Dynamic Analysis of Coupled Fluid-Structure Systems

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Abstract: - This paper deals with developing a computational approach to vibration analysis of coupled fluid-structure systems i.e. liquid containers, a set of parallel /or radial plates used as hydraulic turbine /or turbo-reactor blades subjected to fluid forces. A hybrid fluid-solid element with taking into account simultaneously the membrane and bending effects of plate element is developed. The structural mass and stiffness matrices are determined using exact analytical integration of governing equations on basis of Sanders' shell theory and finite element approach. The Bernoulli equation and velocity potential function are used to describe the liquid pressure applied on the solid-fluid element. Impermeability condition assures a permanent contact at the fluid-structure interface. The applications of this model are presented for both parallel and radial plates as well as the fluid filled cylindrical and rectangular reservoirs. The effect of physical and geometrical parameters on the dynamic behavior of coupled fluid-structure system is investigated. The results are in satisfactorily agreement with those of experiments and other theories.

Keywords: -Fluid, Structure, Vibrations, Added mass

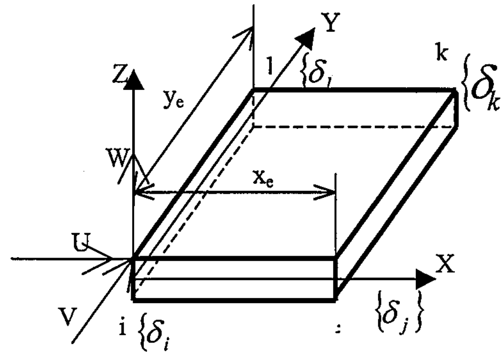
1 Introduction

Shell structures have wide applications in areas such as modern construction engineering, aerospace and aeronautical industries, aircraft construction, shipbuilding, rocket construction, and the components of nuclear power plants to name a few. In the most of industrial applications, these structures are in contact with fluid media. The forces generated by violent fluid-structure contacts can be very high; they are stochastic in nature and thus difficult to describe. They do, however, often constitute the design loading for the structure. Hydrodynamic pressures are generated by the vibrating structure, and these pressures will modify the structural deformation, which, in turn, will modify the hydrodynamic pressures that caused them. The problem is a tightly coupled elasto-dynamic problem in which the structure and the fluid form a single system. Solution of these problems is obviously complex and technically challenging. It is therefore very important that the static and dynamic behaviour of these structures when subjected to different loading conditions be clearly understood so that they may be safely used in these industrial applications. Vibration analysis of plates and shells has received considerable attention and has been the subject of numerous studies, many of which are well documented by Leissa [1 and 2]. The integrity of the mathematical model is a major factor in obtaining a satisfactory modal definition for the structure. If the model is of poor quality, mathematical rigor in the solution of the equations of motion will not improve results. The stiffness and mass distribution as well as the boundary conditions are basic parameters that should be given careful consideration in the synthesis of the mathematical model for structural dynamic analysis. Neglecting any of these parameters may result in a model that is not dynamically similar to the actual structure. It is therefore concluded that the accuracy of solutions reached by the finite element displacement formulation depends on the assumed functions used to model the deformation modes of structure. On the basis of aforementioned discussions, the need for accurate and efficient methods for static/and dynamic analysis of shell structures, which are in contact with fluid media, becomes apparent. To meet requirements related to accuracy of the solution, a new hybrid theory is developed on basis of classical finite element method and Sanders' shell theory [3]. The structural displacement functions are derived using the equations of motion. Then, mass and stiffness matrices required by the finite element method are determined by precise analytical integration. The velocity potential and Bernoulli's equation are adopted to express the fluid pressure acting on the structure. The product of the pressure expression and the developed structural shape function is integrated over the structure-fluid interface to assess the virtual added mass due to the fluid.

2 Structural Modelling

This theory is an extension of that expounded by Lakis and Paidoussis [4, 5]. It is a hybrid formulation, which has its basis in the finite element method, but with displacement functions over an element determined by the exact solution of the equations of static equilibrium of structure instead of the usual and more arbitrary interpolating polynomials. In doing so, the accuracy of the formulation will be less affected as the number of elements used is decreased and as the dynamic characteristics of the structure are required at higher modes, a significant advantage over polynomial interpolation.

The geometry of structure reference surface and the co-ordinate systems used for this analysis are shown in Figure 1. Each element consists of four node and twenty-four degrees of freedom that represent the in-plane and out-of-plane displacement components and their spatial derivatives. To develop the equilibrium equations of the rectangular plates, the Sanders' equations for cylindrical shells are used assuming the radius to be infinite, $\theta = y$ and $rd\theta = dy$.



$$\{\delta_i\} = \{U_i, \quad V_i, \quad W_i, \quad W_{i,x} \quad W_{i,y}, W_{i,xy}\}$$

Figure 1: A typical finite element

2.1 Equilibrium equations

The implicit form of the equilibrium equations of a rectangular plate as a function of the displacement components and structural material properties are given as:

$$L_i(U,V,W,P_{p,q}) = 0 \quad (i = 1,2,3 \;\; and \;\; p = q = 1\,to\,6) \quad (1)$$

Where L_i (i=1 to 3) are three linear differential operators whose forms are fully given in [6], and the components of elasticity matrix [P] are also given in [6].

2.2 Displacement field

The membrane and bending displacement components are presented in terms of bilinear polynomials and exponential function, respectively.

$$U(x,y,t) = C_1 + C_2\frac{x}{A} + C_3\frac{y}{B} + C_4\frac{xy}{AB}$$
$$V(x,y,t) = C_5 + C_6\frac{x}{A} + C_7\frac{y}{B} + C_8\frac{xy}{AB} \quad (2)$$
$$W(x,y,t) = \sum_{j=9}^{24} C_j e^{i\pi\left(\frac{x}{A}+\frac{y}{B}\right)} e^{i\omega t}$$

Where U, V and W are the in-plane and out-of-plane displacement components, and A, B are the plate dimensions in X and Y directions, respectively. The transversal displacement component of the reference surface can be expressed in terms of the Taylor's series as:

$$W(x,y,t)=C_9+C_{10}\frac{x}{A}+C_{11}\frac{y}{B}+C_{12}\frac{x^2}{2A^2}$$
$$+C_{13}\frac{xy}{AB}+C_{14}\frac{y^2}{2B^2}+C_{15}\frac{x^3}{6A^3}+C_{16}\frac{x^2y}{2A^2B} \quad (3)$$
$$+C_{17}\frac{xy^2}{2AB^2}+C_{18}\frac{y^3}{6B^3}+C_{19}\frac{x^3y}{6A^3B}$$
$$+C_{20}\frac{x^2y^2}{4A^2B^2}+C_{21}\frac{xy^3}{6AB^3}+C_{22}\frac{x^3y^2}{12A^3B^2}$$
$$+C_{23}\frac{x^2y^3}{12A^2B^3}+C_{24}\frac{x^3y^3}{36A^3B^3}$$

The displacement vector at each node is expressed as:

$$\{\delta\}=\{\{\delta_i\},\{\delta_j\},\{\delta_k\},\{\delta_l\}\}^T=[A]\{C\} \quad (4)$$

The {C} is the unknown constant vector. Multiplying the equation (4) by $[A]^{-1}$ and then substituting in equation (2) results in the following relations that express the displacement components as a function of the nodal displacement vector:

$$\begin{Bmatrix} U \\ V \\ W \end{Bmatrix}=[R]\{C\}=[R][A]^{-1}\{\delta\}=[N]\{\delta\} \quad (5)$$

Where, [N] expresses the displacement functions, the inverse of matrix [A] and matrix [*R*] are given in [6]. The unknown constants can be defined as a function of twenty-four degrees of freedom of chosen element.

2.3 Deformation vector

Introducing the displacement components into the deformation-displacement relationships given in [6], yields to the following equation that describes the deformation vector as a function of nodal displacements.

$$\{\varepsilon\}=[Q][A]^{-1}\{\delta\}^e \quad (6)$$

For an isotropic material, the stress resultants may be expressed as follows:

$$\{\sigma\}=[P]\{\varepsilon\} \quad (7)$$

Using the procedure of the classical finite element, one obtains the mass and stiffness matrices of a typical element in its local co-ordinates:

$$[k]^e=[[A]^{-1}]^T\left(\int_0^{y_e}\int_0^{x_e}[Q]^T[P][Q]dxdy\right) \quad (8)$$

$$[m]^e=\rho_s h[[A]^{-1}]^T\left(\int_0^{y_e}\int_0^{x_e}[R]^T[R]dxdy\right)[A]^{-1}$$

Where ρ_s is the structural density, x_e and y_e are the element dimensions in X, and Y directions, respectively (see Figure 1). These integrals are analytically calculated to determine the mass and stiffness matrices of each element.

3 Fluid Modelling

Linear potential flow is applied to describe the fluid effect that leads to the fluid dynamic forces. The mathematical model is based on the following assumptions: *i*) the fluid flow is potential, *ii*) the fluid is assumed to be non-viscous, incompressible and irrotational. Based on the aforementioned hypothesis the potential function, which satisfies the Laplace equation, is expressed in the Cartesian coordinate system as:

$$\nabla^2\phi=\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}+\frac{\partial^2\phi}{\partial z^2}=0 \quad (9)$$

The Bernouilli equation for the case of stationary fluid when the fluid velocity is null, is expressed as:

$$\left.\frac{\partial\phi}{\partial t}+\frac{P}{\rho_f}\right|_{z=0}=0 \quad (10)$$

Equation (10) results in the following expression for pressure:

$$P|_{z=0}=-\rho_f\left.\frac{\partial\phi}{\partial t}\right|_{z=0} \quad (11)$$

The following separate variable relation is assumed for the velocity function:

$$\phi(x,y,z,t)=F(z)S(x,y,t) \quad (12)$$

Where F(z) and S (x, y, t) are two separate functions to be determined. The impermeability condition of the structure surface requires that the out-of-plane velocity component of the fluid on the plate surface should match the instantaneous rate of change of the plate displacement in the transversal direction. This condition implies a permanent contact between the plate surface and the peripheral fluid layer, which should be:

$$\left.\frac{\partial\phi}{\partial z}\right|_{z=0}=\frac{\partial W}{\partial t} \quad (13)$$

The following expression may be defined by introducing equation (13) into equation (12)

$$S(x,y,t) = \frac{1}{dF(0)/dz}\frac{\partial W}{\partial t} \qquad (14)$$

Then, substituting equation (14) into relation (12) results in the following expression for the potential function:

$$\phi(x,y,z,t) = \frac{F(z)}{dF(0)/dz}\frac{\partial W}{\partial t} \qquad (15)$$

Fluid boundary conditions are introduced separately for each element, which allows us to study partially or totally submerged plates, i.e. vertical plates or inclined plates as well as floating plates. Substituting equation (15) into relation (9) leads to the following differential equation of second order:

$$\frac{d^2F(z)}{dz^2} - \mu^2 F(z) = 0 \qquad (16)$$

Where;

$$\mu = \pi\sqrt{\frac{1}{A^2} + \frac{1}{B^2}} \qquad (17)$$

The general solution of equation (16) is given as:

$$\mathrm{F(z)} = \mathrm{A_1 e^{\mu z} + A_2 e^{-\mu z}} \qquad (18)$$

Substituting equation (18) into (15), one gets the following expression for the potential function:

$$\phi(x,y,z,t) = \frac{\left(\mathrm{A_1 e^{\mu z} + A_2 e^{-\mu z}}\right)}{dF(0)/dZ}\frac{\partial W}{\partial t} \qquad (19)$$

The potential function 'ϕ' must be verified for given boundary conditions at the fluid-structure interface and the fluid extremity surfaces as well. This function is developed for various cases i.e. submerged and floating structure; structure-fluid model with null-pressure applied at the free surface and are given in [6].

3.1 Calculation of Fluid-Induced Force

Fluid forces are replaced by the added mass, representing inertia force, when the flow velocity is null. The following relation expresses the fluid-induced force vector:

$$\{\mathrm{F}\}^e = \int_A [\mathrm{N}]^T \{\mathrm{P}\} d\mathrm{A} \qquad (20)$$

Where [N] is the shape function matrix of the finite element defined in equation (5) and {P} is a vector expressing the pressure applied by the fluid on the plate, which are given in [6]. Substituting the transversal displacement (given in equation (5)) into the appropriate pressure expressions, which depend on the fluid-structure contact model as well as the boundary conditions, one obtains the fluid pressure that is applied on the plate. The dynamic pressure is then defined as:

$$\{P\} = Z_{fi}[R_f][A]^{-1}\{\ddot{\delta}\} \qquad (21)$$

The coefficients Z_{fi} (i= 1 to 6), which depend on the fluid-structure contact model, are given in [6]. The matrix $[R_f]$ and ($[A]^{-1}$) terms are also given in [6]. Introducing equation (21), and displacement functions (5) into relation (20) leads to the vector of fluid forces as:

$$\{F\}^e = Z_{fi}\int_A \left[[A]^{-1}\right]^T [R]^T [R_f][A]^{-1} dA\{\ddot{\delta}\} \qquad (22)$$

where dA is the fluid-structure interface area.

4 Dynamic Behaviour of Fluid-Structure Interaction

The dynamic responses of a plate are affected by the presence of a fluid media. Generally, the fluid pressure acting upon the structure is expressed as a function of displacement and its derivatives i.e. velocity and acceleration. These three terms represent, respectively, the stiffness, Coriolis and inertia effects of the fluid forces. The fluid force matrices are superimposed onto the structural matrices to form the dynamic equations of a coupled fluid-structure system. The global equations for motion of a structure interacting with a fluid can be represented as follows:

$$([M_s]-[M_f])\{\ddot{\delta}_T\}-[C_f]\{\dot{\delta}_T\}+([K_s]-[K_f])\{\delta_T\}=\{F\} \qquad (23)$$

Where subscripts '*s*' and '*f*' refer to the shell in *vacuo* and fluid filled respectively. *$[M_s]$* and *$[K_s]$* are the global mass and stiffness matrices of the plate in *vacuo*. They have been previously defined by equation (8) for an element. *$[M_f]$, $[C_f]$ and $[K_f]$*, represents respectively the inertial, Coriolis and centrifugal force of the fluid; $\{\delta_T\}$ is the displacement vector, and $\{F\}$ represents the external forces. It is noted that in case of stationary fluid ($V_f = 0$), the terms C_f and K_f would be null. After applying the boundary conditions, these matrices are reduced to square matrices of order 6*(N)-NC, where N and NC are the number of elements and the restrictions imposed.

5 Results and Discussions

The first examples examines the dynamic responses of a fluid-filled rectangular reservoir whose dimensions are given in Figure 2 and the plate thickness is h = 2mm. The mechanical properties used in this example are:

$E = 69\text{GPa}$, $\rho_s = 2700\ \text{kg/m}^3$, $\nu = 0.3$, $\rho_f = 1000\ \text{kg/m}^3$

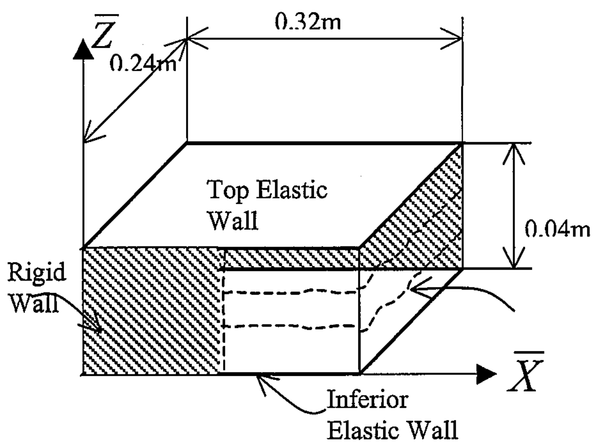


Figure 2: A fluid-filled rectangular reservoir

The results are presented in Table 1 that shows a good agreement between the present theory and that of Jeong et al. [7]. The natural frequencies of the two parallel plates coupled with an incompressible fluid and vibrating in out-of-phase mode (0,1) are listed in Table 2. Only the frequency corresponding to the mode (0,1) is presented since it reserves the mass conservation that was also met by Jeong et al. [7] in the analysis of two circular plates coupled with fluid.

The assembled parallel plates are often used in different sector of industry i.e. tubes support plates in steam generator.

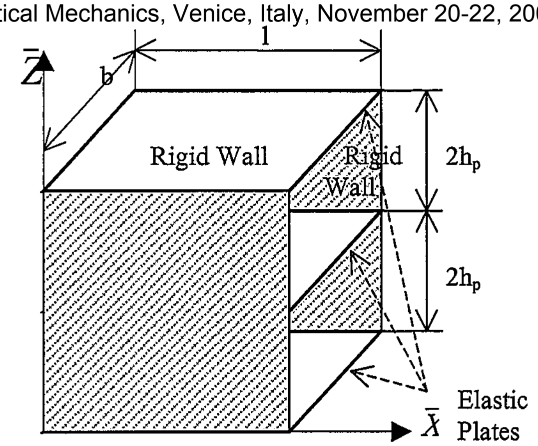


Figure 3: A set of parallel plates

Figure 3 shows a part of a structural system composed of a set of thin plates having each two parallel sides fixed to the lateral rigid walls. All plates have the same properties and they are distributed uniformly. Guo and Paidoussis [8] studied an identical system submitted to a flowing fluid in a channel formed by rectangular plates. Here, it is assumed that the fluid velocity is null and only the inertia force due to the fluid is taken into account. One arranges a dimensionless frequency parameter defined by the following expression:

$$\omega = \frac{1}{b^2}\sqrt{\frac{K}{\rho_s h_p}}\,\overline{\omega}$$, ω: Vibration frequency in rad/sec.

The parameter 'hp', 'b', and 'l' are represented in Figure 3. The parameters η, ζ, and ψ are dimensionless parameters defined by the following relations:

$$\eta = \rho_s b / \rho_s h = 1; \qquad \zeta = l/b = 0.5;$$

$$\psi = h_p / b = 0.05$$

Table 3 lists the natural frequencies of fluid-structure model along with corresponding results of Reference [8]. In out-of-phase mode, the applied pressure on the middle plate is the sum of the calculated pressure acting on the top and bottom plates. The applied pressure on the top and bottom plates is the same. It is for this reason that the analysis of a set of plates comes back to study only one plate subject to the calculated pressure at out-of-phase vibration mode of side walls.

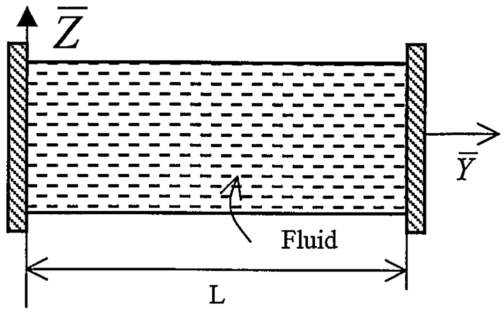


Figure 4: A cylindrical reservoir

Kim and Lee [9] studied the hydro-elastic behaviour of an open rectangular reservoir, (0.5m × 0.7m × 0.4 m), completely filled with water. The material properties are:
$E = 200$ GPa, $\nu = 0.3$, $\rho_s = 7970$ kg/m^3, $h =$ 5mm, $\rho_f = 1000$ kg/m^3
Referring to Kim and Lee's results (Table 4), it is noted that the frequency for in-phase modes is more precise than the frequency of out-of-phase mode. One says that our solution is more exact since when the ratio between the height of the fluid and reservoir dimensions passes at certain value, there is no difference between frequencies of in-phase and out-of-phase modes.

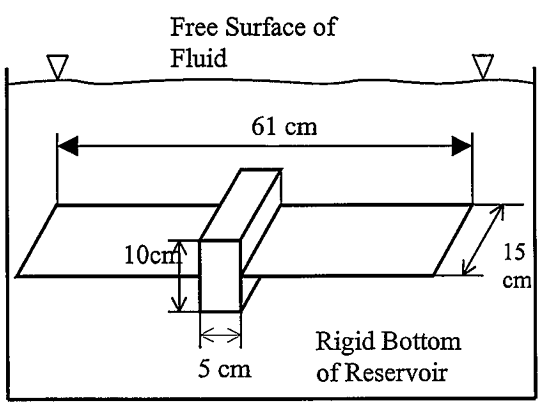


Figure 5: A submerged bird-structure

The dynamic behaviour of a cylindrical shell fixed at its two ends (Figure 4) is investigated in this example. The effect of geometrical parameter on the natural frequency of this structure is also verified. The results are listed in Table 5 along with those of [10] that shows a good agreement between two models.

$E = 206$ GPa, $\nu = 0.3$, $\rho_s = 7680$ kg/m^3, $\rho_f = 1000$ kg/m^3
The dynamic characteristics of a simplified two-wing structure shown in Figure 5 are calculated in this example. The structure (bird) is composed of two plates in aluminium welded to a solid box made of same material whose mechanical properties are:
$E = 72$ GPa, $\nu = 0.3$, $\rho_s = 2720$ kg/m^3, $h =$ 3.2mm
This structure made the object of an experimental work [10] in which the tests firstly were performed in air and with the free-free conditions, a suspension system was conceived to assure these limit conditions. This structure is also studied to the laboratory of the IREQ (Institute of research of hydro Quebec). An experiment is carried out for this structure submerged in an open rectangular reservoir with free surface on top and a rigid wall at the base of vessel. The fluid height was 37.5 cm. The numerical and experimental results are listed in Table 6. It is important to note that at out of this fluid height the structural frequencies do not change when one increases the level of fluid on the plate [10]. We calculated the vibration frequencies of the same structure completely submerged in water while satisfying the same boundary conditions imposed during the experimental tests. The applied water pressure on the walls is the sum of the pressure applied by the fluid on the upper and lower surfaces of structure. The results are presented in Table 6 that shows they are in good agreement with the experimental results.

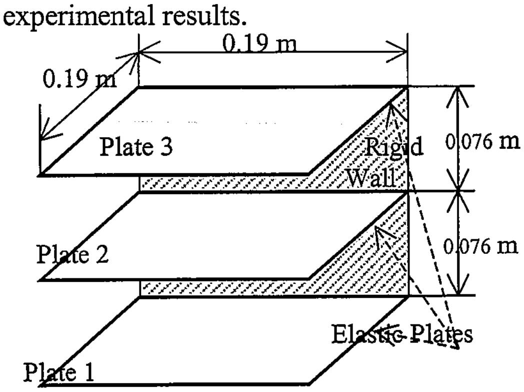


Figure 6: A set of parallel plates fixed at one side

There are some complex structures composed of a set of identical parallel /or radial plates that are in interaction with the fluid i.e. turbine blades. If the height of fluid between plates is low, the fluid transports the kinetic energy from one plate to another. Therefore, one studies the case of three parallel plates fixed to a rigid wall as shown in Figure 6. The material properties are given as:
$E = 69$ GPa, $\nu = 0.3$, $\rho_s = 2700$ kg/m^3, $h = 2$mm
When the system is submerged into a big reservoir, every plate undergoes a different pressure at its two sides caused by the fluid being, respectively, on top and bottom of the plate. In addition, the plates vibrate between them according to in-phase /or out-of-phase modes. Among several possible combinations, one distinguishes three modes of vibration of the system. The remainder cases are only repetitions of one of the three modes. Table 7 presents the vibrational frequencies according to the three distinct modes of plate. According to results, we note that the dynamic behaviour of this system can be studied while considering only one plate that vibrates in out-of-phase mode in relation to the two others since this mode provides the lowest frequency.

The dynamic behaviour of a system composed of several radial plates having one side of each plate welded to a rigid axis as shown in Figure 7. The angle between every two plates is 45 degrees. All plates have the same geometrical dimensions and mechanical properties.

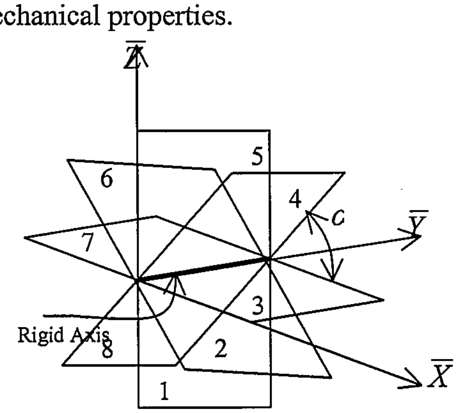


Figure 7: A set of radial plates

$L = 0.655$m, $W = 0.2016$m, $h = 9.36$mm, $E=207$GPa, $\rho_s = 7850$kg/m^3 and $\nu = 0.3$

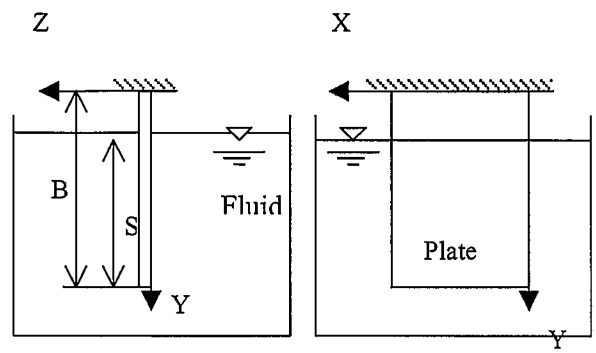


Figure 8: A vertical cantilever plate partially submerged in fluid

In this structural model, the fluid level on top and bottom of plates vary from one to another. Considering the axial symmetry of the system and the uniform distribution of plates in the circumferential direction indicates that the dynamic analysis of such system comes back to study only one plate that vibrates according to three different modes in relations with neighbouring plates. Natural frequencies of this structure without /and with fluid (when totally submerged in a water reservoir) effects according to the three distinct modes are enumerated in Table 8. It is important to note that the dynamic analysis of a set of parallel /or radial plates can be reduced to the dynamic analysis of only one plate when all plates have the same dimensions and the same mechanical properties. In addition, the fluid height has to be even between every two plates and the axis or the wall that attaches all plates together be rigid.

The dynamic behaviour of a vertical plate submerged in fluid reservoir is investigated, Figure 8. Natural frequencies are calculated for a plate vibrating in both air and water and different boundary conditions. In each case, the vibration analysis is carried out for four level of fluid (25%, 50%, 75%, and 100%). The fluid pressure applied on the submerged part of plate is equal to twice of the pressure. To validate this model, the results are compared with those of

experiment [11]. The parameters used for this case are:
$E = 206$ GPa, $\nu = 0.3$, $\rho_s = 7830$ kg/m^3, $\rho_f = 1000$ kg/m^3, $A = 0.2032$m, $B = 1.016$m and $h = 4$mm

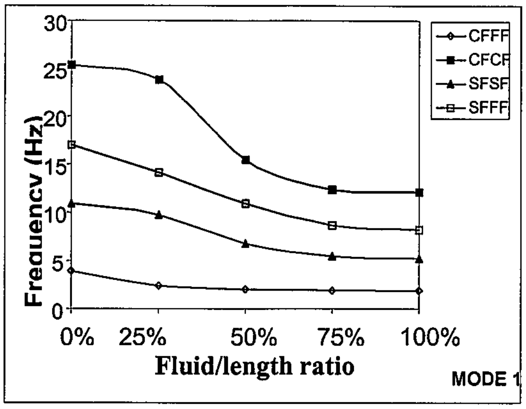


Figure 9: Frequencies of a cantilever plate as a function of fluid height

The obtained frequencies are listed in Table 9. It is noted that results are in good agreement with those of Lindholm [11]. It is underlined that the natural frequency of a cantilever submerged plate significantly decreases when the submerged part is less than of plate's half-length.

To investigate the boundary condition effects on the dynamic behaviour of the vertical submerged plates, the same plate as forgoing example is considered by changing the boundary conditions and calculated results are presented in Figure 9 at the first mode. The plate is supported on two sides (SFSF and CFCF) and undergoes to a significant change in natural frequencies of the first mode when the level of immersion is situated between 25% and 75%. Below 25% the influence of fluid is insignificant. On the other hand for plates supported on only at one side (CFFF and SFFF), the fluid presence has a very important influence on frequencies even at a level of immersion below 25%.

6 Conclusions

A hybrid element is developed to dynamic analysis of a coupled fluid-structure system i.e. 'n' parallel /or radial plates, cylindrical and rectangular reservoirs. The structure may be empty, partially or completely filled with fluid or submerged in a liquid. The structural mass and stiffness matrices are determined by exact analytical integration. The fluid pressure applied on structure is expressed according to the acceleration of the transverse displacement of the structure and the density of the fluid. The major importance in this work is to verify the applicability of this element to represent the hydro-elastic behaviour of different structures. The results are compared with those of other theory and experiments and commercial finite element computer codes i.e. ANSYS. The satisfactory agreement is found. This element can be applied to vibration analysis of non-uniform structures supported by any combinations of various boundary conditions without changing the displacement field.

References:

[1] A. W. Leissa, Vibration of plates, *NASA*, SP-160, 1969.
[2] A. W. Leissa, Vibration of shells, *NASA,* SP-288, 1973.
[3] J. L. Sanders, An improved first approximation theory for thin shel, *NASA TR-24*, 1959.
[4] A.A. Lakis, M.P. Paidoussis, Free vibration of cylindrical shells partially filled with liquid, *Journal of Sound and Vibration*, V19, N1, 1971, pp. 1-15.
[5] A.A. Lakis and M.P. Paidoussis, Dynamic analysis of axially non-uniform thin cylindrical shells, *Journal of Mechanical Engineering Science*, V14, N1, 1972, pp. 49-71.
[6] Y. Kerboua, A.A.Lakis, M. Thomas and L. Marcouiller, Comportement dynamique des plaques rectangulaires submergées, *Ecole Polytechnique of Montreal*, EPM-RT-2005-05.
[7] Kyeong_Hoon Jeong; Gye-Hyoung Yoo and Seong-Cheol Lee, Hydroelastic vibration of two identical rectangular plates, *Journal of Sound and Vibration*, V272, N3-5, 2004, pp.539-555.

[8] C. Q. Guo and M. P. Paidoussis, Analysis of hydroelastic instabilities of rectangular parallel-plate assemblies, *Transactions of the ASME, Journal of Pressure Vessel Technology*, V122, 2000, pp502-508.
[9] M.C. Kim and S.S. Lee, Hydroelastic analysis of rectangular tank, *The aerospace corporation El Segundo*, California 90245, 1997.
[10] K. Abassi and M. Thomas, Analyse modale expérimentale d'une structure vibrant dans l'eau, *Ecole de technologie supérieure, Montreal*, Canada, 2005.
[11] U. S. Lindholm, D. D. Kana, H. N. Abramson. Breathing vibrations of a circular cylindrical shell with an internal liquid, *Journal of the Aerospace Sciences* N 29, 1962, pp1052-1059.

Table 1: In-phase vibrational frequencies (Hz) of a fluid-filled reservoir

Mode # (n, m)	Jeong et al. [7]	Present Theory
(0,0)	113.3	112.9
(0,1)	192.5	188.7
(1,0)	272.4	265.6
(0,2)	326.5	314.0
(1,1)	348.4	332.9
(1,2)	479.6	447.7
(0,3)	516.1	485.6

Table 2: Out-of-phase vibrational frequencies (Hz) of a fluid-filled reservoir

Mode number (n, m)	Jeong et al. [7]	Present Theory
(0,1)	61.2	66.5

Table 3: Non-dimensional out-of-phase vibration frequencies

Mode number	Present Theory	C.Q. Guo and M. Paidoussis [8]
1	16.3	16.6
2	26.5	32.5
3	45.0	48.5

Table 4: Vibrational frequencies (Hz) of a fluid-filled reservoir

Theory	In-Phase	Out-of-Phase
Kim et Lee [9]	49.8	42.2
Present Theory	50.7	49.9

Table 5: Vibration frequencies of a fluid-filled clamped-clamped cylindrical shell

Mode #	Case 1		Case 2	
	PT	[11]	PT	[11]
1	137.9	138.6	135.0	135.9
2	146.2	152.8	149.9	138.8
3	165.9	158.3	154.6	166.9
4	178.0	201.9	195.6	177.7
5	224.4	205.3	212.9	223.3
6	234.3	256.9	228.3	245.7

Case 1: L=0.664m, R=0.175m, h=1mm
Case 2: L=0.9m, R=0.25m, h=2mm
PT: Present Theory

Table 6: Vibrational frequencies (Hz) a simplified wing model (bird structure)

Mode #	In Air		In Fluid	
	[10]	PT	[10]	PT
1	38.0	38.0	16.9	18.3
2	106.0	118.9	42.0	48.2
3	137.0	141.3	N. A.	52.9
4	N. A.	157.3	79.0	82.1
5	218.0	224.3	95.0	98.1
6	317.0	349.8	151.0	159.5
7	N. A.	463.9	N. A.	164.3
8	393.0	471.9	206.0	207.8
9	N. A.	618.1	265.0	248.0

N. A. Not Avaialable
PT : Present Theory

Table 7: Vibrational frequencies (Hz) of a set of three plates

Mode #	Case 1	Case 2	Case 3
1	12.3	9.7	10.5
2	12.3	10.5	11.1
3	13.4	15.7	12.3
4	30.1	23.8	25.7
5	30.1	25.7	27.3
6	32.8	38.5	30.1
7	75.9	60.0	64.6
8	75.9	64.6	68.7
9	82.7	75.7	75.9
10	95.7	81.5	81.5

Case 1: Three submerged plates vibrating in-phase
Case 2: Plate (2) is out-of-phase with plate (1) and (3)
Case 3: Plate (2) is out-of-phase with plate (3) and is in-phase with plate (1)

Table 8: Vibrational frequencies (Hz) of three N radial plates

Mode #	In air	In fluid		
		Case 1	Case 2	Case 3
1	199	140	116	127
2	243	171	142	154
3	362	255	211	230
4	558	394	324	353
5	852	603	492	538
6	1244	885	683	773
7	1251	918	699	781
8	1312	958	732	815
9	1441	1058	785	889
10	1679	1234	911	1034

Case 1: Three plates vibrating in-phase
Case 2: Plate (2) is out-of-phase with plates (1) and (3)
Case 3: Plate (2) is out-of-phase with plate (3) and is in-phase with plate (1)

Table 9: Natural frequencies (Hz) of a vertical cantilever plate gradually submerged in a fluid reservoir

Mode #	Submerged height Ratio (S/B)							
	25%		50%		75%		100%	
	PT	[11]	PT	[11]	PT	[11]	PT	[11]
1	2	2	2	2	2	2	2	2
2	21	21	15	16	12	12	12	12
3	25	30	20	26	19	24	19	24
4	56	57	49	52	36	38	33	34

PT: Present Theory