THE MESHLESS MANIFOLD METHOD BASED ON THE PARTITION OF UNITY

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Abstract: The meshless manifold method based on circle covers is made of the partition of unity method and the finite cover approximation theory which provides a unified framework for solving problems dealing with both continuums and dis-continuums. In the method based on circle covers, two cover systems are employed. The mathematical covers system provides the nodes for forming finite covers of the solution domain and the partition of unity functions. The physical covers system describes geometry of the domain and the discontinuous surfaces in the domain. The shape functions in the method are derived by the partition of unity and the finite covers approximation theory. In meshless method based on circle covers , the mathematical finite cover approximation theory is used to model cracks that lead to interior discontinuities in the displacement. Therefore, the discontinuity is treated mathematically instead of empirically by the existing methods. The approximation functions and the equations of the coupling method are developed in detail in the paper. The validity and accuracy of the meshless manifold method are illustrated by numerical examples.

Key-words: Numerical manifold method; Partition of unity; meshless manifold method; Finite cover approximation; Crack growth

1. Introduction

In fracture problems, for instance, elements edges provide natural lines along which cracks can grow [1]. In these classes of problems, the finite element methods require remeshings to ensure that element boundaries coincide with moving discontinuities such as crack growth; projection or remapping of field quantities accompanies each remeshings, which can degrade the solution further. So an attractive option for such problems is meshless computational methods, due to their flexibility in solving boundary value problems, especially in problems with discontinuities, or with moving boundaries, or with severe deformations, which has been popularized in recent years.

In meshless methods the discretization is purely nodal, and the finite element concept of connectivity between elements is not introduced. The shape functions are constructed on a given nodal arrangement in the solving domain. By basing an analysis on nodes or particles, it demonstrates flexibility in modeling complex discontinuities [2,3]. It also avoids the distortion of mesh when extreme large deformation is encountered [4], and provides an efficient means for addressing high gradient problems such as that occurred in strain localization [5]. The advantage of meshless methods by Belytschko [6,7] is that it is possible to model arbitrary growth of cracks without remeshings and adaptive refinement at the crack tip is easily accomplished. With adequate refinement, stress intensity factors can be computed accurately. But the disadvantage of the method is that test functions have nearly constructed by the original visibility criterion [8] which is empirical method to model cracks leads to interior discontinuities in the displacements, especially arbitrary cracks.

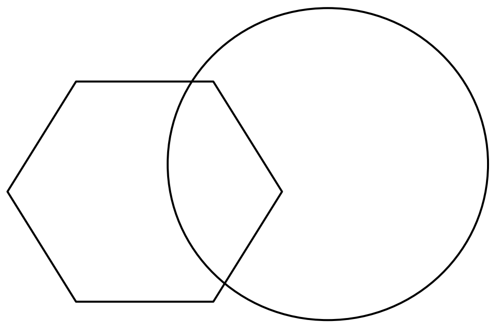
To overcome shortcomings of discontinuities in the meshless method, the meshless method based on circle covers is developed in the paper. The method is based on the partition of unity method and the finite cover approximation theory in the mathematical manifold. The basic idea of the method is derived from the numerical manifold method [9] which is extended to meshless method by circle covers in this paper. Two cover systems are employed in the meshless manifold method. The circle mathematical cover system provides the nodes for forming finite covers of the solution domain and the partition of unity functions; and the physical cover system describes geometry of the domain of the problem and the discontinuous surfaces in the domain. The shape functions in this method are formed by the partition of unity and the finite covers approximation theory, hence the shape functions are not affected by discontinuity of a domain. Therefore crack problems can be treated better. For local problems, the shape functions are more effective than that in other methods. As a result the method can avoid the disadvantages of other meshless methods in which the tip of a discontinuous crack is not considered. Compared with the conventional numerical manifold method [9], the shape of the mathematical covers can be selected easily, and the finite cover approximation and the partition of unity functions are formed with the influence domains of a series of nodes. Compared with the conventional meshless method, the test functions are not affected by the discontinuity in the solving domain because of the finite covers approximation theory. Therefore this method can overcome difficulties in the conventional meshless methods for problems with a discontinuous domain.

2. A Brief Review of Finite Covers in Manifold Method

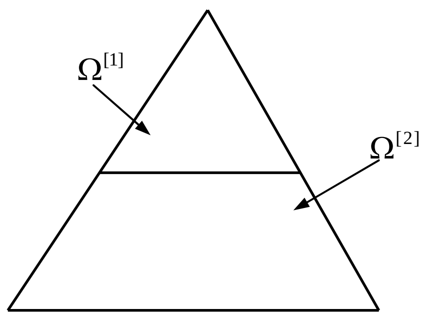
Following the formulation of the manifold method proposed by Shi [9], we present how the use of meshless finite covers instead of finite element covers is effective for modeling discontinuities and its growth in strain and displacement.

2.1 Mathematical and physical covers

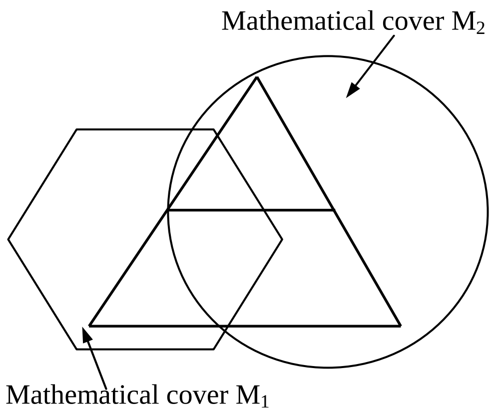
Based on finite cover systems, the newly developed "manifold method" has the potential to meet more engineering requirements. With reference to the schematic presented in Figure 1, we provide the definitions of domains and covers. A domain where mathematical functions independent of physics are introduced is called a mathematical domain, whereas a domain where physical quantities are defined is referred to as a physical domain. The mathematical domain can be constructed as a union of a finite number of patches, which can be overlapped either partially or totally. These patches are called mathematical covers which are chosen by engineers and may consist of finite overlapping covers which occupy the whole material volume. The conventional meshes and regions such as regular grids or finite element meshes can be transferred to finite mathematical covers. In the description of the problem, the mathematical domain or the union of mathematical covers need not coincide with the physical domain which is called the physical mesh includes the boundaries of material volume, joints, blocks and the interfaces of different material zones. The constantly changing water surfaces are also part of the physical meshes. The physical mesh represents material conditions which cannot be chosen artificially. The physical cover system is formed by both mathematical covers and physical meshes. If the joints or block boundary divide a mathematical cover to two or more completely disconnected domains, those domains are defined as physical covers. Therefore, physical covers are subdivision of mathematical covers by discontinuities. The manifold method is more suitable to compute large deformations, moving boundaries of both continuums and jointed or blocky materials. The above concept of cover systems is illustrated in the example shown in Fig. 1. The circle and the hexagon mesh are arbitrarily selected as the mathematical mesh, as shown in Fig. 1a. Fig. 1b shows the structure containing a crack that defines the physical mesh. The common region of the mathematical cover M_I and the physical mesh $\Omega^{[\alpha]}$ forms the physical covers, and is denoted by $P_I^{[\alpha]}$ in Fig. 1.



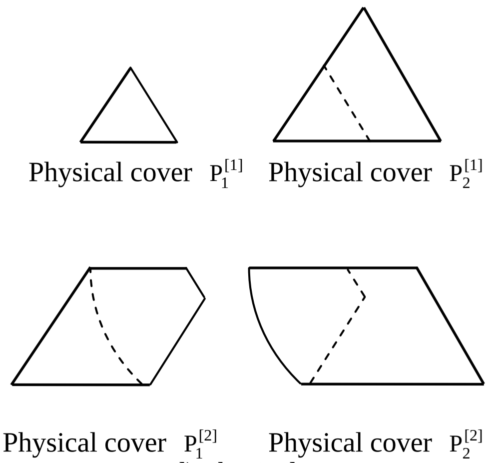
a) Mathematical mesh



b) Physical mesh



c) Overlapping two meshes



d) Physical covers

Fig. 1 Mathematical and physical covers

2.2 Approximation based on finite covers

In the following, the approximation properties facilitated in the meshless manifold method is presented assuming the one-dimensional version of the problem see [10] in detail.

Then the displacement $u(x) \in H^1(\Omega)$ is approximated as

$$u(x) \approx u^h(x) = \sum_{I=1}^{N_M} \sum_{i=1}^{n_i} \phi_I(x) u_I^{[\alpha]}(x) \quad \text{on } \Omega \quad (1)$$

where n_I is the number of physical covers associated with mathematical cover M_I and $u_I^{[\alpha]}(x)$ is cover displacement functions corres -ponding to the physical cover $P_I^{[\alpha]}$.

3. Cover Functions and Weight Funct -ions

In this section, the construction of the basis functions and test functions are described and some properties of these functions are reviewed in the meshless manifold method. One key idea used in the construction of the meshless manifold method is that of a finite cover approximation theory in manifold method and the partition of unity function. The finite cover approximation theory formed the sub-covers which constructed the approximation function by the partition of unity.

3.1 The partition of unity

Let M be open bounded domain in R^n, $n = 1, 2$ or 3 and Q_N denote an arbitrarily chosen set of N points $x_I \in M$ referred to as nodes

$$Q_N = \{x_1, x_2 \cdots, x_N\}, \quad x_I \in M \quad (2)$$

We associate with the set Q_N a finite open sub-cover M_I, $I = 1, \cdots, N$, which denotes a set of solid circles, balls or spheres which called the finite cover hereunder. x_I and h_I are the center and the radius of the sub-cover I, respectively. Hence $\Re_N = \{M_I, I = 1, \cdots, N\}$ constitutes an open cover of M

$$M \subset \bigcup_{I=1}^{N} M_I \quad (3)$$

where

$$M_I = \{y \in R^n : \|x_\alpha - y\| < h_\alpha\}$$

A class of functions $\Phi_N = \{\phi_\alpha, \alpha = 1 \cdots N\}$ subordinated to the sub-cover $\Re_N$ is called a partition of unity if it possesses the following properties:

$$\sum_I \phi_I(\boldsymbol{x}) = 1 \quad (4)$$

$$\phi_I(\boldsymbol{x}) = 0 \quad \boldsymbol{x} \notin \Omega_I \quad (5)$$

$$\phi_I(\boldsymbol{x}) \in C_0^s(\Omega_I) \quad 1 \le I \le N; s \ge 0 \quad (6)$$

The function $\phi_I(\boldsymbol{x})$ is called the partition of unity function.

In numerical manifold method, Shi [15] used the term "weighting function", which has the following properties:

$$\begin{cases} w_I(\boldsymbol{x}) \ge 0 & \boldsymbol{x} \in \Omega_I; \\ w_I(\boldsymbol{x}) = 0 & \boldsymbol{x} \notin \Omega_I \end{cases} \quad (7)$$

$$\sum_{I=1}^{N} w_I(\boldsymbol{x}) = 1 \quad \boldsymbol{x} \in \Omega \quad (8)$$

The properties content with the properties of the partition of unity, hence the numerical manifold method called the partition of unity based on mesh.

3.2 Approximation functions of the meshless manifold

We used the basic idea of the approximation based on finite covers in the section 2.2, for a continuous domain, the numbers of physical covers equal to one of the mathematical covers. The shape functions are the partition of unity functions of the local covers by

Eq. (1), the approximation function, $\boldsymbol{u}^h(\boldsymbol{x})$, can be expressed as

$$\boldsymbol{u}^h(\boldsymbol{x}) = \sum_{i=1}^{N_M} \phi_i(\boldsymbol{x}) \cdot \boldsymbol{u}_i \qquad (9)$$

where $\phi_i(\boldsymbol{x})$ is the partition of unity function of the local cover M_i, $\boldsymbol{u_i}$ is the value of the function at node i.

The same above process, for a discontinuous domain of two-dimensional problem, one cover is divided into two sub-covers at a discontinuity. The partition for sub-covers is also separated so that there is one partition associated with each sub-cover. To facilitate a sub-cover partition, let sub-cover $\overline{M}_i^j$ be the jth sub-cover of M_i. The partition of unity function for $\overline{M}_i^j$, denoted as $\phi_i^j(x)$, is given by

$$\phi_i^j(\boldsymbol{x}) = \phi_i(\boldsymbol{x}) \cdot \delta_i^j(\boldsymbol{x}) \qquad (10)$$

where $\phi_i(\mathbf{x})$ is the partition of unity function for cover i, and

$$\delta_i^j(\boldsymbol{x}) = 1 \qquad \boldsymbol{x} \in \overline{\Omega}_i^j \qquad (11)$$

$$\delta_i^j(\boldsymbol{x}) = 0 \qquad \boldsymbol{x} \notin \overline{\Omega}_i^j \qquad (12)$$

A geometric interpretation of the construction of $\delta_i^j(\boldsymbol{x})$ is illustrated in Fig. 2. The approximate functions are the partition of unity function multiplied $\delta_i^j(\boldsymbol{x})$ to the discontinuous domain.

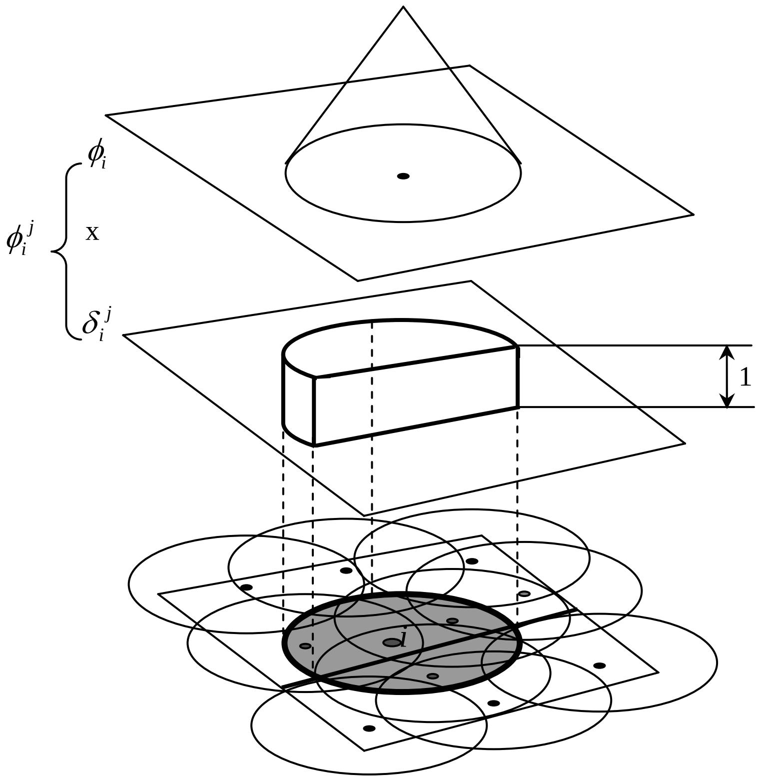


Fig. 2. A sub-cover and its partition of unity function

With the introduction of sub-covers, i.e. more than one covers subordinated to one node, the meshless method based on circle covers can model a discontinuous function over a disjoint local cover by the finite cover technology. To illustrate this, Fig. 4 depicts a single cover M_i with two sub-covers M_i^1 and M_i^2. Since each of the sub-cover has an independent cover with its own partition of unity function, a discontinuous approximation function as depicted in Fig. 3, is obtained by

$$\boldsymbol{u}^h(\boldsymbol{x}) = a_1\phi_i^1(\boldsymbol{x}) + a_2\phi_i^2(\boldsymbol{x}) \qquad (13)$$

where, a_1 and a_2 are the nodal values associated with P_i^1 and P_i^2, respectively. Hence, to the discontinuous domain, $\boldsymbol{u}^h(\boldsymbol{x})$ by the Eq. (1) can be expressed as

$$\boldsymbol{u}^h(\boldsymbol{x}) = \sum_{I=1}^{N_M}\sum_{i=1}^{n_i} \phi_I(\boldsymbol{x})\delta_I^j(\boldsymbol{x})u_I^{[\alpha]}(\boldsymbol{x}) \qquad \text{on } \Omega \qquad (14)$$

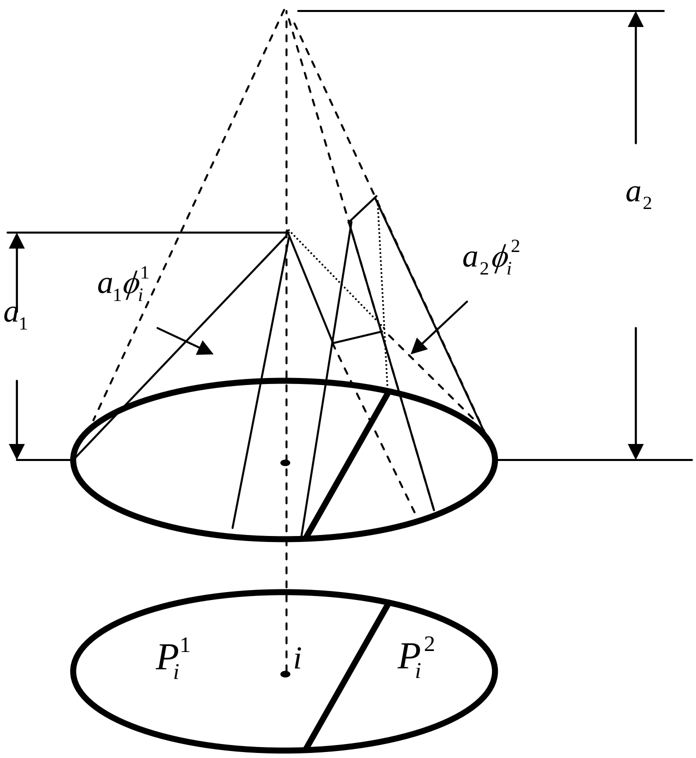


Fig. 3. Dividing sub-cover Ω_i into two sub-covers P_i^1 and P_i^2 and its partition of unity function

Given a cover $\Re_N = \{M_I, I = 1, \cdots, N\}$ we can then define a partition of unity and local approximation functions by using Shepard functions or Moving least square as $\phi_i(\boldsymbol{x})$, and local approximation spaces V_i^m on the cover M_i.

In summary, the above construction can be expressed as follows

$$\begin{pmatrix}\{x_i\}\\ w\\ \{m\}\end{pmatrix}\rightarrow\begin{pmatrix}\{M_i\}\\ \{W_i\}\\ \{V_i^m = span<\boldsymbol{\Psi}_i^n>\}\end{pmatrix}\rightarrow\begin{pmatrix}\{\phi_i\}\\ \{V_i^m\}\end{pmatrix} \quad (15)$$
$$\rightarrow V^h = \sum \phi_i V_i^m \rightarrow \boldsymbol{u}^h = \sum \phi_i \boldsymbol{u}_i$$

where w_i is the weight function. This process is further explained by the following algorithm.

Algorithm:

1. Providing a set of points $x_i\,(i = 1,\cdots,N)$ to the resolve the domain Ω;
2. Forming mathematical covers with the centers x_i and the radii h_i;
3. Forming the local physical cover (the local cover) M_i by the overlapping mathematical cover and the solving domain;
4. Generating the weight function w_i for the local cover M_i;
5. Constructing the partition of unity function ϕ_i for the local cover M_i.
6. Formed physical covers $P_i^{[\alpha]}$.
7. Generating approximation function on each physical cover.

A partition of unity can be generated by Shepard's method, i.e.

$$\phi_i(x) = \frac{w_i(x)}{\sum w_k(x)} \quad (16)$$

If we use MLS to construct the partition of unity ϕ_i, it is given by

$$\phi_i(\boldsymbol{x}) = \boldsymbol{p}^T(\boldsymbol{x})\boldsymbol{A}^{-1}(\boldsymbol{x})\boldsymbol{B}_i(\boldsymbol{x}) \quad (17)$$

where

$$\boldsymbol{p}^T(\boldsymbol{x}) = \{\boldsymbol{p}_1(\boldsymbol{x}), \boldsymbol{p}_2(\boldsymbol{x}), \cdots, \boldsymbol{p}_m(\boldsymbol{x})\} \quad (18)$$
$$\boldsymbol{A}_{ij}(\boldsymbol{x}) = (\boldsymbol{p}_i, \boldsymbol{p}_j)_x \quad (19)$$
$$\boldsymbol{B}_i(\boldsymbol{x}) = w_i(\boldsymbol{x})\boldsymbol{p}(\boldsymbol{x}_i) \quad (20)$$

in which m is the number of terms in the basis; $\boldsymbol{p}^T(\boldsymbol{x})$ are the monomial basis functions.

4. The Discrete Equation of the Meshless Method

Consider a two-dimensional domain Ω bounded by S. The equation of equilibrium is

$$\sigma_{ij,j} + f_i = 0, \quad \text{in } \Omega \quad (21)$$

where σ_{ij} is the stress tensor, f_i is the body force. The boundary conditions are

$$\sigma_{ij} n_j = \bar{t}_i, \quad \text{on } S_\sigma \quad (22)$$
$$u_i = \bar{u}_i, \quad \text{on } S_u \quad (23)$$

where $\bar{u}_i$ and $\bar{t}_i$ denote the prescribed displacements and tractions, respectively, n_j is the unit outward normal to S, and S_σ and S_u are complementary parts of S where essential and nature boundary conditions are prescribed.

The variational (or weak) form for Eq. (21) can be written as

$$\int_\Omega \nabla\delta\boldsymbol{u} : \boldsymbol{\sigma}(\boldsymbol{u})d\Omega - \int_\Omega \delta\boldsymbol{u}\cdot \boldsymbol{f}d\Omega - \int_{S_\sigma} \delta\boldsymbol{u}\cdot\bar{\boldsymbol{T}}dS - \delta W_u(\boldsymbol{u}) = 0 \quad (24)$$

where ∇ is the symmetric gradient operator. The term $\delta W_u(\boldsymbol{u})$ is required for enforcing the essential boundary conditions in a meshless manifold method [15].

For linear elasticity, the strain-displacement equation and the stress-strain law are

$$\boldsymbol{\varepsilon} = \frac{1}{2}(\nabla\boldsymbol{u} + (\nabla\boldsymbol{u})^T) \quad (25)$$
$$\boldsymbol{\sigma} = \boldsymbol{D} : \boldsymbol{\varepsilon} \quad (26)$$

which can be used to write the weak form in Eq. (24) in terms of the displacements $\boldsymbol{u}$. The discrete form can be obtained by Eq. (14) as approximations for $\boldsymbol{u}$ and $\delta\boldsymbol{u}$. This leads to the system of equations

$$\boldsymbol{K}\boldsymbol{u} = \boldsymbol{F} \quad (27)$$

where

$$\boldsymbol{K}_{IJ} = \int_\Omega \boldsymbol{B}_I^T \cdot \boldsymbol{D}\cdot \boldsymbol{B}_J d\Omega \quad (28)$$
$$\boldsymbol{F}_I = \int_{S_\sigma} \phi_I^T \bar{t}_I dS + \int_\Omega \phi_I^T \boldsymbol{f}_I d\Omega \quad (29)$$
$$D = \frac{E}{(1+\nu)(1-2\nu)}\begin{bmatrix}1-\nu & \nu & 0\\ \nu & 1-\nu & 0\\ 0 & 0 & \frac{1-2\nu}{2}\end{bmatrix} \quad (30)$$
$$\boldsymbol{B}_I = \begin{bmatrix}(\phi_I\delta_I^j)_{,x} & 0 & (\phi_I\delta_I^j)_{,y}\\ 0 & (\phi_I\delta_I^j)_{,y} & (\phi_I\delta_I^j)_{,x}\end{bmatrix}^T \quad (31)$$

in which $\boldsymbol{E}$ and ν are Young's modulus and Poisson's ratio, respectively.

5. Numerical Examples

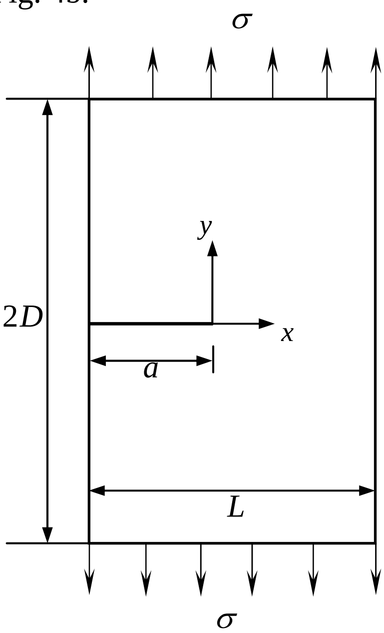
In this section, a classical problem will be solved an edge crack problem in fracture mechanics using the meshless manifold method to model discontinuous problems.

A rectangular plate with an edge crack is shown in Fig.4a. The plate is loaded in tension at the top with $\sigma = 0.2$Gpa and essential boundary conditions are applied at the bottom of the plate. The following parameters are used in the numerical simulations: $L = 52$mm, $D = 20$mm, $a = 12$mm;

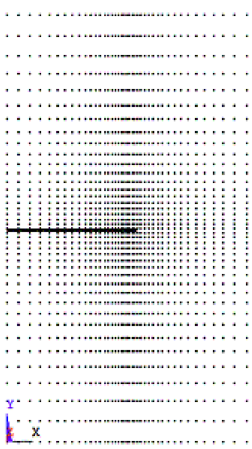
Elastic modulus $E = 76\text{Gpa}$; Poisson's ratio $\mu = 0.286$; Plane strain state of deformation is assumed.

The closed form solution for the crack is obtained by using the well-known near tip field in a domain about the crack tip and prescribing the displacements along the boundary of this field. The computed mode I stress intensity factors are compared with a finite geometry corrected value $K_I = C\sigma\sqrt{a\pi}$ where the correction is given by Ewalds and Wanhill [11].

Linear and quadratic bases with Gaussian weight function and a cover diameter of $d_{\max}$ = 4.0 mm are used in the numerical simulations. For linear and quadratic bases, a node system with 1722 regular nodes is used to form the local covers, as shown in Fig. 4b.



(a) The geometric model



(b) The distributing nodes of the model

Fig. 4 an edge crack model

The stress intensity factor K_I is normalized by $\sigma\sqrt{a\pi}$ and the normalized K_I value by the linear and the quadratic basis functions are calculated to be 1.32 and 1.40, respectively. The analytical solution is $C = 1.44$. The errors for linear and quadratic bases are 8.3% and 2.7%, respectively. It clearly indicated that the quadratic of basis function results in a higher accuracy of solution at the same number of node.

The stress field in front of the crack tip is shown in Fig. 5 and Fig. 6. It can be seen from the figures that the numerical solutions agree well with the exact solution. It can also be seen that the singularity at the crack tip is better modeled by the nonlinear (quadratic) bases method.

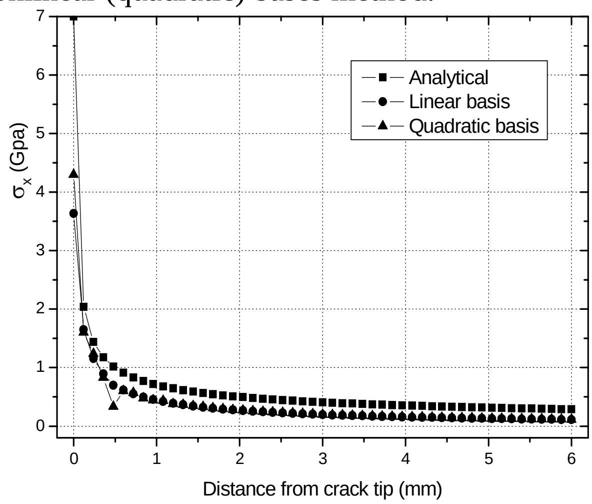


Fig. 5 Stresses σ_x ahead of the crack tip for the edge crack problem

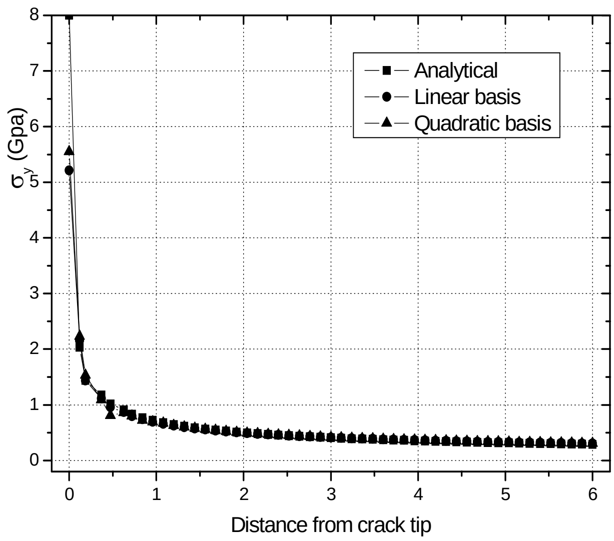


Fig. 6 Stresses σ_y ahead of the crack tip for the edge crack problem

The displacement field around the crack tip is shown in Fig.7. The two solutions agree well with the exact solution. It can be seen that the quadratic bases provides a better solution than that linear bases.

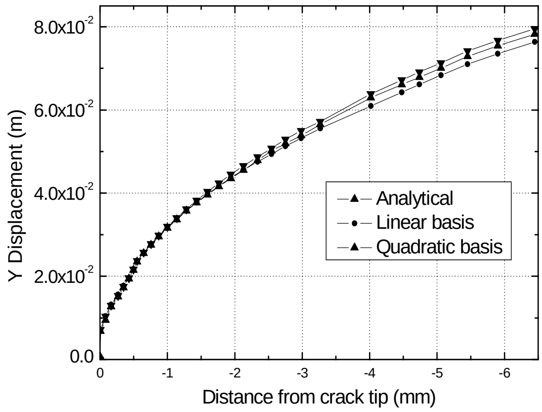


Fig. 7 Y-displacements behind the crack tip for the edge crack problem

6. Conclusions

The meshless manifold method is presented in this paper. The proposed method is based on the mathematics of partition of unity and finite cover approximation theory for the discontinuous problems in the solving domain. It constitutes a new class of meshless method.

When problems of crack are solved with the method, each node in the affected domain is separated into two or more nodes. All nodes that are not affected by the crack remain unchanged. As a result, arbitrary crack can be treated easily.

The meshless manifold method is given a kind of the mathematical method to treat discontinuity in solving domain. The method mainly used the finite covers approximation theory to model cracks that lead to interior discontinuity of displacement. Consequently, the method overcomes the shortcomings of the empirical methods that include three methods of the visibility criterion.

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