, obtained by the action of the group $\mathbf{SE}(3)$, has the following expression

$$\mathcal{H}(\mathbf{z}) = \frac{\mid \mathbf{p}_{\boldsymbol{\lambda}} \mid^2}{2g_1} + \frac{\mid \mathbf{p}_{\boldsymbol{\mu}} \mid^2}{2g_2} + \frac{1}{2}\mathbf{\Pi}\mathbb{I}^{-1}\mathbf{\Pi} + \mathcal{V} \qquad (1)$$

being

$$M_2 = m_1 + m_2, \qquad M_1 = m_1 + m_2 + m_0$$

$$g_1 = \frac{m_1 m_2}{M_2}, \qquad g_2 = \frac{m_0 M_2}{M_1}$$

with $\mathcal{V}$ the potential function of the system. The potential function is given by the formula

$$\mathcal{V}(\boldsymbol{\lambda}, \boldsymbol{\mu}) = -\frac{G m_1 m_2}{\mid \boldsymbol{\lambda} \mid} - \int\limits_{S_0} \frac{G m_1 dm(\mathbf{Q})}{\mid \mathbf{Q} + \boldsymbol{\mu} + \frac{m_2}{M_2}\boldsymbol{\lambda} \mid} - \int\limits_{S_0} \frac{G m_2 dm(\mathbf{Q})}{\mid \mathbf{Q} + \boldsymbol{\mu} - \frac{m_1}{M_2}\boldsymbol{\lambda} \mid} \qquad (2)$$

Let $\mathbf{M} = \mathbb{R}^{15}$, and we consider the Poisson manifold $(M, \{\ ,\ \}, \mathcal{H})$, with Poisson brackets $\{\ ,\ \}$ defined by means of the Poisson tensor

$$\mathbf{B}(\mathbf{z}) = \begin{pmatrix} \widehat{\mathbf{\Pi}} & \widehat{\boldsymbol{\lambda}} & \widehat{\mathbf{p}_{\boldsymbol{\lambda}}} & \widehat{\boldsymbol{\mu}} & \widehat{\mathbf{p}_\mu} \\ \widehat{\boldsymbol{\lambda}} & \mathbf{0} & \mathbf{I}_{\mathbb{R}^3} & \mathbf{0} & \mathbf{0} \\ \widehat{\mathbf{p}_{\boldsymbol{\lambda}}} & -\mathbf{I}_{\mathbb{R}^3} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \widehat{\boldsymbol{\mu}} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I}_{\mathbb{R}^3} \\ \widehat{\mathbf{p}_\mu} & \mathbf{0} & \mathbf{0} & -\mathbf{I}_{\mathbb{R}^3} & \mathbf{0} \end{pmatrix} \qquad (3)$$

In $\mathbf{B}(\mathbf{z})$, $\widehat{\mathbf{v}}$ is considered to be the image of the vector $\mathbf{v} \in \mathbb{R}^3$ by the standard isomorphism between the Lie Algebras $\mathbb{R}^3$ and $\mathfrak{so}(3)$, i.e.

$$\widehat{\mathbf{v}} = \begin{pmatrix} 0 & -v_3 & v_2 \\ v_3 & 0 & -v_1 \\ -v_2 & v_1 & 0 \end{pmatrix}$$

The equations of the motion is given by the following expression

$$\frac{d\mathbf{z}}{dt} = \{\mathbf{z}, \mathcal{H}(\mathbf{z})\}(\mathbf{z}) = \mathbf{B}(\mathbf{z})\boldsymbol{\nabla}_{\mathbf{z}}\mathcal{H}(\mathbf{z}) \qquad (4)$$

with $\boldsymbol{\nabla}_{\mathbf{u}}\mathcal{V}$ is the gradient of $\mathcal{V}$ with respect to an arbitrary vector $\mathbf{u}$.

Developing $\{\mathbf{z}, \mathcal{H}(\mathbf{z})\}$, we obtain the following group of vectorial equations of the motion

$$\begin{aligned} \frac{d\mathbf{\Pi}}{dt} &= \mathbf{\Pi} \times \mathbf{\Omega} + \boldsymbol{\lambda} \times \boldsymbol{\nabla}_{\boldsymbol{\lambda}}\mathcal{V} + \boldsymbol{\mu} \times \boldsymbol{\nabla}_{\boldsymbol{\mu}}\mathcal{V} \\ \frac{d\boldsymbol{\lambda}}{dt} &= \frac{\mathbf{p}_{\boldsymbol{\lambda}}}{g_1} + \boldsymbol{\lambda} \times \mathbf{\Omega}, \quad \frac{d\mathbf{p}_{\boldsymbol{\lambda}}}{dt} = \mathbf{p}_{\boldsymbol{\lambda}} \times \mathbf{\Omega} - \boldsymbol{\nabla}_{\boldsymbol{\lambda}}\mathcal{V} \\ \frac{d\boldsymbol{\mu}}{dt} &= \frac{\mathbf{p}_{\boldsymbol{\mu}}}{g_2} + \boldsymbol{\mu} \times \mathbf{\Omega}, \quad \frac{d\mathbf{p}_{\boldsymbol{\mu}}}{dt} = \mathbf{p}_{\boldsymbol{\mu}} \times \mathbf{\Omega} - \boldsymbol{\nabla}_{\boldsymbol{\mu}}\mathcal{V} \end{aligned} \qquad (5)$$

Important elements of $\mathbf{B}(\mathbf{z})$ are the associate Casimir functions. We consider the total angular momentum $\mathbf{L}$ given by

$$\mathbf{L} = \mathbf{\Pi} + \boldsymbol{\lambda} \times \mathbf{p}_{\boldsymbol{\lambda}} + \boldsymbol{\mu} \times \mathbf{p}_{\boldsymbol{\mu}} \qquad (6)$$

Then the following result is verified (see Vera and Vigueras [5] for details).

Proposition 1 *If φ is a real smooth function no constant, then $\varphi(\frac{\mid \mathbf{L} \mid^2}{2})$ is a Casimir function of the Poisson tensor $\mathbf{B}(\mathbf{z})$. Moreover $Ker\mathbf{B}(\mathbf{z}) =< \boldsymbol{\nabla}_{\mathbf{z}}\varphi >$. Also, we have $\frac{d\mathbf{L}}{dt} = \mathbf{0}$, that is to say the total angular momentum vector remains constant.*

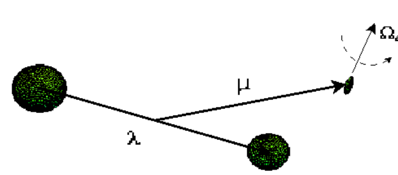

Figure 1: Gyrostat in the three body problem

2.1 Approximate Poisson dynamics

It is outstanding that the integrals of the potential $\mathcal{V}$, except for some geometries of the rigid body S_0, show important difficulties for the calculation. It arises in a natural way to consider the multipolar development of these potentials, supposing that the involved bodies are at much more mutual distances than the individual dimensions of the same ones. Under additional hypothesis we will be able to develop the potential in quickly convergent series. Considering the potentials truncated until an first order, then we will be able to study the approximated Poisson dynamics.

For a triaxial rigid body at great distance the following formula is verified with great accuracy

$$\mathcal{V} = \mathcal{V}_1 + \mathcal{V}_2 \tag{7}$$

where

$$\mathcal{V}_1 = -\left(\frac{Gm_1m_2}{|\,\boldsymbol{\lambda}\,|} + \frac{Gm_1m_0}{|\,\boldsymbol{\mu} - \frac{m_2}{M_2}\boldsymbol{\lambda}\,|} + \frac{Gm_2m_0}{|\,\boldsymbol{\mu} + \frac{m_1}{M_2}\boldsymbol{\lambda}\,|}\right)$$

$$\mathcal{V}_2 = -\frac{1}{2}\left(\frac{Gm_1\alpha}{|\,\boldsymbol{\mu} - \frac{m_2}{M_2}\boldsymbol{\lambda}\,|^3} + \frac{Gm_2\alpha}{|\,\boldsymbol{\mu} + \frac{m_1}{M_2}\boldsymbol{\lambda}\,|^3}\right) +$$

$$\frac{3}{2}\left(\frac{Gm_1\beta_1}{|\,\boldsymbol{\mu} - \frac{m_2}{M_2}\boldsymbol{\lambda}\,|^5} + \frac{Gm_2\beta_2}{|\,\boldsymbol{\mu} + \frac{m_1}{M_2}\boldsymbol{\lambda}\,|^5}\right)$$

and

$$\alpha = I_1 + I_2 + I_3$$

$$\beta_1(\boldsymbol{\lambda}, \boldsymbol{\mu}) = \boldsymbol{\mu} \cdot \mathbb{I}\boldsymbol{\mu} - \frac{2m_2}{M_2}\boldsymbol{\lambda} \cdot \mathbb{I}\boldsymbol{\mu} + \left(\frac{m_2}{M_2}\right)^2 \boldsymbol{\lambda} \cdot \mathbb{I}\boldsymbol{\lambda}$$

$$\beta_2(\boldsymbol{\lambda}, \boldsymbol{\mu}) = \boldsymbol{\mu} \cdot \mathbb{I}\boldsymbol{\mu} + \frac{2m_1}{M_2}\boldsymbol{\lambda} \cdot \mathbb{I}\boldsymbol{\mu} + \left(\frac{m_1}{M_2}\right)^2 \boldsymbol{\lambda} \cdot \mathbb{I}\boldsymbol{\lambda}$$

with I_1, I_2 and I_3 the principal moments of inertia of S_0 in the appropriate orientation of the body frame $\mathfrak{I}$.

Definition 2 *Let be* $\mathbf{M} = \mathbb{R}^{15}$ *and the Poisson manifold* $(M, \{\ ,\ \}, \mathcal{H}^0)$, *with brackets* $\{\ ,\ \}$ *defined by means of the Poisson tensor (3). We call* approximate dynamics of order zero *to the differential equations of motion given by the following expression*

$$\frac{d\mathbf{z}}{dt} = \{\mathbf{z}, \mathcal{H}^0(\mathbf{z})\}(\mathbf{z}) = \mathbf{B}(\mathbf{z})\boldsymbol{\nabla}_{\mathbf{z}}\mathcal{H}^0(\mathbf{z})$$

being

$$\mathcal{H}^0(\mathbf{z}) = \frac{|\,\mathbf{p}_{\boldsymbol{\lambda}}\,|^2}{2g_1} + \frac{|\,\mathbf{p}_{\boldsymbol{\mu}}\,|^2}{2g_2} + \frac{1}{2}\Pi\mathbb{I}^{-1}\Pi + \mathcal{V}_1(\boldsymbol{\lambda}, \boldsymbol{\mu})$$

Similarly the approximate dynamics of order one *is given by* $(M, \{\ ,\ \}, \mathcal{H}^1)$ *with* $\mathcal{H}^1 = \mathcal{H}^0 + \mathcal{V}_2$.

On the other hand, it is easy to verify that

$$\nabla_{\mathbf{z}}(|\,\boldsymbol{\Pi}\,|^2))\mathbf{B}(\mathbf{z})\nabla_{\mathbf{z}}\mathcal{H}^0(\mathbf{z}) = 0$$

and similarly when the rigid body is of revolution

$$\nabla_{\mathbf{z}}(\boldsymbol{\pi}_3)\mathbf{B}(\mathbf{z})\nabla_{\mathbf{z}}\mathcal{H}^0(\mathbf{z}) = 0$$

where $\boldsymbol{\pi}_3$ is the third component of the rotational angular momentum of the rigid body. It is verified the following property.

Theorem 1 *In the approximate dynamics of order zero,* $|\ \boldsymbol{\Pi}\ |^2$ *is an integral of motion and also when the gyrostat is of revolution* $\boldsymbol{\pi}_3$ *is another integral of motion.*

In what continues $\mathcal{H} = \mathcal{H}^1$.

2.2 Relative Equilibria

The relative equilibria are the equilibria of the twice reduced problem whose Hamiltonian function is obtained in Vera and Vigueras [5] for the case $n = 2$. If we denote by $\mathbf{z}_e = (\boldsymbol{\Pi}_e, \boldsymbol{\lambda}^e, \mathbf{p}^e_{\boldsymbol{\lambda}}, \boldsymbol{\mu}^e, \mathbf{p}^e_{\mu})$ a generic relative equilibrium of an approximate dynamics of order one, then this verifies the equations

$$\boldsymbol{\Pi}_e \times \boldsymbol{\Omega}_e + \boldsymbol{\lambda}^e \times (\boldsymbol{\nabla}_{\boldsymbol{\lambda}}\mathcal{V})_e + \boldsymbol{\mu}^e \times (\boldsymbol{\nabla}_{\boldsymbol{\mu}}\mathcal{V})_e = \mathbf{0}$$

$$\frac{\mathbf{p}^e_{\boldsymbol{\lambda}}}{g_1} + \boldsymbol{\lambda}^e \times \boldsymbol{\Omega}_e = \mathbf{0}, \quad \mathbf{p}^e_{\boldsymbol{\lambda}} \times \boldsymbol{\Omega}_e = (\boldsymbol{\nabla}_{\boldsymbol{\lambda}}\mathcal{V})_e$$

$$\frac{\mathbf{p}^e_{\boldsymbol{\mu}}}{g_2} + \boldsymbol{\mu}^e \times \boldsymbol{\Omega}_e = \mathbf{0}, \quad \mathbf{p}^e_{\boldsymbol{\mu}} \times \boldsymbol{\Omega}_e = (\boldsymbol{\nabla}_{\boldsymbol{\mu}}\mathcal{V})_e \tag{8}$$

Also by virtue of the relationships obtained in Vera and Vigueras [5], we have the following result.

Lemma 3 *If* $\mathbf{z}_e = (\boldsymbol{\Pi}_e, \boldsymbol{\lambda}^e, \mathbf{p}^e_{\boldsymbol{\lambda}}, \boldsymbol{\mu}^e, \mathbf{p}^e_{\mu})$ *is a relative equilibrium of an approximate dynamics of order one the following relationships are verified*

$$|\,\boldsymbol{\Omega}_e\,|^2\,|\,\boldsymbol{\lambda}^e\,|^2 - (\boldsymbol{\lambda}^e \cdot \boldsymbol{\Omega}_e)^2 = \frac{1}{g_1}(\boldsymbol{\lambda}^e \cdot (\boldsymbol{\nabla}_{\boldsymbol{\lambda}}\mathcal{V})_e)$$

$$|\,\boldsymbol{\Omega}_e\,|^2\,|\,\boldsymbol{\mu}^e\,|^2 - (\boldsymbol{\mu}^e \cdot \boldsymbol{\Omega}_e)^2 = \frac{1}{g_2}(\boldsymbol{\mu}^e \cdot (\boldsymbol{\nabla}_{\boldsymbol{\mu}}\mathcal{V})_e)$$

The last two previous identities will be used to obtain necessary conditions for the existence of relative equilibria in this approximate dynamics.

We will study certain relative equilibria in the approximate dynamics supposing that the vectors $\boldsymbol{\Omega}_e$, $\boldsymbol{\lambda}^e$, $\boldsymbol{\mu}^e$ satisfy special geometric properties.

Definition 4 *We say that* $\mathbf{z}_e$ *is a* Eulerian equilibrium *in an approximate dynamics of order one when,* $\boldsymbol{\lambda}^e$, $\boldsymbol{\mu}^e$ *are proportional and* $\boldsymbol{\Omega}_e$ *is perpendicular to the straight line that these generate.*

From the equations of motion, after some easy calculations with $\mathcal{V}$, the following property is deduced.

Proposition 5 *In a Eulerian equilibrium for any approximate dynamics, moments are not exercised on the rigid body. The vector* $\boldsymbol{\lambda}^e$ *is a eigenvector of the tensor of inertia* $\mathbb{I}$.

Next we obtain necessary and sufficient conditions for the existence of Eulerian relative equilibria.

3 Eulerian relative equilibria

According to the relative position of the rigid body S_0 with respect to S_1 and S_2 there are three possible equilibrium configurations: a) $S_0S_2S_1$, b) $S_2S_0S_1$ and c) $S_2S_1S_0$.

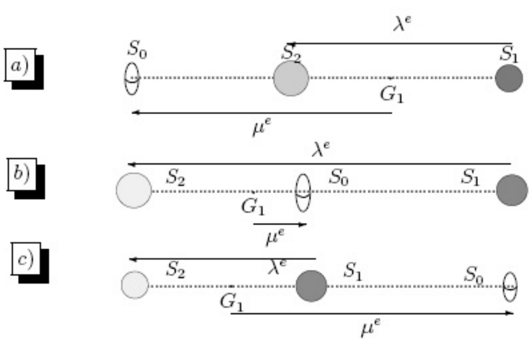

Figure 2: Eulerian configuration $S_2S_1S_0$

3.1 Necessary condition of existence

Lemma 6 *If* $\mathbf{z}_e = (\boldsymbol{\Pi}_e, \boldsymbol{\lambda}^e, \mathbf{p}^e_{\boldsymbol{\lambda}}, \boldsymbol{\mu}^e, \mathbf{p}^e_{\mu})$ *is a relative equilibrium of Euler type, then for the configuration* $S_0S_2S_1$ *we have*

$$\mid \boldsymbol{\mu}^e + \frac{m_1}{M_2}\boldsymbol{\lambda}^e \mid = \mid \boldsymbol{\lambda}^e \mid + \mid \boldsymbol{\mu}^e - \frac{m_2}{M_2}\boldsymbol{\lambda}^e \mid$$

In a similar way, for the configuration $S_2S_0S_1$ *we have*

$$\mid \boldsymbol{\lambda}^e \mid = \mid \boldsymbol{\mu}^e - \frac{m_1}{M_2}\boldsymbol{\lambda}^e \mid + \mid \boldsymbol{\mu}^e + \frac{m_2}{M_2}\boldsymbol{\lambda}^e \mid$$

Finally, for the configuration $S_2S_1S_0$ *we have*

$$\mid \boldsymbol{\mu}^e - \frac{m_2}{M_2}\boldsymbol{\lambda}^e \mid = \mid \boldsymbol{\mu}^e + \frac{m_1}{M_2}\boldsymbol{\lambda}^e \mid + \mid \boldsymbol{\lambda}^e \mid$$

Next we study necessary and sufficient conditions for the existence of relative equilibria of Euler type for the configuration $S_0S_2S_1$; the other configurations are studied in a similar way. If $\mathbf{z}_e$ is a relative equilibrium of Euler type, in the configuration $S_0S_2S_1$ in an approximate dynamics of order one, we have

$$g_1 \mid \boldsymbol{\Omega}_e \mid^2 \mid \boldsymbol{\lambda}^e \mid^2 = \boldsymbol{\lambda}^e \cdot (\boldsymbol{\nabla}_{\boldsymbol{\lambda}}\mathcal{V})_e$$

$$g_2 \mid \boldsymbol{\Omega}_e \mid^2 \mid \boldsymbol{\mu}^e \mid^2 = \boldsymbol{\mu}^e \cdot (\boldsymbol{\nabla}_{\boldsymbol{\mu}}\mathcal{V})_e$$

and

$$\boldsymbol{\mu}^e - \frac{m_1}{M_2}\boldsymbol{\lambda}^e = \rho\boldsymbol{\lambda}^e, \qquad \boldsymbol{\mu}^e + \frac{m_2}{M_2}\boldsymbol{\lambda}^e = (1+\rho)\boldsymbol{\lambda}^e,$$

$$\boldsymbol{\mu}^e = \frac{((1+\rho)m_1 + \rho m_2)}{M_2}\boldsymbol{\lambda}^e$$

where $\rho \in (0, +\infty)$ in the case a), $\rho \in (-1, 0)$ in the case b) and $\rho \in (-\infty, -1)$ in the case c). And it is possible to obtain the following expressions

$$(\boldsymbol{\nabla}_{\boldsymbol{\lambda}}\mathcal{V})_e = f_1(\rho)\boldsymbol{\lambda}^e, \qquad (\boldsymbol{\nabla}_{\boldsymbol{\mu}}\mathcal{V})_e = f_2(\rho)\boldsymbol{\lambda}^e$$

where

$$f_1(\rho) = \frac{Gm_1m_2}{\mid \boldsymbol{\lambda}^e \mid^3} + \frac{Gm_1m_2}{M_2}\left(\frac{m_0}{\mid \boldsymbol{\lambda}^e \mid^3}\left(\frac{1+\rho}{\mid 1+\rho \mid^3} - \frac{\rho}{\mid \rho \mid^3}\right) + \frac{\beta_1}{\mid \boldsymbol{\lambda}^e \mid^5}\left(\frac{1+\rho}{\mid 1+\rho \mid^5} - \frac{\rho}{\mid \rho \mid^5}\right)\right) \tag{9}$$

$$f_2(\rho) = \frac{Gm_0}{\mid \boldsymbol{\lambda}^e \mid^3}\left(\frac{m_1(1+\rho)}{\mid 1+\rho \mid^3} + \frac{m_2\rho}{\mid \rho \mid^3}\right) + \frac{G\beta_1}{\mid \boldsymbol{\lambda}^e \mid^5}\left(\frac{m_1(1+\rho)}{\mid 1+\rho \mid^5} + \frac{m_2\rho}{\mid \rho \mid^5}\right) \tag{10}$$

Remark 7 *The parameter* β_1 *takes the following values*

$$\beta_1 = \frac{3(A+B-2C)}{2}, \qquad \beta_1 = \frac{3(A+C-2B)}{2}$$

$$\beta_1 = \frac{3(B+C-2A)}{2}$$

according to the orientation of the body frame $\mathfrak{J}$.

Restricting us to the case $a)$ we have

$$f_1(\rho) = \frac{Gm_1m_2}{\mid \boldsymbol{\lambda}^e \mid^3} +$$

$$\frac{Gm_1m_2}{M_2}\left(\frac{m_0}{\mid \boldsymbol{\lambda}^e \mid^3}\left(\frac{1}{\mid 1+\rho \mid^2} - \frac{1}{\mid \rho \mid^2}\right) + \right. \tag{11}$$

$$\left.\frac{\beta_1}{\mid \boldsymbol{\lambda}^e \mid^5}\left(\frac{1}{\mid 1+\rho \mid^4} - \frac{1}{\mid \rho \mid^4}\right)\right)$$

$$f_2(\rho) = \frac{Gm_0}{\mid \boldsymbol{\lambda}^e \mid^3}\left(\frac{m_1}{\mid 1+\rho \mid^2} + \frac{m_2}{\mid \rho \mid^2}\right) +$$

$$\frac{G\beta_1}{\mid \boldsymbol{\lambda}^e \mid^5}\left(\frac{m_1}{\mid 1+\rho \mid^4} + \frac{m_2}{\mid \rho \mid^4}\right) \tag{12}$$

Now, from the identities

$$\boldsymbol{\lambda}^e \cdot (\boldsymbol{\nabla}_{\boldsymbol{\lambda}}\mathcal{V})_e = \mid \boldsymbol{\lambda}^e \mid^2 f_1(\rho)$$

$$\boldsymbol{\mu}^e \cdot (\boldsymbol{\nabla}_{\boldsymbol{\mu}}\mathcal{V})_e = \frac{((1+\rho)m_1 + \rho m_2)}{M_2} \mid \boldsymbol{\lambda}^e \mid^2 f_2(\rho)$$

we deduce the following equations

$$\mid \boldsymbol{\Omega}_e \mid^2 = \frac{(m_1+m_2)f_1(\rho)}{m_1m_2}$$

$$\mid \boldsymbol{\Omega}_e \mid^2 = \frac{(m_0+m_1+m_2)f_2(\rho)}{m_0\left((1+\rho)m_1+\rho m_2\right)}$$

Then for a relative equilibrium of Euler type ρ must be a positive real root of the following equation

$$\frac{m_0(m_1+m_2)\left((1+\rho)m_1+\rho m_2\right)f_2(\rho)}{m_1m_2(m_0+m_1+m_2)} = f_1(\rho) \tag{13}$$

We summarize all these results in the following proposition.

Proposition 8 *If $\mathbf{z}_e = (\boldsymbol{\Pi}_e, \boldsymbol{\lambda}^e, \mathbf{p}^e_{\boldsymbol{\lambda}}, \boldsymbol{\mu}^e, \mathbf{p}^e_{\mu})$ is an Eulerian relative equilibrium in the configuration $S_0S_2S_1$, the equation (13) has, at least, a positive real root; where the functions $f_1(\rho)$ and $f_2(\rho)$ are given by (11) and (12).*

The modulus of the angular velocity of the rigid body is

$$\mid \boldsymbol{\Omega}_e \mid^2 = \frac{Gm_1m_2}{\mid \boldsymbol{\lambda}_e \mid^3}h_1(\rho)$$

with

$$h_1(\rho) = 1 + \frac{m_0}{m_1+m_2}\left(\frac{1}{(1+\rho)^2} - \frac{1}{\rho^2}\right) +$$

$$\beta_1\left(\frac{1}{(1+\rho)^2} - \frac{1}{\rho^2}\right)$$

Remark 9 *If a solution of relative equilibrium of Euler type exists, in an approximate dynamics of order one, fixed $\mid \boldsymbol{\lambda}_e \mid$, the equation (13) has positive real solutions. The number of real roots of the equation (13) will depend, obviously, of the numerous parameters that exist in our system. Similar results would be obtained for the other two cases.*

3.2 Sufficient condition of existence

The following proposition indicates how to find solutions of the eq. (8).

Proposition 10 *Fixed $\mid \boldsymbol{\lambda}^e \mid$, let ρ be a solution of the equation (13) where the functions $f_1(\rho)$ and $f_2(\rho)$ are given for the case $a)$ with the relationships (11) and (12), then $\mathbf{z}_e = (\boldsymbol{\Pi}_e, \boldsymbol{\lambda}^e, \mathbf{p}^e_{\boldsymbol{\lambda}}, \boldsymbol{\mu}^e, \mathbf{p}^e_{\mu})$ given by*

$$\begin{array}{ll} \boldsymbol{\lambda}^e = (\lambda^e, 0, 0), & \boldsymbol{\mu}^e = (\mu^e, 0, 0), \\ \mathbf{p}^e_{\boldsymbol{\lambda}} = (0, \pm g_1\omega_e\lambda^e, 0), & \mathbf{p}^e_{\boldsymbol{\mu}} = (0, \pm g_2\omega_e\mu^e, 0), \\ \boldsymbol{\Omega}_e = (0, 0, \pm\omega_e) & \boldsymbol{\Pi}_e = (0, 0, \pm C\omega_e) \end{array} \tag{14}$$

or

$$\begin{array}{ll} \boldsymbol{\lambda}^e = (\lambda^e, 0, 0), & \boldsymbol{\mu}^e = (\mu^e, 0, 0), \\ \mathbf{p}^e_{\boldsymbol{\lambda}} = (0, 0, \mp g_1\omega_e\lambda^e), & \mathbf{p}^e_{\boldsymbol{\mu}} = (0, 0, \mp g_2\omega_e\mu^e), \\ \boldsymbol{\Omega}_e = (0, \pm\omega_e, 0) & \boldsymbol{\Pi}_e = (0, \pm C\omega_e, 0) \end{array} \tag{15}$$

where

$$\boldsymbol{\mu}^e = \frac{((1+\rho)m_1+\rho m_2)}{M_2}\boldsymbol{\lambda}^e, \qquad \omega_e^2 = \frac{Gm_1m_2}{\mid \boldsymbol{\lambda}_e \mid^3}h_1(\rho)$$

is a solution of relative equilibrium of Euler type, in an approximate dynamics of order one, in the configuration $S_0S_2S_1$. The total angular momentum of the system is given by

$$\mathbf{L} = (0, 0, \pm C\omega_e \pm g_1\omega_e\lambda^e \pm g_2\omega_e\mu^e)$$

or

$$\mathbf{L} = (0, \pm C\omega_e \mp g_1\omega_e\lambda^e \mp g_2\omega_e\mu^e, 0)$$

Let us see the existence and number of solutions for the approximate dynamics of order zero and one respectively. For superior order it is possible to use a similar technical.

4 Eulerian relative equilibria in an approximate dynamics of order zero and one

For the configuration $S_0S_2S_1$, in an approximate dynamics of order zero, we have

$$f_1(\rho) = \frac{Gm_1m_2}{\mid \boldsymbol{\lambda}_e \mid^3}\left(1 + \frac{m_0}{M_2}\left(\frac{1}{(1+\rho)^2} - \frac{1}{\rho^2}\right)\right)$$
$$f_2(\rho) = \frac{Gm_0}{\mid \boldsymbol{\lambda}_e \mid^3}\left(\frac{m_1}{(1+\rho)^2} + \frac{m_2}{\rho^2}\right)$$

The equation (13) is equivalent to the following polynomial equation

$$\begin{aligned} p_0(\rho) = (m_1+m_2)\rho^5 + (3m_1+2m_2)\rho^4 + \\ (3m_1+m_2)\rho^3 - (3m_0+m_2)\rho^2 - \\ (3m_0+2m_2)\rho - (m_0+m_2) = 0 \end{aligned} \quad (16)$$

This equation has an unique positive real solution. On the other hand, one has

$$\mid \boldsymbol{\Omega}_e \mid^2 = \frac{G(m_1+m_2)}{\mid \boldsymbol{\lambda}^e \mid^3} h_0(\rho)$$

with

$$h_0(\rho) = 1 + \frac{m_0}{m_1+m_2}\left(\frac{1}{(1+\rho)^2} - \frac{1}{\rho^2}\right)$$

being ρ the only one positive solution of the equation (16).

The following proposition gathers the results about relative equilibria of Euler type in an approximate dynamics of order zero in any of the cases previously mentioned $a)$, $b)$ or $c)$.

Proposition 11 *1. If ρ is the unique positive root of the equation (16) with*

$$\mid \boldsymbol{\Omega}_e \mid^2 = \frac{G(m_1+m_2)}{\mid \boldsymbol{\lambda}_e \mid^3} h_0(\rho)$$
$$h_0(\rho) = 1 + \frac{m_0}{m_1+m_2}\left(\frac{1}{(1+\rho)^2} - \frac{1}{\rho^2}\right)$$

then $\mathbf{z}_e = (\boldsymbol{\Pi}_e, \boldsymbol{\lambda}^e, \mathbf{p}^e_{\boldsymbol{\lambda}}, \boldsymbol{\mu}^e, \mathbf{p}^e_\mu)$, given by (14) or (15) is a relative equilibrium of Euler type in the configuration $S_0S_2S_1$.

2. If $\rho \in (-1, 0)$ is the unique root of the equation

$$\begin{aligned} p_0(\rho) = (m_1+m_2)\rho^5 + (3m_1+2m_2)\rho^4 + \\ (3m_1+m_2)\rho^3 + (3m_0+2m_1+m_2)\rho^2 + \\ (3m_0+2m_2)\rho + (m_0+m_2) = 0 \end{aligned}$$

with

$$\mid \boldsymbol{\Omega}_e \mid^2 = \frac{G(m_1+m_2)}{\mid \boldsymbol{\lambda}_e \mid^3} h_0(\rho)$$
$$h_0(\rho) = 1 + \frac{m_0}{m_1+m_2}\left(\frac{1}{\rho^2} - \frac{1}{(1+\rho)^2}\right)$$

then $\mathbf{z}_e = (\boldsymbol{\Pi}_e, \boldsymbol{\lambda}^e, \mathbf{p}^e_{\boldsymbol{\lambda}}, \boldsymbol{\mu}^e, \mathbf{p}^e_\mu)$, given by (14) or (15) is a relative equilibrium of Euler type in the configuration $S_2S_0S_1$.

3. If $\rho \in (-\infty, -1)$ is the unique root of the equation

$$\begin{aligned} p_0(\rho) = (m_1+m_2)\rho^5 + (3m_1+2m_2)\rho^4 + \\ (2m_0+3m_1+m_2)\rho^3 + (3m_0+m_2)\rho^2 + \\ (3m_0+2m_2)\rho + (m_0+m_2) = 0 \end{aligned}$$

with

$$\mid \boldsymbol{\Omega}_e \mid^2 = \frac{G(m_1+m_2)}{\mid \boldsymbol{\lambda}_e \mid^3} h_0(\rho)$$
$$h_0(\rho) = 1 + \frac{m_0}{m_1+m_2}\left(\frac{1}{\rho^2} + \frac{1}{(1+\rho)^2}\right)$$

then $\mathbf{z}_e = (\boldsymbol{\Pi}_e, \boldsymbol{\lambda}^e, \mathbf{p}^e_{\boldsymbol{\lambda}}, \boldsymbol{\mu}^e, \mathbf{p}^e_\mu)$, given by (14) or (15) is a relative equilibrium of Euler type in the configuration $S_2S_1S_0$.

Remark 12 *If $m_0 \to 0$ then $\mid \boldsymbol{\Omega}_e \mid^2 = \dfrac{G(m_1+m_2)}{\mid \boldsymbol{\lambda}_e \mid^3}$ and the equations that determine the Eulerian equilibria are the same ones of the Restricted Three Body Problem.*

4.1 Bifurcation of Eulerian relative equilibria in an approximate dynamics of order one

For the approximate dynamics of order one, after carrying out the appropriate calculations, the equation (13) corresponding to the configuration $S_0S_2S_1$, is reduced to the study of the positive real roots of the polynomial

$$p_1(\rho) = m_0 a^2(m_1+m_2)\rho^9 +$$
$$m_0a^2(5m_1+4m_2)\rho^8 + m_0a^2(10m_1+6m_2)\rho^7 +$$
$$3m_0a^2(3m_1+m_2-m_0)\rho^6 +$$
$$3m_0a^2(m_1-m_2-3m_0)\rho^5 -$$
$$(6m_0m_2a^2+10m_0^2a^2+\beta_1(m_1+m_2+5m_0))\rho^4 -$$
$$(4m_0m_2a^2+5m_0^2a^2+\beta_1(10m_0+4m_2))\rho^3 -$$
$$(m_0m_2a^2+m_0^2a^2+\beta_1(6m_2+10m_0))\rho^2 -$$
$$\beta_1(5m_0+4m_2)\rho - \beta_1(m_0+m_2) \qquad (17)$$

where $a = \mid \boldsymbol{\lambda}_e \mid$ and $\beta_1 = 3(A+B-2C)/2$, $3(A+C-2B)/2$ or $3(B+C-2A)/2$.

To study the positive real roots of this equation, after a detailed analysis of the same one, it can be expressed in the following way

$$\beta_1 = R_1(\rho) = \frac{m_0a^2\rho^2(\rho+1)^2 p_0(\rho)}{q_0(\rho)}$$

being p_0 the polynomial of grade five that determines the relative equilibria in the approximate dynamics of order zero, that is given by the formula (16), and the polynomial q_0 comes determined by the following expression

$$q_0(\rho) = (m_1+m_2+5m_0)\rho^4 + (4m_2+10m_0)\rho^3$$
$$+(6m_2+10m_0)\rho^2 + (4m_2+5m_0)\rho + (m_0+m_2)$$

The rational function $R_1(\rho)$, for any value of m_0, m_1, m_2, always presents a minimum ξ_1 located among 0 and ρ_0, being this last value the only one positive zero of the polynomial $p_0(\rho)$.

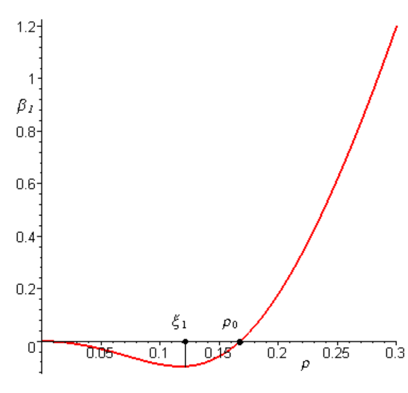

Figure 3: Function $R_1(\rho)$

By virtue of these statements the following result is obtained.

Proposition 13 *In the approximate dynamics of order one, if $\beta_1 < 0$, we have:*

1. *$\beta_1 < R_1(\xi_1)$, then relative equilibria of Euler type don't exist.*
2. *$\beta_1 = R_1(\xi_1)$, then there exists an unique relative equilibrium of Euler type.*
3. *$R_1(\xi_1) < \beta_1 < 0$, then two 1-parametric families of relative equilibria of Euler type exist.*

If $\beta_1 > 0$, then there exists an unique 1-parametric family of relative equilibria of Euler type.

Similarly for the configuration $S_2S_0S_1$ we obtain the following result.

Proposition 14 *In the approximate dynamics of order one, if $m_1 \neq m_2$ and $\beta_1 > 0$, then there exists an unique 1-parametric family of relative equilibria of Euler type; on the other hand, if $\beta_1 < 0$ we have:*

1. *$\beta_1 < R_1(\xi_1)$, then there exists an unique 1-parametric family of relative equilibria of Euler type.*
2. *$\beta_1 = R_1(\xi_1)$, then there exists an unique relative equilibrium of Euler type.*
3. *$R_1(\xi_1) < \beta_1 < 0$, then three 1-parametric families of relative equilibria of Euler type exist. If $m_1 = m_2$ and $\beta_1 > 0$, then relative equilibria of Euler type don't exist; but if $\beta_1 < 0$ we have:*
4. *$R_1(-1/2) < \beta_1 < 0$, then two 1-parametric families of relative equilibria of Euler type exist.*
5. *$\beta_1 = R_1(-1/2)$, then there exists an unique relative equilibrium of Euler type.*
6. *$\beta_1 < R_1(-1/2)$, then relative equilibria of Euler type don't exist.*

The results for the configuration $S_2S_1S_0$ are similar to the configuration $S_0S_2S_1$.

5 Conclusions and future works

The approximate dynamics of a rigid body in Newtonian interaction with two spherical or punctual rigid bodies is considered. For order zero approximate dynamics and one a complete study of Eulerian relative equilibria is made. Diverse results, which had been obtained by means of classic methods in previous works, have been obtained and generalized in a different way. And other results, not previously considered, have been studied. The bifurcations of the Eulerian relative equilibria is completely determined for an approximate dynamics of order one.

The methods employed in this work are susceptible of being used in similar problems. Numerous problems are open, and among them it is necessary to consider the study of the "inclined" relative equilibria, in which $\mathbf{\Omega}_e$ form an angle $\alpha \neq 0$ and $\pi/2$ with the vector $\boldsymbol{\lambda}^e$.

A The function $\mathbf{R}_1(\rho)$ in approximate dynamics of order one for the configuration $S_2S_0S_1$

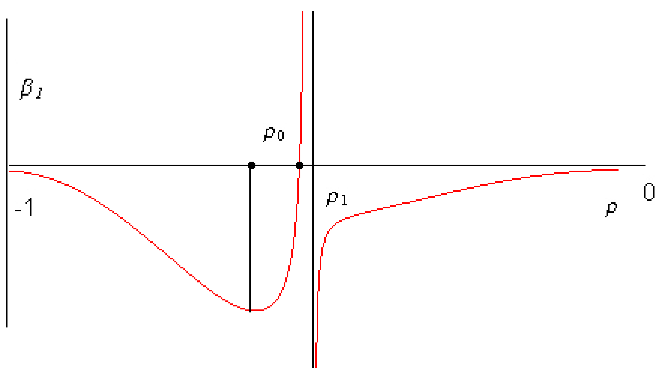

Figure 4: Function $R_1(\rho)$ for $m_1 \neq m_2$

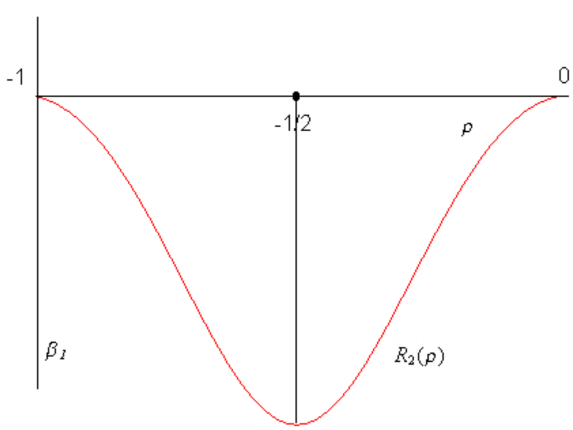

Figure 5: Function $R_1(\rho)$ for $m_1 = m_2$

Acknowledgements

This research was partially supported by the Spanish Ministerio de Ciencia y Tecnología (Project BFM2003-02137) and by the Consejería de Educación y Cultura de la Comunidad Autónoma de la Región de Murcia (Project Séneca 2002: PC-MC/3/00074/FS/02).

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