Thermal stresses in an infinitely long solid cylinder using Green's function and the hyperbolic heat equation

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Abstract: - We compute the thermal stresses arising in an homogeneous isotropic infinitely long solid circular cylinder when a constant heat flux is acting on its boundary. We use the hyperbolic heat equation and Green's function to compute the temperature. A comparison with analogous results obtained with the classical Fourier heat equation is made.

Key-words: - Green's function, integral transforms, hyperbolic heat equation, thermal stresses.

1 Introduction

The purpose of this communication is to compute the thermal stresses arising in an infinitely long solid circular cylinder when a constant heat flux is applied to its boundary using the hyperbolic model of heat conduction, and to compare this results with the computed ones in the case of using the classical Fourier heat conduction model.

We consider an homogeneous isotropic solid infinitely long circular cylinder of radius $R > 0$ at initial temperature T_0 with constant and independent on the temperature physical parameters thermal conductivity k, density d, specific heat c, relaxation parameter τ, diffusivity $\alpha := \frac{k}{c\,d}$, coefficient of linear dilatation $\overline{\alpha}$, Young's modulus E and Poisson's coefficient ν. Starting in the time $t = 0$ a constant heat flux Q_0 is normally applied in all the points of its surface. The first step in order to compute the thermal stresses developed in the cylinder is to find the induced temperature field. Choosing the z-cartesian coordinate axis along the axis of the cylinder, the temperature will be axisymmetric with respect to the z-axis (independent on the polar angle θ) and independent on the z-coordinate, thus the governing hyperbolic heat equation in cylindrical coordinates will be (see [5])

$$-\frac{\partial^2 T}{\partial \rho^2} - \frac{1}{\rho}\frac{\partial T}{\partial \rho} + \frac{1}{\alpha}\left(\frac{\partial T}{\partial t} + \tau\frac{\partial^2 T}{\partial t^2}\right) = 0 \quad (1)$$

in the domain $(\rho, t) \in]0, R[\times]0, \infty[$, with the initial and boundary conditions

$$T(\rho, 0) = T_0 \quad \frac{\partial T}{\partial t}(\rho, 0) = 0 \quad \rho \in]0, R[, \quad (2)$$

$$\frac{\partial T}{\partial \rho}(0, t) = 0 \quad t > 0 \quad (3)$$

and

$$\frac{\partial T}{\partial \rho}(R, t) = \frac{Q_0}{k}\Big(H(t) + \tau\delta(t)\Big) \quad t > 0. \quad (4)$$

Following [1] we introduce the dimensionless

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variables

$$V(\rho,t) = k\frac{T(\rho,t)-T_0}{R\,Q_0}\ , \tag{5}$$

$$\eta = \frac{\rho}{R},\ \ \xi = \frac{\alpha\, t}{4\,R^2},\ \ \Lambda = \frac{\alpha\,\tau}{4\,R^2} \tag{6}$$

which yield to the problem

$$-\frac{\partial^2 V}{\partial \eta^2} - \frac{1}{\eta}\frac{\partial V}{\partial \eta} + \frac{1}{4}\left(\frac{\partial V}{\partial \xi} + \Lambda\frac{\partial^2 V}{\partial \xi^2}\right) = 0 \tag{7}$$

in the new domain $(\eta,\xi) \in]0,1[\times]0,\infty[$,

$$V(\eta,0) = \frac{\partial V}{\partial \xi}(\eta,0) = 0 \qquad \eta \in]0,1[, \tag{8}$$

$$\frac{\partial V}{\partial \eta}(0,\xi) = 0 \qquad \xi > 0 \tag{9}$$

and

$$\frac{\partial V}{\partial \eta}(R,\xi) = H(\xi) + \Lambda\ \delta(\xi) \quad \xi > 0. \tag{10}$$

2 Green's function

Problem (7)-(10) is solved in [1] using the residue and convolution theorems to find the Laplace inverse transform of V. We follow a less involved method, the use of Green's function, furthermore providing the key to solve many thermal problems concerning the quoted geometry in an automatic way (see [3]).

In cylindrical dimensionless coordinates (5) and (6) Green's function corresponding to our problem is the distribution $G(\eta,\xi) := G(\eta,\xi|\eta_0,\xi_0)$ verifying for every $(\eta_0,\xi_0) \in]0,1[\times\mathbb{R}$ the boundary value problem

$$-\frac{\partial^2 G}{\partial \eta^2} - \frac{1}{\eta}\frac{\partial G}{\partial \eta} + \frac{1}{4}\left(\frac{\partial G}{\partial \xi} + \Lambda\frac{\partial^2 G}{\partial \xi^2}\right) = \\ = \frac{1}{\eta}\ \delta(\eta-\eta_0)\ \delta(\xi-\xi_0) \quad (\eta,\xi) \in]0,1[\times\mathbb{R}, \tag{11}$$

$$\frac{\partial G}{\partial \eta}(0,\xi) = \frac{\partial G}{\partial \eta}(1,\xi) = 0 \quad \xi \in \mathbb{R} \tag{12}$$

and

$$\lim_{\xi\to\infty} G(\eta,\xi) = \lim_{\xi\to\infty}\frac{\partial G}{\partial \xi}(\eta,\xi) = 0 \quad \eta \in]0,1[. \tag{13}$$

Denoting by $\mathfrak{L}(\eta,p)$ the Schwartz-Laplace transform with respect to ξ of $G(\eta,\xi)$ we obtain (see [6])

$$-\frac{\partial^2 \mathfrak{L}}{\partial \eta^2}(\eta,p) - \frac{1}{\eta}\frac{\partial \mathfrak{L}}{\partial \eta}(\eta,p) + \\ + \frac{1}{4}(p+\Lambda p^2)\ \mathfrak{L}(\eta,p) = \frac{1}{\eta}\ \delta(\eta-\eta_0)\ e^{-p\,\xi_0}, \tag{14}$$

$$\frac{\partial \mathfrak{L}}{\partial \eta}(0,p) = \frac{\partial \mathfrak{L}}{\partial \eta}(1,p) = 0 \tag{15}$$

and (16)

$$\lim_{p\to 0}\mathfrak{L}(\eta,p) = \lim_{p\to 0} p\ \mathfrak{L}(\eta,p) = 0. \tag{17}$$

In order to verify (15), condition (17) allow us to take finite Hankel transforms of second kind with respect to η

$$\mathcal{H}_m(p) := \int_0^1 \eta\ J_0(\beta_m\,\eta)\ \mathfrak{L}(\eta,p)\ d\eta$$

($m \in \mathbb{N}\cup\{0\}$) where $\{\beta_m\}_{m=0}^{\infty}$ is the strictly increasing sequence of non negative zeros of the equation $J_1(\beta) = 0$, the Bessel function of first kind and order 1. Thus, using equation (14) and the inversion formula of finite Hankel transforms (see [7]) we obtain

$$\mathfrak{L}(\eta,p) = \\ = 8\sum_{m=0}^{\infty}\frac{J_0(\beta_m\,\eta_0)\ J_0(\beta_m\,\eta)}{\left(4\,\beta_m^2 + p + \Lambda\,p^2\right)\ J_0(\beta_m)^2}\ e^{-p\,\xi_0}.$$

In order to simplify the exposition we assume that the material of the cylinder is such that $16\,\lambda\,\beta_m^2 - 1 \neq 0$ for every $m \in \mathbb{N}\cup\{0\}$. Thus, by inversion of Schwartz-Laplace transforms we obtain the adimensional Green's function

$$G(\eta,\xi|\eta_0,\xi_0) = 16\ H(\xi-\xi_0)\ \times \\ \times\sum_{m=0}^{\infty}\frac{J_0(\beta_m\,\eta_0)\ J_0(\beta_m\,\eta)}{J_0(\beta_m)^2}\ F_m(\xi,\xi_0) \tag{18}$$

where, for every $m \in \mathbb{N}\cup\{0\}$

$$F_m(\xi,\xi_0) = e^{-\frac{\xi-\xi_0}{2\,\Lambda}}\times \\ \times\left(\frac{H\left(1-16\,\Lambda\,\beta_m^2\right)}{\sqrt{1-16\,\Lambda\,\beta_m^2}}\ \sinh\ \gamma_m(\xi-\xi_0) + \right.$$

$$+ \frac{H\big(16\,\Lambda\,\beta_m^2 - 1\big)}{\sqrt{16\,\Lambda\,\beta_m^2 - 1}}\;\sin\,\Gamma_m(\xi - \xi_0)\Bigg)$$

where we have defined, for every $m \in \mathbb{N}$

$$\gamma_m := \frac{\sqrt{1 - 16\Lambda\beta_m^2}}{2\Lambda}\,,$$
$$\Gamma_m := \frac{\sqrt{16\Lambda\beta_m^2 - 1}}{2\Lambda}\,. \qquad (19)$$

3 Temperature in the case of boundary constant flux

According to the theory of Green's functions, the solution of problem (7)-(10) is given directly by

$$\forall\,(\eta,\xi) \in]0,1[\times]0,\infty[\quad V(\eta,\xi) =$$
$$= \int_{-\infty}^{\infty} G(\eta,\xi|1,\xi_0)\,\big(H(\xi_0) + \Lambda\,\delta(\xi_0)\big)\,d\xi_0$$

and after elementary computations and the application of the equality (see [1])

$$\sum_{m=1}^{\infty} \frac{J_0(\eta\beta_m)}{\beta_m^2\,J_0(\beta_m)} = \frac{\eta^2}{4} - \frac{1}{8}$$

we obtain

$$= 16\sum_{m=0}^{\infty} \frac{J_0(\beta_m\,\eta)}{J_0(\beta_m)} \int_0^{\xi} F_m(\xi,\xi_0)\,d\xi_0 +$$
$$+ 16\,\Lambda\,\sum_{m=0}^{\infty} \frac{J_0(\beta_m\,\eta)}{J_0(\beta_m)}\,F_m(\xi,0) =$$
$$= 8\,\xi + \frac{2\,\eta^2 - 1}{4}$$
$$- e^{-\frac{\xi}{2\,\Lambda}}\,\sum_{m=1}^{\infty} \frac{J_0(\beta_m\eta)}{\beta_m^2 J_0(\beta_m)}\,R_m(\xi) \qquad (20)$$

where

$$\forall\,m \in \mathbb{N}\quad R_m(\xi) = H\big(1 - 16\Lambda\beta_m^2\big)\times$$
$$\times\left(\cosh\gamma_m\xi + \frac{2 - 16\Lambda\beta_m^2}{\sqrt{1 - 16\Lambda\beta_m^2}}\sinh\gamma_m\xi\right) +$$
$$+ H\big(16\Lambda\beta_m^2 - 1\big)\times$$
$$\times\left(\cos\Gamma_m\xi + \frac{2 - 16\Lambda\beta_m^2}{\sqrt{16\Lambda\beta_m^2 - 1}}\sin\Gamma_m\xi\right).$$

It can be checked, after elementary computations, that this function coincides with the obtained one in [1].

4 Thermal stresses

We start computing the actual value of some expressions which will appear frequently in the sequel. From (5), (6) and (20), by the well known equality (see [8], page 45)

$$\forall\,\gamma > 0 \quad \int_0^{\rho} r\,J_0(\gamma\,r)\,dr = \frac{\rho}{\gamma}J_1(\gamma\,\rho)$$

we obtain

$$\int_0^{\rho} \big(T(\rho,t) - T_0\big)\,rdr =$$
$$= \frac{R\,Q_0}{k}\left(\frac{\alpha\rho^2}{R^2}t + \frac{\rho^4}{8R^2} - \frac{\rho^2}{8}\right.$$
$$\left. - R\rho\,e^{-\frac{t}{2\tau}}\sum_{m=1}^{\infty} \frac{J_1\left(\frac{\beta_m}{R}\rho\right)}{\beta_m^3 J_0(\beta_m)}\,R_m\left(\frac{\alpha t}{4R^2}\right)\right) \qquad (21)$$

and thus

$$\int_0^{R} \big(T(\rho,t) - T_0\big)\,\rho\,d\rho = \frac{\alpha R Q_0}{k}\,t. \qquad (22)$$

We are now in a position to compute the thermal stresses in the cylinder. Since we are concerned with an infinitely long cylinder we have to deal with a state of plane strain. It is well known that in this case the components of the stress tensor in cylindrical coordinates are given by (see [4], page 206)

$$\sigma_{\rho\theta}(\rho,t) = 0$$
$$\sigma_{\rho\rho}(\rho,t) = \frac{\overline{\alpha}E}{1-\nu}\left(-\frac{1}{\rho^2}\int_0^{\rho} r\big(T(r,t) - T_0\big)\,dr +\right.$$
$$\left. + \frac{1}{R^2}\int_0^{R} r\big(T(r,t) - T_0\big)\,dr\right), \qquad (23)$$

$$\sigma_{\theta\theta}(\rho,t) = \frac{\overline{\alpha}E}{1-\nu}\left(\frac{1}{\rho^2}\int_0^\rho r\big(T(r,t)-T_0\big)\,dr + \frac{1}{R^2}\int_0^R r\big(T(r,t)-T_0\big)\,dr - \big(T(\rho,t)-T_0\big)\right) \tag{24}$$

and

$$\sigma_{zz}(\rho,t) = \frac{\overline{\alpha}E}{1-\nu}\left(\frac{2}{R^2}\int_0^R r\big(T(r,t)-T_0\big)\,dr - \big(T(\rho,t)-T_0\big)\right) \tag{25}$$

having in mind the existence of an axial strain of magnitude

$$\varepsilon_0(t) = \frac{2\alpha\overline{\alpha}Q_0}{kR}\,t \tag{26}$$

(see [4], page 205).

After substitution of previous results and elementary computations we obtain

$$\sigma_{\rho\rho}(\rho,t) = \frac{\overline{\alpha}ERQ_0}{k(1-\nu)}\left(\frac{1}{8}-\frac{\rho^2}{8R^2} + \frac{R}{\rho}\,e^{-\frac{t}{2\tau}}\sum_{m=1}^{\infty}\frac{J_1\left(\frac{\beta_m}{R}\rho\right)}{\beta_m^3 J_0(\beta_m)}\,R_m\left(\frac{\alpha t}{4R^2}\right)\right),$$

$$\sigma_{\theta\theta}(\rho,t) = \frac{\overline{\alpha}ERQ_0}{k(1-\nu)}\left(\frac{1}{8}-\frac{3\rho^2}{8R^2} - R\,e^{-\frac{t}{2\tau}} \times \sum_{m=1}^{\infty}\left(\frac{J_1\left(\frac{\beta_m\rho}{R}\right)}{\rho\,\beta_m^3} - \frac{J_0\left(\frac{\beta_m\rho}{R}\right)}{R\,\beta_m^2}\right)\frac{R_m\left(\frac{\alpha t}{4R^2}\right)}{J_0(\beta_m)}\right),$$

$$\sigma_{\rho\theta}(\rho,t) = 0$$

and

$$\sigma_{zz}(\rho,t) = \frac{\overline{\alpha}ERQ_0}{k(1-\nu)}\left(\frac{1}{4}-\frac{\rho^2}{2\,R^2} + e^{-\frac{t}{2\tau}}\sum_{m=1}^{\infty}\frac{J_0\left(\frac{\beta_m}{R}\rho\right)}{\beta_m^2 J_0(\beta_m)}\,R_m\left(\frac{\alpha t}{4R^2}\right)\right).$$

5 Comparison with the classical case

If we use the classical parabolic Fourier heat equation to find the temperature (denoted by $T_F(\rho,t)$ to distinguish) we arrive (see [1]) to

$$T_F(\rho,t) = T_0 + \frac{RQ_0}{k}\,V_F\left(\frac{\rho}{R},\frac{\alpha t}{4R^2}\right)$$

where

$$V_F(\eta,\xi) = = 8\,\xi + \frac{2\eta^2-1}{4} - 2\sum_{m=1}^{\infty}\frac{J_0(\beta_m\eta)}{\beta_m^2 J_0(\beta_m)}\,e^{-4\beta_m^2\xi}.$$

It can be seen that this temperature field produce the same axial strain ε_0 given by (26) and that formulas (22) and $\sigma_{\rho\theta}(\rho,t)=0$ still holds. However in this case we have

$$\sigma_{\rho\rho}(\rho,t) = \frac{\overline{\alpha}ERQ_0}{k(1-\nu)}\left(\frac{1}{8}-\frac{\rho^2}{8R^2} + 2\,\frac{R}{\rho}\sum_{m=1}^{\infty}\frac{J_1\left(\frac{\beta_m}{R}\rho\right)}{\beta_m^3 J_0(\beta_m)}\,e^{-\frac{\alpha\beta_m^2}{R^2}\,t}\right),$$

$$\sigma_{\theta\theta}(\rho,t) = \frac{\overline{\alpha}ERQ_0}{k(1-\nu)}\left(\frac{1}{8}-\frac{3\rho^2}{8R^2} - 2R \times \sum_{m=1}^{\infty}\left(\frac{J_1\left(\frac{\beta_m\rho}{R}\right)}{\rho\,\beta_m^3} - \frac{J_0\left(\frac{\beta_m\rho}{R}\right)}{R\,\beta_m^2}\right)\frac{e^{-\frac{\alpha\beta_m^2 t}{R^2}}}{J_0(\beta_m)}\right)$$

and

$$\sigma_{zz}(\rho,t) = \frac{\overline{\alpha}ERQ_0}{k(1-\nu)}\left(\frac{1}{4}-\frac{\rho^2}{2\,R^2} + 2\sum_{m=1}^{\infty}\frac{J_0\left(\frac{\beta_m}{R}\rho\right)}{\beta_m^2\,J_0(\beta_m)}\,e^{-\frac{\alpha\beta_m^2 t}{R^2}}\right). \tag{27}$$

The quantitative differences arising in the stresses values given by the hyperbolic and the parabolic temperature models are numerically considerable and thus very important concerning practical industrial applications. Remark that these differences are proportional to the applied heat flux Q_0 (see the position of the term Q_0 in all previous formulas of stresses) and that high fluxes applied in small temporal intervals are more and more frequently used in modern laser technology. As a simple illustration, working with the dimensionless stresses deduced from (6) and ignoring the factor $\frac{\overline{\alpha} E R Q_0}{k(1-\nu)}$ we have plotted in figures 1, 2 and 3 the radial variation of tension $\sigma_{\rho\rho}$ in the dimensionless time $\xi = 0.15, \xi = 0.158$ and $\xi = 0.17$ respectively, taking $\Lambda = 0.1$ and using the dashed line for the tension computed with Fourier heat equation and the solid line in the case of using the hyperbolic heat equation.

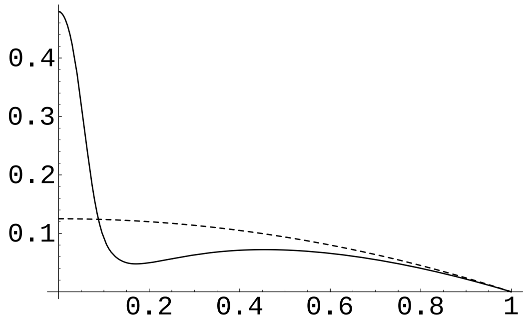

Figure 1: Stress $\sigma_{\rho\rho}$ corresponding to $\xi = 0.14$ and $\Lambda = 0.1$

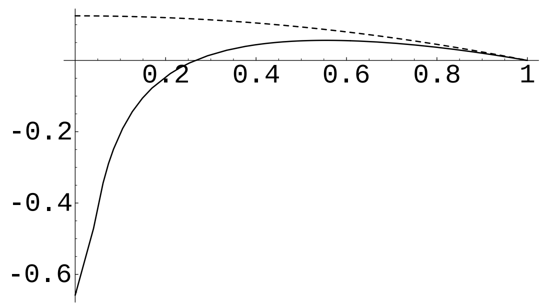

Figure 2: Stress $\sigma_{\rho\rho}$ corresponding to $\xi = 0.15$ and $\Lambda = 0.1$

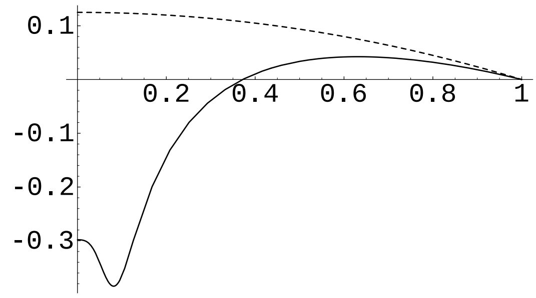

Figure 3: Stress $\sigma_{\rho\rho}$ corresponding to $\xi = 0.159$ and $\Lambda = 0.1$

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