Dynamics Modeling Analysis of the Mechanism System Based on Rigid body Motion and Elastic Motion

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Abstract: Dynamics modeling of the mechanical system with flexible deformation and rigid body motion are discussed. Regard generalized coordinate as rigid body motion degree of freedom and elastic deformation degree of freedom, utilize the finite element method to describe motion and deformation of elastic connecting rod, use Kane equation to derive the movement equation of the elastic connecting rod organization.

Key words: Elastic Deformation; The Finite Element Method; Kane Equation; Dynamics Analysis

1 Introduce

Dynamic analysis of mechanism system has been based on the assumption that the links behave as rigid bodies. Stresses in the members are assumed to be only due to inertia forces and external loads. Based on these stresses calculations the mechanism is designed, built, and tested. This design procedure is reasonably accurate if the links behave as rigid bodies. However, as speeds of operation become higher, the inertia forces become quite large and the links undergo considerable deformation. Under these conditions the rigid body assumption is no longer valid. The movements of mechanism system can be accurate simulated by taking into account the elasticity of the links during simulation and design process. Since elastic behavior in mechanism systems cannot be completely eliminated, mechanism systems would have to be actively controlled in order to further minimize effects due to elastic deflections. For such control application it is necessary to develop accurate models, which more realistically represent the actual mechanism systems.

The modeling of mechanical system with the elastic links has been paid attention by people all the time. The work can be divided into three respects at present, according to its modeling way:

The first approach [1-5] which originated earlier, models the elastic links as continuous systems possessing infinite degrees of freedom. The equations of motion obtained are nonlinear partial differential equations. This approach has been used to derive equations of motion, analyze and determine the dynamic response of the slider crank mechanism, which has an elastic connecting rod and rigid rod.

In the second approach [6-18], the elastic links represented as discrete systems possessing finite elastic degrees of freedom by using methods like the finite element method. The advantage of using the finite element method to model the elastic links is that it provides a systematic modeling technique for complex mechanisms and lays the groundwork for a general approach for the modeling of mechanisms. In these works the net motion or the total motion of the system is considered to be a superposition of the rigid body motion and the elastic motion.

A third approach [19-31] uses the Lagrange Multiplier technique to incorporate joint constraints into the equation of motion. This approach is broadly applicable to a large class of dynamic systems rather than just mechanism systems and results in a formulation that is general concise, and conveniently implemented on the computer.

In this work, analyzes the manipulator system of operation with elastics in systematic way, dynamics modeling of the mechanical systems with rigid body movement and flexibility deformation are discussed. Regard generalized coordinate as rigid body between degree of freedom and elastic deformation degree of freedom, utilize finite element method to describe motion and deformation of elastic connecting rod, use Kane equation to derive the equation of motion of the elastic connecting rod. This kind of equation of motion can be used for analyzing industry's machinery operates hands. Because use the finite element method to modeling to the elastic pole, no matter which kind of complicated forms it has.

2 Description of deformation of the element

As shown fig.1, is an elastic mechanism system,

$oxyz$ is a system of coordinates of the inertia, $o_i x_i y_i z_i$ is the system of coordinates fixed at link. The deformation of the operation manipulator system, can express with the following relation:

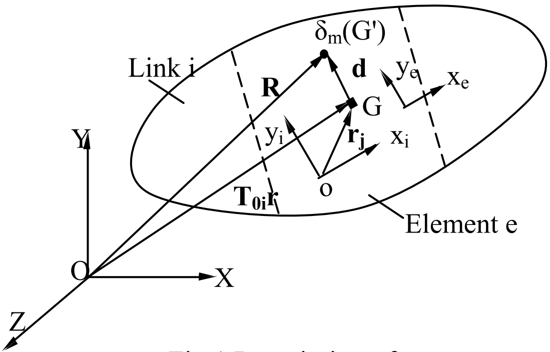

Fig.1 Description of deformation of the element

$$\mathbf{r_i} = \mathbf{T_{0i}} \mathbf{r} \quad (\text{where} \quad \mathbf{T_{0i}} = \mathbf{T_{01}} \mathbf{T_{12}} \mathbf{T_{23}} \wedge \mathbf{T_{i-1,i}}\ ) \quad (1)$$

$\mathbf{T_{0i}}$ represents the homogeneous coordinate transformation matrix from link 1 to link i, is a 4×4 matrix that represents the rigid body translation and rotation translation of link i with respect to the reference coordinate system $oxyz$,and is of the following form:

$$\mathbf{T_{0i}} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ x_{ooi} & \cos\theta_{xx_i} & \cos\theta_{xy_i} & \cos\theta_{xz_i} \\ y_{ooi} & \cos\theta_{yx_i} & \cos\theta_{yy_i} & \cos\theta_{xz_i} \\ z_{ooi} & \cos\theta_{zx_i} & \cos\theta_{zy_i} & \cos\theta_{zz_i} \end{bmatrix}$$

If only existed rotation translation between joints, we have

$$\mathbf{T_{i-1,i}} = \begin{bmatrix} 1 & 0 & -0 & 0 \\ 0 & \cos\phi_i & -\cos\phi_i & 0 \\ 0 & \sin\phi_i & \cos\phi_i & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The first time derivative of $\mathbf{T_{i-1,i}}$ is:

$$\dot{\mathbf{T}}_{\mathbf{i-1,i}} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -\dot{\phi}_i \sin\phi_i & -\dot{\phi}_i \cos\phi_i & 0 \\ 0 & \dot{\phi}_i \cos\phi_i & -\dot{\phi}_i \sin\phi_i & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\dot{\mathbf{T}}_{\mathbf{i-1,i}} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\phi_i & -\sin\phi_i & 0 \\ 0 & \sin\phi_i & \cos\phi_i & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \mathbf{Q_i T_{i-1,i}} \dot{\phi}_i$$

Thus

$$\begin{aligned} \dot{\mathbf{T}}_{\mathbf{0i}} &= \mathbf{Q_1 T_{01}} \dot{\phi}_1 \mathbf{T_{12}} \wedge \mathbf{T_{i-1,i}} + \wedge + \mathbf{T_{01} T_{12}} \wedge \mathbf{Q_{i-1} T_{i-1,i}} \dot{\phi}_i \\ &= \mathbf{Q_1 T_{01} T_{12}} \wedge \mathbf{T_{i-1,i}} \dot{\phi}_1 + \wedge + \mathbf{Q_2 T_{01} T_{12}} \wedge \mathbf{T_{i-1,i}} \dot{\phi}_2 + \wedge + \mathbf{Q_i T_{01} T_{12}} \wedge \mathbf{T_{i-1,i}} \dot{\phi}_i \\ &= \sum_{j=1}^{i-1} \mathbf{Q_j} \dot{\phi}_j \mathbf{T_{0i}} = \mathbf{w_i} \cdot \mathbf{T_{0i}} \end{aligned} \quad (2\text{-a})$$

where $\mathbf{w_i} = \sum_{j=1}^{i-1} \mathbf{Q_j} \dot{\phi}_j$ is the operator matrix $\mathbf{Q_j}$ is the constant transformation matrix.

Different $\dot{\mathbf{T}}_{0i}$ by time t, we have

$$\begin{aligned} \ddot{\mathbf{T}}_{0i} &= \sum_{j=1}^{i-1} (\mathbf{Q_j} \ddot{\phi}_j) \mathbf{T_{0i}} + \sum_{j=1}^{i-1} (\mathbf{Q_j} \dot{\phi}_j) \dot{\mathbf{T}}_{\mathbf{0i}} \\ &= \sum_{j=1}^{i-1} (\mathbf{Q_j} \ddot{\phi}_j) \mathbf{T_{0i}} + \sum_{j=1}^{i-1} (\mathbf{Q_j} \dot{\phi}_j) \sum_{j=1}^{i-1} (\mathbf{Q_k} \dot{\phi}_k) \mathbf{T_{0i}} \end{aligned} \quad (3)$$

Where $\dot{\phi}_j$ can be written by following: To the operating system of the general manipulator, its equation of geometry restrains can be written: $\phi_i(\varphi_1, \cdots \varphi_f) = 0 \quad (i = 1, \cdots, N)$ (4)

Here $\varphi_1, \cdots \varphi_f$ show f degrees of freedom of system, i show systematic number of object. The partial derivatives of the rigid body constraint values ϕ_i with respect to the value φ of the rigid body degrees of freedom is definite as following: $\phi'_{ik} = \delta\phi_i / \delta\varphi_k$ .Therefore

$$\dot{\phi}_l = \sum_{k=1}^{f} \phi'_{ik} \dot{\varphi}_k \quad (5)$$

Formula (5) can regard as the expression formula of the generalized speed that is derivated by inclusive condition of dynamics, so $\mathbf{w_i}$ can be written as following

$$\mathbf{w_i} = \sum_{j=1}^{i-1} \sum_{k=1}^{f} \mathbf{Q_j} \phi'_{ik} \dot{\varphi}_k = \sum_{k=1}^{f} [\sum_{j=1}^{i-1} \mathbf{Q_j} \phi'_{ik}] \dot{\varphi}_k = \sum_{k=1}^{f} \overline{\mathbf{W}}_{ik} \dot{\varphi}_k \quad (6)$$

where $\overline{\mathbf{W}}_{ik} = \sum_{j=1}^{i-1} \mathbf{Q_j} \phi'_{jk}$. Thus

$$\dot{\mathbf{T}}_{\mathbf{0i}} = \sum_{j=1}^{i-1} \sum_{k=1}^{f} \phi'_{jk} \mathbf{Q_j} \dot{\varphi}_k \mathbf{T_{0i}} = \sum_{k=1}^{f} \mathbf{W}_{ik} \dot{\varphi}_k \mathbf{T_{0i}} \quad (2\text{-b})$$

The second derivative of derivatives of ϕ_i is

$$\ddot{\phi}_i = \sum_{k=1}^{f} \sum_{l=1}^{f} \frac{\partial^2 \phi_i}{\partial\varphi_k \partial\varphi_l} \dot{\varphi}_k \dot{\varphi}_l + \sum_{k=1}^{f} \phi'_{ik} \ddot{\varphi}_k = \nabla^2 \phi_i \dot{\varphi}_k \dot{\varphi}_l + \sum_{k=1}^{f} \phi'_{ik} \ddot{\varphi}_k \quad (7)$$

So formula (3) can be shown as following:

$$\begin{aligned} \ddot{\mathbf{T}}_{\mathbf{0i}} &= \sum_{j=1}^{i-1} \mathbf{Q_j} (\nabla^2 \phi_i \dot{\varphi}_k \dot{\varphi}_l + \sum_{k=1}^{f} \phi'_{ik} \ddot{\varphi}_k) \mathbf{T_{0i}} + [\sum_{j=1}^{i-1} (\mathbf{Q_j} \dot{\phi}_j)][\sum_{k=1}^{i-1} (\mathbf{Q_k} \dot{\phi}_k)] \mathbf{T_{0i}} \\ &= (\sum_{j=1}^{i-1} \mathbf{Q_j} \nabla^2 \phi_i \dot{\varphi}_k \dot{\varphi}_l + \sum_{k=1}^{f} \overline{\mathbf{W}}_{ij} \ddot{\varphi}_k) T_{0i} + \overline{\mathbf{W}}_{ij} \overline{\mathbf{W}}_{ik} \dot{\varphi}_j \dot{\varphi}_k \mathbf{T_{0i}} \end{aligned} \quad (8)$$

3 Description of motion of the element

To the mechanism system with the elastic links, shown as fig.2, the elemental mass δm in element e on the link i. Coordinate system $oxyz$ is the fixed reference coordinate system, $o_i x_i y_i z_i$ is local coordinate system, which is usually the rigid body position of its center of gravity. $x_e y_e z_e$ is the elemental coordinate system built in the center of gravity of the element e. G indicates the rigid body position of the elemental e.

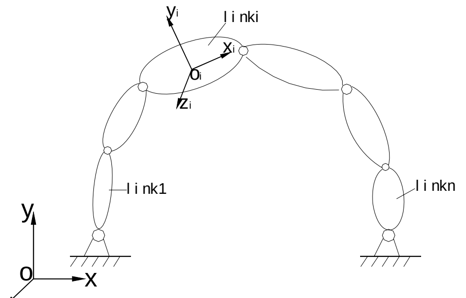

Fig.2 System with the elastic links

A additional 4×4 transformation matrix $\mathbf{R}^e$ is defined for elemental e. $\mathbf{R}^e$ depends on the constant orientation of element coordinate system $x_e y_e z_e$ with respect to the link coordinate system $x_i y_i z_i$.Thus $\mathbf{R}^e$ is a constant matrix and is of the following form:

$$\mathbf{R}^e = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \cos\theta_{x_i x_0} & \cos\theta_{x_i y_e} & \cos\theta_{x_i z_e} \\ 0 & \cos\theta_{y_i x_e} & \cos\theta_{y_i y_e} & \cos\theta_{y_i z_e} \\ 0 & \cos\theta_{z_i x_e} & \cos\theta_{z_i y_e} & \cos\theta_{z_i z_e} \end{bmatrix}$$

The product of the transformation matrices $\mathbf{T}_{oi}$ and $\mathbf{R}^e$ is used to transform the elastic displacement $\mathbf{d}$ which is measure in the element coordinate system to the reference coordinate system. This transformation matrix requires only the relative angular orientation of the element coordinate system with respect to the reference coordinate system. Therefore the first diagonal element of $\mathbf{R}^e$ is set to zero.

The position of the elemental mass δm in the reference coordinate system is specified by vector **R.** Vector R is made up of two components, the rigid body position of δm and the elastic displacement of δm. Vector r locates rigid body position of δm in link coordinate system $x_i y_i z_i$. Vector r is a constant vector. The rigid body position of δm in the reference coordinate system. is represented by $\mathbf{T}_{0i}\mathbf{r}$. Vector d is the elastic displacement of δm in the coordinate system. $x_e y_e z_e$. The elastic displacement of δm is represented in the coordinate system $oxyz$ as the matrix product $\mathbf{T}_{0i}\mathbf{R}^e\mathbf{d}$.The position of δm in the reference coordinate system is given by:

$$\mathbf{R} = \mathbf{T}_{0j}\mathbf{r} + \mathbf{T}_{0j}\mathbf{R}^e\mathbf{d} \tag{9}$$

Based on the theory of the finite element method, the elastic displacement $\mathbf{d}$ of δm in the coordinate system $x_e y_e z_e$ can be expressed as a linear function of the nodal elastic displacement vector $\mathbf{u}^e$ as shown form: $\mathbf{d} = \mathbf{N}^e\mathbf{u}^e$ (10)

Matrix $\mathbf{N}^e$ contains the finite element shape functions, which relate the nodal elastic displacement, $\mathbf{u}^e$ the elastic displacement vector $\mathbf{d}$. The nodal rigid body position vector $\mathbf{p}^e$ is defined that is the rigid body position of the nodal of the element e as measured in the link coordinate system $x_i y_i z_i$. This is a constant vector as the rigid body position of the nodal in the link coordinate system $x_i y_i z_i$ are fixed. Using this vector and the shape function $\mathbf{N}^e$, the rigid body position of the mass δm is expressed as

$$\mathbf{r} = \mathbf{N}^e\mathbf{p}^e \tag{11}$$

Equation (11) holds for isoparametric finite elements, which are the most commonly used type finite element. Substituting equations (10) and (11) in equation (9) the position of the elemental mass δm is expressed as: $\mathbf{R} = \mathbf{T}_{0i}\mathbf{N}^e\mathbf{p}^e + \mathbf{T}_{0i}\mathbf{R}^e\mathbf{N}^e\mathbf{u}^e$ (12)

The above equation is differentiated with respect to time to determine the velocity the elemental mass δm in the reference coordinate system :

$$\dot{\mathbf{R}} = \dot{\mathbf{T}}_{0i}\mathbf{N}^e\mathbf{p}^e + \dot{\mathbf{T}}_{0i}\mathbf{R}^e\mathbf{N}^e\mathbf{u}^e + \mathbf{T}_{0i}\mathbf{R}^e\mathbf{N}^e\dot{\mathbf{u}}^e \tag{13-a}$$

$\dot{\mathbf{T}}_{0i}$ is the time derivative of $\mathbf{T}_{0i}$ and $\dot{\mathbf{u}}^e$ is the velocity vector of the elastic degree of freedom measured in the element coordinate system $x_e y_e z_e$. The shape function $\mathbf{N}^e$ and the 4×4 matrix $\mathbf{R}^e$ are constant matrices and are unaffected by the differentiation with respect to time. Using equation (2), the above equation is expressed as:

$$\begin{aligned} \dot{\mathbf{R}} &= \mathbf{w}_i\mathbf{T}_{0i}\mathbf{N}^e\mathbf{p}^e + \mathbf{w}_i\mathbf{T}_{0i}\mathbf{R}^e\mathbf{N}^e\mathbf{u}^e + \mathbf{T}_{0i}\mathbf{R}^e\mathbf{N}^e\dot{\mathbf{u}}^e \\ &= \sum_{k=1}^{f}\bar{\mathbf{W}}_{ik}\dot\varphi_k\mathbf{T}_{0i}\mathbf{N}^e\mathbf{p}^e + \sum_{k=1}^{f}\bar{\mathbf{W}}_{ik}\dot\varphi_k\mathbf{T}_{0i}\mathbf{R}^e\mathbf{N}^e\mathbf{u}^e + \mathbf{T}_{0i}\mathbf{R}^e\mathbf{N}^e\dot{\mathbf{u}}^e \\ &= [\sum_{k=1}^{f}\bar{\mathbf{W}}_{ik}\mathbf{T}_{0i}(\mathbf{N}^e\mathbf{p}^e + \mathbf{R}^e\mathbf{N}^e\mathbf{u}^e)]\dot\varphi_k + \mathbf{T}_{0i}\mathbf{R}^e\mathbf{N}^e\dot{\mathbf{u}}^e \end{aligned} \tag{13-b}$$

The second differentiation of R can be differentiated with respect to equation (13-a)

$$\ddot{\mathbf{R}} = \ddot{\mathbf{T}}_{0i}\mathbf{N}^e\mathbf{p}^e + \ddot{\mathbf{T}}_{0i}\mathbf{R}^e\mathbf{N}^e\mathbf{u}^e + 2\dot{\mathbf{T}}_{0i}\mathbf{R}^e\mathbf{N}^e\dot{\mathbf{u}}^e + \mathbf{T}_{0i}\mathbf{R}^e\mathbf{N}^e\ddot{\mathbf{u}}^e \tag{14}$$

Substituting equations (2) and (8) in equation (14), we have:

$$\begin{aligned} \ddot{\mathbf{R}} &= [(\sum_{j=1}^{i-1}\mathbf{Q}_j\nabla^2\phi_i\dot\varphi_k\dot\varphi_l + \sum_{k=1}^{f}\bar{\mathbf{W}}_{ij}\ddot\varphi_k)\mathbf{T}_{0i} + \bar{\mathbf{W}}_{ij}\bar{\mathbf{W}}_{ik}\dot\varphi_j\dot\varphi_k\mathbf{T}_{0i}]\mathbf{N}^e\mathbf{p}^e + \\ &[(\sum_{j=1}^{i-1}\mathbf{Q}_j\nabla^2\phi_i\dot\varphi_k\dot\varphi_l + \sum_{k=1}^{f}\bar{\mathbf{W}}_{ik}\varphi_k)\mathbf{T}_{0i} + \bar{\mathbf{W}}_{ij}\bar{\mathbf{W}}_{ik}\dot\varphi_j\dot\varphi_k\mathbf{T}_{0i}]\mathbf{R}^e\mathbf{N}^e\mathbf{u}^e \\ &+ 2\sum_{j=1}^{i-1}\bar{\mathbf{W}}_{ij}\dot\varphi_j\mathbf{T}_{0i}\mathbf{R}^e\mathbf{N}^e\dot{\mathbf{u}}^e + \mathbf{T}_{0i}\mathbf{R}^e\mathbf{N}^e\ddot{\mathbf{u}}^e \end{aligned} \tag{15}$$

4 The generalized partial velocity, the generalized partial acceleration and the generalized inertial force

The rigid body degree of freedom φ and the elastic displacement degree of freedom u are regarded as the generalized coordinate, and regard $\dot{\varphi}_i$ and $\dot{u}$ as the generalized speed. Research equation(13-b)and(15) determine generalized partial speeds of the element mass δm :

$$\mathbf{v}_\mathrm{i}^\mathbf{P}=\sum_{k=1}^{f}\bar{\mathbf{W}}_{ik}\mathbf{T}_{0\mathrm{i}}(\mathbf{N}^\mathrm{e}\mathbf{p}^\mathrm{e}+\mathbf{R}^\mathrm{e}\mathbf{N}^\mathrm{e}\mathbf{u}^\mathrm{e})+\mathbf{T}_{0\mathrm{i}}\mathbf{R}^\mathrm{e}\mathbf{N}^\mathrm{e} \tag{16}$$

The acceleration of the element δm $\boldsymbol{a}_o^N=\ddot{\mathbf{R}}$ is given by formula (15).

The generalized inertial force related with the ith. Generalized speed $F_i^*=\mathbf{V}_i^N\boldsymbol{F}^*$ where $\mathbf{F}^*=-\int_B\rho\,\boldsymbol{a}_o^N\,dx_1dx_2$ so The generalized inertial force is expressed as below

$$F_i^*=-\int_B\rho\;\mathbf{V}_i^P\;\boldsymbol{a}_o^P\;dx_1dx_2 \tag{17}$$

Using the equations (15) and (16), Set $\mathbf{T}_{0\mathrm{i}}\mathbf{T}_{0i}^\mathrm{T}\mathbf{N}^\mathrm{e}\mathbf{N}^{\mathrm{eT}}\mathbf{p}^\mathrm{e}\mathbf{p}^{\mathrm{eT}}$ =[**TNP**], $\mathbf{T}_{0\mathrm{i}}\mathbf{T}_{0i}^\mathrm{T}\mathbf{R}^\mathrm{e}\mathbf{N}^\mathrm{e}\mathbf{N}^{\mathrm{eT}}\mathbf{p}^\mathrm{e}$ =[**TrNp**], $\mathbf{T}_{0\mathrm{i}}\mathbf{T}_{0i}^\mathrm{T}\mathbf{R}^\mathrm{e}\mathbf{R}^{\mathrm{eT}}\mathbf{N}^\mathrm{e}\mathbf{N}^{\mathrm{eT}}$ =[**TRN**], the equation (17) can be expanded shown as following form:

$$\begin{aligned}
F_i^*=&-\int_B\rho\sum_{k=1}^{f}\sum_{j=1}^{i-1}\sum_{l=1}^{l}\mathbf{Q}_j\bar{\mathbf{W}}_{ik}\nabla^2\phi_i\dot{\varphi}_k\dot{\varphi}_l[\mathbf{TNP}]dx_1dx_2\\
&-\int_B\rho\sum_{k=1}^{f}\sum_{l=1}^{f}\sum_{j=1}^{i-1}\mathbf{Q}_j\bar{\mathbf{W}}_{ik}\nabla^2\phi_i\dot{\varphi}_k\dot{\varphi}_l[\mathbf{TrNp}]\mathbf{u}^\mathrm{e}dx_1dx_2\\
&-\int_B\rho\sum_{k=1}^{f}\sum_{j=1}^{f}\bar{\mathbf{W}}_{ij}\bar{\mathbf{W}}_{ik}\ddot{\varphi}_k[\mathbf{TNP}]dx_1dx_2\\
&-\int_B\rho\sum_{k=1}^{f}\sum_{j=1}^{f}\bar{\mathbf{W}}_{ij}\bar{\mathbf{W}}_{ik}\ddot{\varphi}_k[\mathbf{TrNp}]\mathbf{u}^\mathrm{e}dx_1dx_2\\
&-\int_B\rho\sum_{k=1}^{f}\bar{\mathbf{W}}_{ik}\sum_{j=1}^{f}\bar{\mathbf{W}}_{ij}\sum_{k=1}^{f}\bar{\mathbf{W}}_{ik}\dot{\varphi}_j\dot{\varphi}_k[\mathbf{TNP}]dx_1dx_2\\
&-\int_B\rho\sum_{k=1}^{f}\bar{\mathbf{W}}_{ik}\sum_{j=1}^{f}\bar{\mathbf{W}}_{ij}\sum_{k=1}^{f}\bar{\mathbf{W}}_{ik}\dot{\varphi}_j\dot{\varphi}_k[\mathbf{TrNp}]\mathbf{u}^\mathrm{e}dx_1dx_2)\\
&-\int_B\rho\sum_{k=1}^{f}\sum_{j=1}^{i-1}\mathbf{Q}_j\bar{\mathbf{W}}_{ik}\nabla^2\phi_i\dot{\varphi}_k\dot{\varphi}_l[\mathbf{TrNp}]\mathbf{u}^\mathrm{e}dx_1dx_2\\
&-\int_B\rho\sum_{k=1}^{f}\sum_{j=1}^{i-1}\mathbf{Q}_j\bar{\mathbf{W}}_{ik}\nabla^2\phi_i\dot{\varphi}_k\dot{\varphi}_l[\mathrm{TRN}]\mathbf{u}^\mathrm{e}\mathbf{u}^{\mathrm{eT}}dx_1dx_2\\
&-\int_B\rho\sum_{k=1}^{f}\sum_{j=1}^{f}\bar{\mathbf{W}}_{ij}\bar{\mathbf{W}}_{ik}\varphi_k[\mathbf{TrNp}]\mathbf{u}^\mathrm{e}dx_1dx_2\\
&-\int_B\rho\sum_{k=1}^{f}\sum_{k=1}^{f}\bar{\mathbf{W}}_{ij}\bar{\mathbf{W}}_{ik}\varphi_k[\mathrm{TRN}]\mathbf{u}^\mathrm{e}\mathbf{u}^{\mathrm{eT}}dx_1dx_2\\
&-\int_B\rho\sum_{k=1}^{f}\bar{\mathbf{W}}_{ik}\bar{\mathbf{W}}_{ij}\bar{\mathbf{W}}_{ik}\dot{\varphi}_j\dot{\varphi}_k[\mathbf{TrNp}]\mathbf{u}^\mathrm{e}dx_1dx_2\\
&-\int_B\rho\sum_{k=1}^{f}\bar{\mathbf{W}}_{ik}\bar{\mathbf{W}}_{ij}\bar{\mathbf{W}}_{ik}\dot{\varphi}_j\dot{\varphi}_k[\mathrm{TRN}]\mathbf{u}^\mathrm{e}\mathbf{u}^{\mathrm{eT}}dx_1dx_2\\
&-2\int_B\rho\sum_{k=1}^{f}\sum_{j=1}^{f}\bar{\mathbf{W}}_{ij}\bar{\mathbf{W}}_{ik}\dot{\varphi}_j[\mathbf{TrNp}]\dot{\mathbf{u}}^\mathrm{e}dx_1dx_2\\
&-2\int_B\rho\sum_{k=1}^{f}\sum_{j=1}^{f}\bar{\mathbf{W}}_{ij}\bar{\mathbf{W}}_{ik}\dot{\varphi}_j[\mathrm{TRN}]\mathbf{u}^\mathrm{e}\dot{\mathbf{u}}^\mathrm{e}dx_1dx_2\\
&-\int_B\rho\sum_{k=1}^{f}\bar{\mathbf{W}}_{ik}[\mathbf{TrNp}]\ddot{\mathbf{u}}^\mathrm{e}dx_1dx_2\\
&-\int_B\rho\sum_{k=1}^{f}\bar{\mathbf{W}}_{ik}[\mathrm{TRN}]\mathbf{u}^\mathrm{e}\ddot{\mathbf{u}}^\mathrm{e}dx_1dx_2)\\
&-\int_B\rho\sum_{j=1}^{i-1}\mathbf{Q}_j\nabla^2\phi_i\dot{\varphi}_k\dot{\varphi}_l[\mathbf{TrNp}]dx_1dx_2\\
&-\int_B\rho\sum_{k=1}^{f}\bar{\mathbf{W}}_{ij}\ddot{\varphi}_k[\mathbf{TrNp}]dx_1dx_2)\\
&-\int_B\rho\bar{\mathbf{W}}_{ij}\bar{\mathbf{W}}_{ik}\dot{\varphi}_j\dot{\varphi}_k[\mathbf{TrNp}]dx_1dx_2\\
&-\int_B\rho\sum_{j=1}^{i-1}\mathbf{Q}_j\nabla^2\phi_i\dot{\varphi}_k\dot{\varphi}_l[\mathrm{TRN}]^\mathrm{T}\mathbf{u}^\mathrm{e}dx_1dx_2\\
&-\int_B\rho\sum_{k=1}^{f}\bar{\mathbf{W}}_{ik}\varphi_k[\mathrm{TRN}]\mathbf{u}^\mathrm{e}dx_1dx_2\\
&-\int_B\rho\bar{\mathbf{W}}_{ij}\bar{\mathbf{W}}_{ik}\dot{\varphi}_j\dot{\varphi}_k[\mathrm{TRN}]\mathbf{u}^\mathrm{e}dx_1dx_2\\
&-2\int_B\rho\sum_{j=1}^{i-1}\bar{\mathbf{W}}_{ij}\dot{\varphi}_j[\mathrm{TRN}]\dot{\mathbf{u}}^\mathrm{e}dx_1dx_2\\
&-\int_B\rho[\mathrm{TRN}]\ddot{\mathbf{u}}^\mathrm{e}dx_1dx_2
\end{aligned} \tag{18}$$

Neglect the little quantity of the second-order differentiation and keep the term of intersect , have :

$$\begin{aligned}
F_i^*=&-\int_B\rho\sum_{k=1}^{f}\bar{\mathbf{W}}_{ik}[\mathbf{TrNp}]\ddot{\mathbf{u}}^\mathrm{e}dx_1dx_2\\
&-2\int_B\rho\sum_{k=1}^{f}\sum_{j=1}^{f}\bar{\mathbf{W}}_{ij}\bar{\mathbf{W}}_{ik}\dot{\varphi}_j[\mathbf{TrNp}]\dot{\mathbf{u}}^\mathrm{e}dx_1dx_2\\
&-\int_B\rho\sum_{k=1}^{f}\sum_{j=1}^{f}\bar{\mathbf{W}}_{ij}\bar{\mathbf{W}}_{ik}\ddot{\varphi}_k[\mathbf{TNP}]dx_1dx_2\\
&-\int_B\rho\sum_{k=1}^{f}\sum_{j=1}^{f}\bar{\mathbf{W}}_{ij}\bar{\mathbf{W}}_{ik}\ddot{\varphi}_k[\mathbf{TrNp}]\mathbf{u}^\mathrm{e}dx_1dx_2\\
&-\int_B\rho\sum_{k=1}^{f}\sum_{j=1}^{f}\bar{\mathbf{W}}_{ij}\bar{\mathbf{W}}_{ik}\varphi_k[\mathbf{TrNp}]\mathbf{u}^\mathrm{e}dx_1dx_2\\
&-\int_B\rho\sum_{k=1}^{f}\bar{\mathbf{W}}_{ij}\ddot{\varphi}_k[\mathbf{TrNp}]dx_1dx_2)\\
&-\int_B\rho\sum_{k=1}^{f}\bar{\mathbf{W}}_{ik}\varphi_k[\mathrm{TRN}]\mathbf{u}^\mathrm{e}dx_1dx_2\\
&-2\int_B\rho\sum_{j=1}^{i-1}\bar{\mathbf{W}}_{ij}\dot{\varphi}_j[\mathrm{TRN}]\dot{\mathbf{u}}^\mathrm{e}dx_1dx_2\\
&-\int_B\rho[\mathrm{TRN}]\ddot{\mathbf{u}}^\mathrm{e}dx_1dx_2
\end{aligned} \tag{19}$$

Arrangement the above formula, we have:

(1) The generalized inertial force obtained by the rigid body displacement φ :

$$\begin{aligned}
F_i^*=&\\
&-\int_B\rho\sum_{k=1}^{f}\sum_{j=1}^{i-1}\bar{\mathbf{W}}_{ij}\{\bar{\mathbf{W}}_{ik}([\mathbf{TNP}]+[\mathbf{TrNp}]\mathbf{u}^\mathrm{e})+[\mathbf{TrNp}]\}\ddot{\varphi}_jdx_1dx_2)\\
&-2\int_B\rho\sum_{k=1}^{f}\bar{\mathbf{W}}_{ij}\{\sum_{j=1}^{f}\bar{\mathbf{W}}_{ik}[\mathbf{TrNp}]-[\mathrm{TRN}]\}\dot{\mathbf{u}}^\mathrm{e}\dot{\varphi}_jdx_1dx_2
\end{aligned}$$

$$-\int_B \rho \sum_{j=1}^{f} \bar{\mathbf{W}}_{ij} \{\sum_{k=1}^{i-1} \bar{\mathbf{W}}_{ik} [\mathbf{TrNp}] - [\mathrm{TRN}]\} \mathbf{u}^{\mathrm{e}} \varphi_j dx_1 dx_2$$

$$-\int_B \rho \{\sum_{k=1}^{f} \bar{\mathbf{W}}_{ik} [\mathbf{TrNp}] - [\mathrm{TRN}]\} \ddot{\mathbf{u}}^{\mathrm{e}} dx_1 dx_2 \qquad (20)$$

(2)The generalized inertial force obtained by the elastic displacement φ:

$$F_i^* = -\int_B \rho \{\sum_{k=1}^{f} \bar{\mathbf{W}}_{ik} [\mathbf{TrNp}] + [\mathrm{TRN}]\} \ddot{\mathbf{u}}^{\mathrm{e}} dx_1 dx_2$$

$$-2\int_B \rho \sum_{j=1}^{f} \bar{\mathbf{W}}_{ij} \{\sum_{k=1}^{f} \bar{\mathbf{W}}_{ik} [\mathbf{TrNp}] + [\mathrm{TRN}]\} \dot{\varphi}_j \dot{\mathbf{u}}^{\mathrm{e}} dx_1 dx_2$$

$$-\int_B \rho \sum_{k=1}^{f} \bar{\mathbf{W}}_{ik} \{\sum_{j=1}^{f} \bar{\mathbf{W}}_{ij} \ddot{\varphi}_k [\mathbf{TrNp}] + \sum_{j=1}^{f} \bar{\mathbf{W}}_{ij} \varphi_k [\mathbf{TrNp}] -$$

$$+\varphi_k [\mathrm{TRN}]\} \mathbf{u}^{\mathrm{e}} dx_1 dx_2$$

$$-\int_B \rho \sum_{j=1}^{i-1} \bar{\mathbf{W}}_{ij} \{\sum_{k=1}^{f} \bar{\mathbf{W}}_{ik} [\mathbf{TNP}] + [\mathbf{TrNp}]\} \ddot{\varphi}_k dx_1 dx_2 \qquad (21)$$

5 The generalized active force and the dynamics equation of motion [32,33]

By taking into account an arbitrary point of an arbitrary element e of body j, the stress-strain relation can be expressed as

$$\boldsymbol{\varepsilon}_{\alpha,\beta} = \tfrac{1}{2}(W_{\alpha,\beta} + W_{\beta,\alpha} + \sum_{r=1}^{3} W_{\gamma,\alpha} W_{\gamma,\beta}) \qquad (22)$$

Where $W_{\alpha,\beta} = \dfrac{\partial W_\alpha}{\partial x_\beta}$ W_a represents displacement component while X_a is the positional coordinate component.

Using equation(22), we have

$$\dot{\boldsymbol{\varepsilon}}_{\alpha,\beta} = \tfrac{1}{2}[\dot{w}_{\alpha,\beta} + \dot{w}_{\beta,\alpha} + \sum_{r=1}^{3} (\dot{w}_{r,\alpha} w_{r,\beta} + w_{r,\alpha} \dot{w}_{r,\beta}] \qquad (23)$$

The stress is denoted by: $\boldsymbol{\sigma} = [\sigma_{11}, \sigma_{22}, \sigma_{33}, \sigma_{12}, \sigma_{23}, \sigma_{31}]^{\mathrm{T}}$.

The generalized active force thus induced (σ ε) can be written as following two competent:

$F_e = \sum_{j=1}^{M_j} \mathbf{\Phi}^{\mathrm{T}} \mathbf{K}_{\mathrm{e}} \mathbf{\Phi} \mathbf{u}$ (φ is the characteristic vector matrix) $F_G = \sum_{j=1}^{M_j} \mathbf{\Phi}^T \mathbf{K}_{\mathrm{G}} \mathbf{\Phi} \mathbf{u}$ .Where $\mathbf{K}_{\mathrm{e}}$ is the elastic stiffness matrix, $\mathbf{K}_{\mathrm{G}}$ is the nonlinear constructive stiffness matrix. The generalized active force can be written as below:

$$F = F_{\mathrm{e}} + F_{\mathrm{G}} \qquad (24)$$

According to Kane's equation, the dynamic equation is written as: $F^* + F = 0$. (25)

6 Conclusions

This concludes the application of Kane equation to derive the equations of motion at the element. This development allows for the interdependence of the rigid body and the elastic motion. The elastic links are modeled by using the finite element method. These equations in their final form can be used for realistic modeling of links mechanisms with the rigid body motion and the elastic motion having closed and opened loop multiple degree of freedom chains and geometrically complex elastic links.

The system equations will be published separately.

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