Stabilizer set of linear systems over commutative rings under feedback equivalence

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Abstract:- Let $\mathcal{M}$ the set of linear systems $\mathcal{M} = \{\dot{x} = Ax + Bu\}$ with scalars in a commutative ring R. In this paper we study the stabilizer set of the group action of the full feedback group $\mathcal{G} = \{(P, Q, K)\}$ with $P \in Gl_n(R)$, $Q \in Gl_m(R)$ and $K \in R^{m\times n}$ on the set of m-input n-dimensional linear systems $\mathcal{M} = R^{n\times n} \times R^{n\times m}$ under feedback equivalence.

Key-Words:- Linear systems, Feedback equivalence.

1 Introduction

Linear systems over commutative rings have largely studied in recent last years. This systems appear in literature when for example, one studies linear systems depending on a parameter or linear systems with delays. We recall that in practice, the control requires time to respond, so the feedback equation is in the form $\dot{x}(t) = Ax(t)+BFx(t-a)+BFu(t)$, the case $a = 0$ is given only if the mechanism respond instantaneously. This kind of systems are called systems with delays and they can be studied as a systems with coefficients in the ring of polynomials in one determinate over a field.

Let R be a commutative ring. We denote by $\mathcal{M}$ the set of pairs of matrices $\{x = (A, B) \in R^{n\times n} \times R^{n\times m}\}$ representing m-input linear systems $\dot{x} = Ax + Bu$ over R.

One of the topics in linear systems is analyze the possibility of adjusting the eigenvalues of a given system by feedback.

The feedback group acting on such systems is the group generated by the following elementary actions:

A1 Change of basis $P^{-1} \in Gl_n(R)$ in the state-space R^n, which transforms:

$$\begin{array}{rcl} A & \to & A_1 = PAP^{-1} \\ B & \to & B_1 = PB \end{array}$$

A2 Change of basis $Q \in Gl_m(R)$ in the input-space R^m, which transforms:

$$\begin{array}{rcl} A & \to & A_1 = A \\ B & \to & B_1 = BQ \end{array}$$

A3 Feedback action $F \in R^{m\times n}$, which transforms:

$$\begin{array}{rcl} A & \to & A_1 = A + BF \\ B & \to & B_1 = B \end{array}$$

Definition 1 *We say that the systems x and x_1 are feedback equivalent if x can be transformed to x_1 by one ore more elements of the feedback group acting on $\mathcal{M}$:*

Two systems $x = (A, B)$ and $x_1 = (A_1, B_1)$ in $\mathcal{M}$ are equivalent if and only if

$$(A_1, B_1) = (P^{-1}AP + P^{-1}BK, P^{-1}BQ)$$

for some $P \in Gl_n(R)$, $Q \in Gl_m(R)$, $K \in R^{m\times n}$.

The stabilizer is the set of the actions leaving fixed the given system. So the equivalent systems under feedback equivalence can be observed as a quotient space of the feedback group by the stabilizer.

The aim of this paper is the characterization of stabilizer set of a pair of matrices under equivalence relation considered in terms of the set of solutions of a matrix equation.

2 Equivalence relation as a group action

The equivalence relation defined can be seen as an action over $\mathcal{M}$ by a certain group.

Let $\mathcal{G} = Gl_n(R) \times Gl_m(R) \times R^{m\times n}$ be the feedback group. Using short notations $g = (P, Q, K) \in \mathcal{G}$ and $x = (A, B) \in \mathcal{M}$, we note that multiplication in $\mathcal{G}$, action of the group $\mathcal{G}$, and equivalence condition are as follows

$$
\begin{aligned}
& g_1 g_2 = (P_1P_2, Q_1Q_2, K_1P_2 + Q_1K_2) \in \mathcal{G}, \\
& g \circ x = P^{-1} \begin{pmatrix} A & B \end{pmatrix} \begin{pmatrix} P & \mathbf{0} \\ K & Q \end{pmatrix} = \\
& = (P^{-1}AP + P^{-1}BK, P^{-1}BQ) \in \mathcal{M}, \\
& x_2 = g \circ x_1.
\end{aligned}
$$

Unit element of $\mathcal{G}$ has the form $e = (\mathbf{1}_n, \mathbf{1}_m, 0)$, where $\mathbf{1}_n$ and $\mathbf{1}_m$ are the identity matrices. The inverse element of $g = (P, Q, K)$ is $g^{-1} = (P^{-1}, Q^{-1}, -Q^{-1}KP^{-1})$.

We observe that the operation $\circ$ is an action over $\mathcal{M}$ by the group $\mathcal{G}$, because verifies the necessary and sufficient conditions:

1. associative: $g_2 \circ (g_1 \circ x) = (g_1g_2) \circ x$,
2. the unit element is the unit operator: $e \circ x = x \quad \forall x \in \mathcal{M}$.

This action permit us to describe the equivalence classes as orbits.

3 Orbit and stabilizer

Let us fix some pair of matrices $x_0 = (A_0, B_0)$ and define the mapping $\alpha_{x_0} : \mathcal{G} \longrightarrow \mathcal{M}$, $\alpha_{x_0}(g) = g \circ x_0$, $g \in \mathcal{G}$. Then, the orbit $\mathcal{O}(x_0)$ and stabilizer set $\mathcal{S}(x_0)$ of the pair x_0 are defined as follows:

$$
\begin{aligned}
\mathcal{O}(x_0) &= \{x \in \mathcal{M} | \exists g \in \mathcal{G} \text{ s. t. } \alpha_{x_0}(g) = x\}, \\
\mathcal{S}(x_0) &= \{g \in \mathcal{G} | \alpha_{x_0}(g) = x_0\}.
\end{aligned} \tag{1}
$$

The orbit of x_0 is the set of pairs equivalent to x_0. It is clear that $\mathcal{S}_{x_0}$ is a subgroup of $\mathcal{G}$.

Proposition 1 *Let* $x_1 \in \mathcal{O}(x_0)$. *Then, the stabilizer sets* $\mathcal{S}(x_0)$ *and* $\mathcal{S}(x_1)$ *are related in the form:*

$$\mathcal{S}(x_1) = g_1 \mathcal{S}(x_0) g_1^{-1},$$

where $g_1 \in \mathcal{G}$ *is such that* $g_1 \circ x_0 = x_1$.

Proof. For all $g \in \mathcal{S}(x_0)$ we have $g \circ x_0 = x_0$, then $g \circ (g_1^{-1} \circ x_1) = g_1^{-1} \circ x_1$ and $(g_1 g g_1^{-1}) \circ x_1 = x_1$. □

So, it suffices to compute the stabilizer set of a selected element in the orbit.

Moreover, the map α_{x_0} induces a bijection $\beta : \mathcal{G}/\mathcal{S}_{x_0} \longrightarrow \mathcal{O}(x_0)$ in the following manner. Let $\mathcal{S}(x_0)$, we can model the $\mathcal{G}$-action on the orbit $\mathcal{O}(x_0)$ as follows, we denote by $\widetilde{g} = g\mathcal{S}(x_0) = \{gg_1 \mid \forall g_1 \in \mathcal{S}(x_0)\}$, and we consider the left translation:

$$
\begin{aligned}
t : \mathcal{G} \times \mathcal{G}/\mathcal{S}(x_0) &\longrightarrow \mathcal{G}/\mathcal{S}(x_0) \\
(g, \widetilde{g_1}) &\longrightarrow \widetilde{gg_1}
\end{aligned}
$$

now, we define $\beta : \mathcal{G}/\mathcal{S}(x_0) \longrightarrow \mathcal{O}(x_0)$ by $\beta(\widetilde{g}) = \alpha_{x_0}(g)$. The map β is a bijection and the diagram

$$
\begin{array}{ccc}
\mathcal{G} \times \mathcal{G}/\mathcal{S}(x_0) & \xrightarrow{t} & \mathcal{G}/\mathcal{S}(x_0) \\
{\scriptstyle id\times\beta} \downarrow & & {\scriptstyle \beta} \downarrow \\
\mathcal{G} \times \mathcal{O}(x_0) & \longrightarrow & \mathcal{O}(x_0)
\end{array} \tag{2}
$$

is commutative.

We are interested in computing the stabilizer sets for any pair of matrices (A, B) in $\mathcal{M}$, and we have the following proposition.

Proposition 2 $\mathcal{S}(A, B)$ *is the subset of elements* $g = (P, Q, K) \in \mathcal{G}$ *solution of the following generalized Sylvester equation with scalars in a commutative ring* R:

$$
\left.
\begin{aligned}
[A, P] + BK &= \mathbf{0} \\
BQ - PB &= \mathbf{0}
\end{aligned}
\right\} \tag{3}
$$

Proof. It suffices to observe that $\mathcal{S}(A,B)$ is the set of elements $(P,Q,K) \in \mathcal{G}$ such that

$$P\begin{pmatrix} A & B \end{pmatrix} = \begin{pmatrix} A & B \end{pmatrix}\begin{pmatrix} P & \mathbf{0} \\ K & Q \end{pmatrix}.$$

□

This equation has been solved by M.A. Beitia, J.M. Gracia, I. de Hoyos [1], in the case where R is the field of complex numbers.

Notice that, the generalized Sylvester equation (3), can be seen as the Ker of the following R-module morphism defined over the algebra $\widetilde{\mathcal{G}} = R^{n\times n} \times R^{n\times n} \times R^{m\times n}$ associate to the group $\mathcal{G}$ to the space of pairs of matrices

$$\begin{aligned} \widetilde{\alpha}_{x_0} : \widetilde{\mathcal{G}} &\longrightarrow \mathcal{M} \\ (X,Y,Z) &\longrightarrow X\begin{pmatrix} A & B \end{pmatrix} - \begin{pmatrix} A & B \end{pmatrix}\begin{pmatrix} X & \mathbf{0} \\ Z & Y \end{pmatrix} \end{aligned}$$

Corollary 1

$$\mathcal{S}(A,B) = \operatorname{Ker}\widetilde{\alpha}_{x_0} \cap \mathcal{G}.$$

Suppose now that R is a continuous field, so $\mathcal{G}$ is an analytic group and $\mathcal{M}$ an analytic variety. In this case $\widetilde{\mathcal{G}}$ is the tangent space to $\mathcal{G}$ and

1. $T_{x_0}\mathcal{O}(x_0) = \operatorname{Im}\widetilde{\alpha}_{x_0} \subset \mathcal{M}$,
2. $T_e\mathcal{S}(x_0) = \operatorname{Ker}\widetilde{\alpha}_{x_0} \subset T_e\mathcal{G}$.

4 Explicit description of stabilizer set

Proposition 1, ensures that it suffices to solve system 3, for reduced canonical forms of the orbits to obtain explicit description of stabilizer sets of pairs of matrices.

Let R be a commutative ring such that finitely generated projective R-modules are free (for example, $R = k$ a field or R a principal ideal domain, a local ring or a polynomial ring over a field $R = k[x_1, ..., x_n]$). Let $x = (A,B) \in R^{n\times n} \times R^{n\times m}$ be a linear system such that the feedback invariant module $M_1^x = Coker\,(B) = R^n/Im\,(B)$ is projective and hence free. Then, by [2, Th. 1.11] x is feedback equivalent to a system on the form

$$\left(\begin{pmatrix} \mathbf{0} & \mathbf{0} \\ B_1 & A_1 \end{pmatrix}, \begin{pmatrix} \mathbf{1}_{\tau_1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}\right),$$

where $\tau_1 = n - rg\,(M_1^x)$.

Theorem 1 *The stabilizer set $\mathcal{S}\left((A,B)\right)$ of a system (A,B) of the above form is*

$$\mathcal{S}\left((A_1,B_1)\right) \times Gl_{m-\tau_1}(R) \times R^{(m-\tau_1)\times\tau_1} \times \\ \times R^{(m-\tau_1)\times\tau_1} \times R^{(m-\tau_1)\times(n-\tau_1)}$$

in the sense of an element $g = ((P,Q,F), U, K, L, M) \in \mathcal{S}\left((A,B)\right)$ acts on the linear system (A,B) as the action

$$\left(\begin{pmatrix} Q & F \\ \mathbf{0} & P \end{pmatrix}, \begin{pmatrix} Q^{-1} & \mathbf{0} \\ K & U \end{pmatrix}, \begin{pmatrix} FB_1 & FA_1 \\ L & M \end{pmatrix}\right),$$

which stabilize (A,B); that is, $g \circ (A,B) = (A,B)$.

Proof. Consider the equations $PB = BQ$ and $[P,A] = BF$ and solve for P,Q,F :

$$\begin{pmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{pmatrix}\begin{pmatrix} \mathbf{1}_{\tau_1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} = \begin{pmatrix} \mathbf{1}_{\tau_1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}\begin{pmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{pmatrix}$$

that is to say

$$\begin{pmatrix} P_{11} & \mathbf{0} \\ P_{21} & \mathbf{0} \end{pmatrix} = \begin{pmatrix} Q_{11} & Q_{12} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}$$

equivalently we have that,

$$P_{11} = Q_{11} \text{ and } P_{12} = \mathbf{0} \text{ and } Q_{12} = \mathbf{0}$$

Now from the equality $[P,A] = BF$ one has

$$\begin{pmatrix} P_{11} & P_{12} \\ \mathbf{0} & P_{22} \end{pmatrix}\begin{pmatrix} \mathbf{0} & \mathbf{0} \\ B_1 & A_1 \end{pmatrix} - \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ B_1 & A_1 \end{pmatrix}\begin{pmatrix} P_{11} & P_{12} \\ \mathbf{0} & P_{22} \end{pmatrix}$$

$$= \begin{pmatrix} P_{12}B_1 & P_{12}A_1 \\ P_{22}B_1 - B_1P_{11} & P_{22}A_1 - B_1P_{12} - A_1P_{22} \end{pmatrix}$$

$$= \begin{pmatrix} \mathbf{1}_{\tau_1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}\begin{pmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{pmatrix} = \begin{pmatrix} F_{11} & F_{12} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}.$$

That is to say

$$\begin{aligned} F_{11} &= P_{12}B_1 \\ F_{12} &= P_{12}A_1 \\ P_{22}B_1 - B_1P_{11} &= \mathbf{0} \\ P_{22}A_1 - B_1P_{12} - A_1P_{22} &= \mathbf{0} \end{aligned}$$

And the last two equalities correspond to the stabilizer of system (A_1, B_1). To obtain the notation of the Theorem we rename:

$$F = P_{12} \; Q = P_{11} \; U = P_{22}$$

The rest of indeterminate are freely chosen. □

Theorem 1 *The stabilizer set of the system* $\left((\mathbf{0}_{n\times n}), \left(\mathbf{1}_n, \mathbf{0}_{n\times(m-n)}\right)\right)$ *is*

$$R^{(m-n)\times n} \times Gl_{m-n}(R) \times Gl_n(R) \times R^{(m-n)\times n}$$

in the sense of an element $g = (X, U, V, Y) \in \mathcal{S}\left((\mathbf{0}_{n\times n}), \left(\mathbf{1}_n, \mathbf{0}_{n\times(m-n)}\right)\right)$ *acts on the linear system* $(\mathbf{0}_{n\times n}), \left(\mathbf{1}_n, \mathbf{0}_{n\times(m-n)}\right)$ *as the action*

$$\left(V, \begin{pmatrix} V^{-1} & \mathbf{0} \\ X & U \end{pmatrix}, \begin{pmatrix} \mathbf{0} \\ Y \end{pmatrix}\right)$$

which stabilize $\left((\mathbf{0}_{n\times n}), \left(\mathbf{1}_n, \mathbf{0}_{n\times(m-n)}\right)\right)$*; that is,* $g\left((\mathbf{0}_{n\times n}), \left(\mathbf{1}_n, \mathbf{0}_{n\times(m-n)}\right)\right) = \left((\mathbf{0}_{n\times n}), \left(\mathbf{1}_n, \mathbf{0}_{n\times(m-n)}\right)\right)$.

Proof. It is analogous to the above result. □

As a consequence of the above results we can give the stabilizer set of a Brunovsky system over a projective-free ring. The description of the stabilizer set depends on the invariants $M_i = Coker\left(B, AB, ..., A^{i-1}B\right)$.

Note that if R is a commutative ring such that finitely generated projective R-modules are free then a m-input reachable linear system $x = (A, B)$ over R^n is of Brunovsky type (feedback equivalent to a Brunovsky canonical form) if and only if all invariant modules $M_i^x = R^n/\mathrm{Im}\left(B, AB, ..., A^{i-1}B\right)$ are projective and hence free. For this class of linear systems Theorem 1 can be applied recursively; that is to say

$$\mathcal{S}(A, B) = \mathcal{S}(A_1, B_1) \times Gl_{m-\tau_1} \times R^{(m-\tau_1)\times\tau_1} \times R^{(m-\tau_1)\times\tau_1} \times R^{(m-\tau_1)\times(n-\tau_1)},$$

and τ_1-input reachable linear system (A_1, B_1) over $R^{n-\tau_1}$ is newly of Brunovsky type. Therefore one has a new reachable linear system (A_2, B_2) with $(\tau_2 - \tau_1)$-inputs over $R^{n-\tau_2}$ such that the stabilizer set $\mathcal{S}(A_1, B_1)$ splits as a direct product where a direct factor is the stabilizer set $\mathcal{S}(A_2, B_2)$. Our next result deals with the problem of giving the stabilizer set $\mathcal{S}(x)$ of a Brunovsky linear system x in terms of the controllability indices of x.

Example 1 A linear system $x = (A, B)$ with $m = 6, n = 5$ and $M_1 = R^2$ and $M_2 = 0$ are equivalent to the Brunovsky canonical form

$$\left(\left(\begin{array}{cc|cc} \multicolumn{2}{c|}{\mathbf{0}_{3\times3}} & \multicolumn{2}{c}{\mathbf{0}_{2\times2}} \\ \hline \mathbf{1}_2 & 0 & 0 & 0 \end{array}\right), \left(\begin{array}{c|c} \mathbf{1}_3 & \mathbf{0}_{3\times3} \\ \hline \mathbf{0}_{2\times3} & \mathbf{0}_{2\times3} \end{array}\right)\right)$$

$$= \left(\begin{pmatrix} 0&0&0&0&0 \\ 0&0&0&0&0 \\ 0&0&0&0&0 \\ 1&0&0&0&0 \\ 0&1&0&0&0 \end{pmatrix}, \begin{pmatrix} 1&0&0&0&0&0 \\ 0&1&0&0&0&0 \\ 0&0&1&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&0&0&0&0 \end{pmatrix}\right)$$

Or in its (Usual) Canonical form

$$x_c = \left(\begin{pmatrix} 0&0&0&0&0 \\ 1&0&0&0&0 \\ 0&0&0&0&0 \\ 0&0&1&0&0 \\ 0&0&0&0&0 \end{pmatrix}, \begin{pmatrix} 1&0&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&1&0&0&0&0 \\ 0&0&0&0&0&0 \\ 0&0&1&0&0&0 \end{pmatrix}\right)$$

The stabilizer set is, By Theorem 1

$$\mathcal{S}\left((\mathbf{0}_{2\times2}), (\mathbf{1}_2, \mathbf{0}_{2\times1})\right) \times Gl_{6-3}(R) \times R^{(6-3)\times3} \times R^{(6-3)\times3} \times R^{(6-3)\times(5-3)} =$$

$$= \mathcal{S}\left((\mathbf{0}_{2\times2}), (\mathbf{1}_2, \mathbf{0}_{2\times1})\right) \times Gl_3(R) \times R^{3\times3} \times R^{3\times3} \times R^{3\times2} = (a)$$

By Theorem 2

$$(a) = \left(R^{(3-2)\times2} \times Gl_{3-2}(R) \times Gl_2(R) \times R^{(3-2)\times2}\right) \times Gl_3(R) \times R^{3\times3} \times R^{3\times3} \times R^{3\times2} =$$

$$R^{1\times2} \times Gl_1(R) \times Gl_2(R) \times R^{1\times2} \times Gl_3(R) \times R^{3\times3} \times R^{3\times3} \times R^{3\times2}.$$

In the general case, let R be a projective-free ring. Let $x = (A, B)$ be a m-input Brunovsky system over R^n. It follows by [4] that (1) the feedback invariants $N_i^x = \mathrm{Im}\left(B, AB, ..., A^{i-1}B\right)$, $M_i^x = R^n/N_i^x$ and N_i^x/N_{i-1}^x are free for all $i \geq 0$ (we set $N_0^x = 0$);

and (2) there exists an index $s > 0$ such that $N_s^x = R^n$ and consequently $M_s^x = 0$ because of x is reachable.

Put

$$N_i^x \cong R^{\tau_i},\ M_i^x \cong R^{x_i}, \text{ and } N_i^x/N_{i-1}^x \cong R^{\xi_i}$$

and note that one has:

(1) $0 = \tau_0 < \tau_1 < \cdots < \tau_s = n$

(2) $x_i = n - \tau_i$

(3) $\xi_i = \tau_i - \tau_{i-1}$ and that $\xi_{i+1} \leq \xi_i$.

Theorem 3 *With the above notations the decomposition of system x is obtained in the following way:*

(a) System (A, B) can be brought (by a suitable feedback action) to

$$\left(\left(\begin{array}{c|c} \mathbf{0} & \mathbf{0} \\ \hline B_1 & A_1 \end{array}\right), \left(\begin{array}{c|c} \mathbf{1}_{\xi_1} & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0} \end{array}\right)\right)$$

where (A_1, B_1) is a ξ_1-input Brunovsky system over R^{x_1}

(b) For $i = 1, ..., s-1$; system (A_i, B_i) can be brought (by a suitable feedback action on the whole system (A, B)) to

$$\left(\left(\begin{array}{c|c} \mathbf{0} & \mathbf{0} \\ \hline B_{i+1} & A_{i+1} \end{array}\right), \left(\begin{array}{c|c} \mathbf{1}_{\xi_{i+1}} & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0} \end{array}\right)\right)$$

where (A_{i+1}, B_{i+1}) is a $\xi_{i+1}-$input Brunovsky system over $R^{x_{i+1}}$

Proof. (**Sketch**, see [2] for details). It suffices to complete bases taking in mind that, because of invariant modules N_i^x , M_i^x, and N_i^x/N_{i-1}^x are free, one has the following decompositions of R^m and R^n

$$\begin{array}{rcl} R^m & \cong & N_1^x \oplus M_1^x \\ R^n & \cong & N_1^x \oplus N_2^x/N_1^x \oplus \cdots \oplus N_s^x/N_{s-1}^x \end{array} \qquad \square$$

As a consequence of the above result it is clear that we may give the stabilizer set of a Brunovsky linear system over a projective-free ring only from the knowledge of its feedback invariants: Suppose that we have the ranks

$$rk\left(B, AB, ..., A^{i-1}B\right) = \tau_i$$

for all $i \geq 1$. Since $x = (A, B)$ is Brunovsky it follows that there exists a index s such that $\tau_{s-1} < \tau_s = n$. We may obtain $\xi_1 = \tau_1$ and $\xi_i = \tau_i - \tau_{i-1}$. Then

$$\mathcal{S}\left(A_s, B_s\right) = \\ Gl_{\xi_{s-1}-\xi_s} \times R^{(\xi_{s-1}-\xi_s)\times\xi_s} \times R^{(\xi_{s-1}-\xi_s)\times\xi_s},$$

and

$$\mathcal{S}\left(A_{s-1}, B_{s-1}\right) = \\ \mathcal{S}\left(A_s, B_s\right) \times Gl_{\xi_{s-2}-\xi_{s-1}} \times R^{(\xi_{s-2}-\xi_{s-1})\times\xi_{s-1}} \\ \times R^{(\xi_{s-2}-\xi_{s-1})\times\xi_{s-1}} \times R^{(\xi_{s-2}-\xi_{s-1})\times\sigma_{s-1}},$$

and in a finite steps obtain the sequence $\mathcal{S}\left(A_s, B_s\right)$, $\mathcal{S}\left(A_{s-1}, B_{s-1}\right)$, ,..., $\mathcal{S}\left(A_1, B_1\right)$, $\mathcal{S}\left(A, B\right)$. In particular we have the following result:

Corollary 2 *The stabilizer set of a Canonical Controller form is $R^\times = Gl_1(R) = Units(R)$ and it is independent of the dimension n of system.*

5 Conclusions

In the set of linear systems $\mathcal{M} = \{\dot{x} = Ax + Bu\}$ with scalars in a commutative ring R, we study the stabilizer set of the group action of the full feedback group $\mathcal{G} = \{(P, Q, K)\}$ with $P \in Gl_n(R)$, $Q \in Gl_m(R)$ and $K \in R^{m\times n}$ on the set of m-input n-dimensional linear systems $\mathcal{M} = R^{n\times n} \times R^{n\times m}$ under feedback equivalence.

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