Controllability indices and Segre characteristic for multi-input standardizable singular systems

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Abstract:- Let (E, A, B) be a triple of matrices with $E, A \in M_n(\mathbb{C})$, $B \in M_{n\times m}(\mathbb{C})$, representing a singular time-invariant linear system, $E\dot{x} = Ax + Bu$. In this paper we present a collection of invariants that we will call controllability indices, and Segre characteristic for singular systems in terms of ranks of certain matrices, that can permit us to describe reduced canonical form for standardizable singular systems.

Key-Words:- Singular systems, feedback and derivative feedback, controllability, canonical forms

1 Introduction

Let us consider a finite-dimensional singular linear time-invariant system $E\dot{x}(t) = Ax(t) + Bu(t)$ where $(E, A, B) \in M = M_n(\mathbb{C}) \times M_n(\mathbb{C}) \times M_{n\times m}(\mathbb{C})$.

Different useful and interesting equivalence relations between singular systems have been defined. We deal with the equivalence relation $(E', A', B') = (QEP + QBF_E, QAP + QBF_A, QBR)$. with $P, Q \in Gl(n; \mathbb{C})$, $R \in Gl(m; \mathbb{C})$, $F_A, F_E \in M_{m\times n}(\mathbb{C})$, that is to say the equivalence relation accepting one or more, of the following standard transformations: basis change in the state space, input space, feedback, derivative feedback and premultiplication by an invertible matrix.

One of the topics largely studied in control theory for standard systems (systems with $E = I_n$), is the controllability, for singular systems L. Dai [1] gives the following definition of controllable singular systems.

Definition 1 *The system (E, A, B) is called controllable if and only if, for any $t_1 > 0$, $x(0) \in \mathbb{C}^n$ and $w \in \mathbb{C}^n$, there exists a control $u(t)$ such that $x(t_1 = w$.*

Equivalently,

Proposition 1 *A system $(E, A, B) \in M$ is controllable if and only if*

$$\begin{array}{l} rank\begin{pmatrix} E & B\end{pmatrix} = n \\ rank\begin{pmatrix} sE - A & B\end{pmatrix} = n, \text{ for all } s \in \mathbb{C} \end{array}$$

It is not difficult to prove that the controllability character is invariant under equivalence relation considered.

Systems (E, A, B) verifying the first condition for controllability: rank $\begin{pmatrix} E & B\end{pmatrix} = n$, are called standardizable systems, these are systems (E, A, B), for which there exist a derivative feedback F_E making invertible the matrix $E + BF_E$, and as a consequence, the equivalent system $((E + BF_E)^{-1}E + (E + BF_E)^{-1}BF_E, (E+BF_E)^{-1}A, (E+BF_E)^{-1}B)$ is standard. Then, the triple (E, A, B) can be reduced to (I_n, A_2, B_2), where (A_2, B_2) is in its Kronecker reduced form.

Notice that, if (I_n, A_1, B_1), $(I_n, A_2, B_2) \in M$ are equivalent triples, the pairs (A_1, B_1) and (A_2, B_2) are equivalent under block-similarity.

In this paper, we present a collection of structural invariants which we will call *controllability indices*, *eigenvalues* and for each eigenvalue the *Segre characteristic of the triple* in terms of ranks of certain matrices associated to the triple, that permit us to give the explicit form of reduced triple (I_n, A_2, B_2) without knowing the transformation matrices reducing the triple. Observe that in general, if (E, A, B) is equivalent to (I_n, A_1, B_1), the pairs (A, B), and (A_1, B_1) are not block similar equivalent, so, the collection of invariants of (A, B) are not a collection of invariants for (E, A, B).

As we say before, we remark that the collection of controllability indices as well eigenvalues and Segre characteristic of the triple (E, A, B) obtained are not related with the controllability indices eigenvalues and Segre characteristic of the pair (A, B) but related the controllability indices and the Segre characteristic of the unknown pair (A_2, B_2) corresponding to reduced form (I_n, A_2, B_2), as we can see in the following example.

Example 1 Let (E, A, B) be the triple $\left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right)$. The canonical reduced form for the triple is t $(I_2, A_1, B_1) = \left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right)$, but the Kronecker canonical form for the pair (A, B) is $\left(\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right)$. Notice that (A_1, B_1) is a controllable pair, (A, B) is not controllable and the triple (E, A, B) is controllable.

Controllability indices for triples of matrices, were obtained by García-Planas, [2], in this paper an improvement in the method is presented.

We recall that the authors L. Dai [1], and W. Ratz [6] studied the controllability character for singular systems but they do not consider feedback and derivative feedback in the equivalence relation, considering only, basis change in the state space, input space and premultiplication the system for invertible matrices.

The problem to obtain structural invariants permitting us to conclude conditions for controllability, was largely studied for standard linear systems under several equivalence relations that can be considered ([3],[5], [7] for example).

2 Collection of invariants

First of all, we remember the equivalence relation considered over the space M of triples of matrices.

Definition 2 *Two triples (E', A', B') and (E, A, B) in M are called equivalent if, and only if, there exist matrices $P, Q \in Gl(n; \mathbb{C})$, $R \in Gl(m; \mathbb{C})$, $F_E, F_A \in M_{m\times n}(\mathbb{C})$, such that*

$$\begin{pmatrix} E' & A' & B' \end{pmatrix} = Q \begin{pmatrix} E & A & B \end{pmatrix} \begin{pmatrix} P & 0 & 0 \\ 0 & P & 0 \\ F_E & F_A & R \end{pmatrix}$$

It is easy to check that this relation is an equivalence relation.

Now, we consider a list of ranks o a certain matrices associated to the matrices E, A, B in the triple $(E, A, B) \in M$.

Definition 3 *We consider the following numbers*

1. $r_1 = \operatorname{rank} B$
2. $r_2 = \operatorname{rank} \begin{pmatrix} E & B \end{pmatrix}$
3. $r_3 = \operatorname{rank} \begin{pmatrix} A & B \end{pmatrix}$
4. $r_4 = \operatorname{rank} \begin{pmatrix} E & A & B \end{pmatrix}$
5. $r_5 = (r_5^1, \dots, r_5^\ell, \dots)$, *where*

1) $r_5^1 = \operatorname{rank} M_5^1$ *with* $M_5^1 = \begin{pmatrix} E & B & 0 \\ A & 0 & B \end{pmatrix} \in M_{2n\times(n+2m)}(\mathbb{C})$

2) $r_5^2 = \operatorname{rank} M_5^2$ *with* $M_5^2 = \begin{pmatrix} E & B & 0 & 0 & 0 \\ A & 0 & E & B & 0 \\ 0 & 0 & A & 0 & B \end{pmatrix} \in M_{3n\times(2n+4m)}(\mathbb{C})$

$\vdots$

ℓ) $r_5^\ell = \operatorname{rank} M_5^\ell$ *with* $M_5^\ell = \begin{pmatrix} E & B & 0 & 0 & \dots & 0 & 0 & 0 \\ A & 0 & E & B & \dots & 0 & 0 & 0 \\ 0 & 0 & A & 0 & \dots & 0 & 0 & 0 \\ & & & & \ddots & & & \\ 0 & 0 & 0 & 0 & & E & B & 0 \\ 0 & 0 & 0 & 0 & & A & 0 & B \end{pmatrix} \in M_{(\ell+1)n\times(\ell n+2\ell m)}(\mathbb{C})$.

Proposition 2 *In the set M of singular systems, the r_i numbers are invariant under the equivalence relation considered.*

Proof. Let (E, A, B), (E', A', B') be two equivalent triples in M, then, there exist matrices $P, Q \in Gl(n;\mathbb{C})$, $R \in Gl(m;\mathbb{C})$, $F_E, F_A \in M_{m\times n}(\mathbb{C})$ such that

$$\begin{pmatrix} E' & A' & B' \end{pmatrix} = Q \begin{pmatrix} E & A & B \end{pmatrix} \begin{pmatrix} P & 0 & 0 \\ 0 & P & 0 \\ F_E & F_A & R \end{pmatrix}$$

So,

1. $r_1' = \operatorname{rank} B' = \operatorname{rank} QBR = \operatorname{rank} B = r_1$,
2. $r_2' = \operatorname{rank} \begin{pmatrix} E' & B' \end{pmatrix} = \operatorname{rank} Q \begin{pmatrix} E & B \end{pmatrix} \begin{pmatrix} P & 0 \\ F_E & R \end{pmatrix} = \operatorname{rank} \begin{pmatrix} E & B \end{pmatrix} = r_2$,
3. $r_3' = \operatorname{rank} \begin{pmatrix} A' & B' \end{pmatrix} = \operatorname{rank} Q \begin{pmatrix} A & B \end{pmatrix} \begin{pmatrix} P & 0 \\ F_A & R \end{pmatrix} = \operatorname{rank} \begin{pmatrix} E & B \end{pmatrix} = r_3$,
4. Obvious after equivalence relation definition.
5. ${r^1}'_5 = \operatorname{rank} \begin{pmatrix} E' & B' & 0 \\ A' & 0 & B' \end{pmatrix} = \begin{pmatrix} Q & 0 \\ 0 & Q \end{pmatrix} \begin{pmatrix} E & B & 0 \\ A & 0 & B \end{pmatrix} \begin{pmatrix} P & 0 & 0 \\ F_E & R & 0 \\ F_A & 0 & R \end{pmatrix} = \operatorname{rank} \begin{pmatrix} E & B & 0 \\ A & 0 & B \end{pmatrix} = r_5^1$.

Analogously for the other r_5-numbers. □

It is easy to compute the r_5^ℓ-numbers in the case where the system $(E, A, B) \in M$, is standardizable. As we say at the introduction, if a system (E, A, B) is standardizable, there exists a matrix F_E such that $E + BF_E$ is an invertible matrix. Taking into account that the ranks are invariant under equivalence relation we can suppose that the system is in the form (I_n, A_1, B_1). Making elementary block rows and columns transformations we have

$$r_5^\ell = \operatorname{rank} \begin{pmatrix} I_n & 0 & & & & \\ 0 & 0 & & & & \\ 0 & 0 & & & & \\ & & \ddots & & & \\ & & & I_n & 0 & \\ 0 & -A_1^\ell B_1 & & 0 & -A_1 B_1 & B_1 \end{pmatrix} =$$

$= \ell n + \operatorname{rank} \begin{pmatrix} B_1 & A_1 B_1 & \dots & A_1^\ell B_1 \end{pmatrix} = \ell n + \bar{r}_{\ell+1}$.

Notice that, if we call $\bar{r}_0 = \operatorname{rank} B$, the numbers $\bar{r}_1 - \bar{r}_0, \dots, \bar{r}_\ell - \bar{r}_{\ell-1}, \dots$ are the r-numbers of the pair of matrices (A_1, B_1) defining the triple (I_n, A_1, B_1).

It is easy to prove the following.

Proposition 3 *Let $(E, A, B) \in M$ a standardizable triple. Then there exists $s \in \mathbb{N}$ such that*

$$\begin{aligned} &r_5^n + n < r_5^1 \\ &r_5^1 + n < r_5^2 \\ &\vdots \\ &r_5^{s-2} + n < r_5^{s-1} \\ &r_5^{s-1} + n = r_5^s \\ &\vdots \\ &r_5^{s+\ell} + n = r_5^{s+\ell+1}, \ \forall \ell \in \mathbb{N}. \end{aligned}$$

A first application of the r_5^i numbers is a condition for controllability of the systems.

Let (E, A, B) be a triple, as we say at the introduction a triple (E, A, B) is controllable if and only if

$$\begin{aligned} &\operatorname{rank} \begin{pmatrix} E & B \end{pmatrix} = n, \\ &\operatorname{rank} \begin{pmatrix} sE - A & B \end{pmatrix} = n, \quad \forall s \in \mathbb{C} \end{aligned}$$

We will go to show that the controllability can be obtained computing the rank of a certain matrix.

Theorem 1 *A triple (E, A, B) is controllable if and only if $r_5^{n-1} = n^2$.*

Remark 1 Dai in [1] gives a condition for controllability in terms of a rank of a certain matrix, but he do not obtain controllability indices.

3 Controllability indices

We are going to introduce a collection of invariant numbers, which will permit us to deduce the canonical reduced form for a standardizable system.

Now, we call $r_5^0 = \operatorname{rank} B$. We define the ρ-numbers in the following manner

$$\begin{aligned} \rho_0 &= r_5^0 \\ \rho_1 &= r_5^1 - r_5^0 - n \\ \rho_2 &= r_5^2 - r_5^1 - n \\ &\vdots \\ \rho_s &= r_5^{s-1} - r_5^s - n \end{aligned}$$

It is obvious the following proposition.

Proposition 4 *The ρ-numbers are invariant under equivalence relation considered.*

Finally we define the *controllability indices* $k_1, \ldots, k_p$ for singular systems as the integers $k_1 \geq \ldots \geq k_p$ such that $[k_1, \ldots, k_p]$ is the conjugate partition of $[\rho_0, \rho_1, \ldots, \rho_s]$.

We observe that if the (E, A, B) is controllable then $k_1 + \ldots + k_p = n$ and $p = \rho_0 = \operatorname{rank} B$.

Theorem 2 *Let (E, A, B) be a controllable triple with controllability indices $(k_1, \ldots, k_p)$. Then the triple can be reduced to (I_n, A_1, B_1) with*

$$A_1 = \begin{pmatrix} N_1 & & \\ & \ddots & \\ & & N_p \end{pmatrix}, B_1 = \begin{pmatrix} B_1^1 & & \\ & \ddots & \\ & & B_p^1 \end{pmatrix}$$

and $N_i = \begin{pmatrix} 0 & 1 & & 0 \\ & \ddots & \ddots & \\ & & & 1 \\ & & & 0 \end{pmatrix} \in M_{k_i}(\mathbb{C}), B_i^1 = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} \in M_{k_i \times 1}(\mathbb{C})$.

Proof. It suffices to observe that the controllability indices for (E, A, B) coincide with controllability indices (also called Kronecker indices) of the pair (A_1, B_1). □

Example 2 Let (E, A, B) be a triple with

$$E = \begin{pmatrix} 2 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 \\ 2 & 3 & 3 & 2 \\ 0 & 1 & 0 & 2 \end{pmatrix}, \ A = \begin{pmatrix} 1 & 2 & 1 & 1 \\ 1 & 1 & 2 & 0 \\ 3 & 2 & 1 & -1 \\ 2 & 1 & 3 & 5 \end{pmatrix},$$

$$B = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 5 & 0 \\ 0 & 2 \end{pmatrix}.$$

Computing the r_5^i numbers we obtain, $r_5^0 = 2$, $r_5^1 = 7$, $r_5^2 = 12$, $r_5^3 = 16 = r_5^2 + n$. Then $\rho_0 = 2$, $\rho_1 = 1$, $\rho_2 = 1$ and the controllability indices are $k_1 = 3$, $k_2 = 1$. So (E, A, B) is equivalent to (I_4, A_1, B_1) with

$$A_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \ B_1 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

4 Segre Characteristic

Let now, a standardizable but not controllable system (E, A, B), then there exist $\lambda \in \mathbb{C}$ such that rank $\begin{pmatrix} \lambda E + A & B \end{pmatrix} < n$, otherwise the system was controllable.

Let $\lambda_1, \ldots, \lambda_s$ be the different values such that rank $\begin{pmatrix} -\lambda E + A & B \end{pmatrix} < n$. This values are invariant under equivalence relation considered, and we will call *continuous invariants* or *eigenvalues* of the triple.

Proposition 5 *For each $\lambda \in \mathbb{C}$ The following numbers are invariant under equivalence relation considered*

$$\tau_l^n = \operatorname{rank} \begin{pmatrix} E & B & & & & C_n & 0 \\ -\lambda E + A & 0 & & & & C_{n-1} & 0 \\ & & \ddots & & & & \\ & & & E & B & C_2 & 0 \\ & & & -\lambda E + A & 0 & C_1 & B \end{pmatrix}$$

where $C_i = \begin{cases} -\lambda E + A & \text{if } i = l \\ 0 & \text{if } i \neq l \end{cases}$, *for* $l = 1, 2, \ldots$.

Proof.

$$\operatorname{rank}\begin{pmatrix} E' & B' & & & & C'_n & 0 \\ -\lambda E'+A' & 0 & & & & C'_{n-1} & 0 \\ & & \ddots & & & & \\ & & & E' & B' & C'_2 & 0 \\ & & & -\lambda E'+A' & 0 & C'_1 & B' \end{pmatrix} =$$

$$\operatorname{rank}\begin{pmatrix} Q & & \\ & \ddots & \\ & & Q \end{pmatrix}\cdot\begin{pmatrix} E & B & & & & C_n & 0 \\ -\lambda E+A & 0 & & & & C_{n-1} & 0 \\ & & \ddots & & & & \\ & & & E & B & C_2 & 0 \\ & & & -\lambda E+A & 0 & C_1 & B \end{pmatrix}\cdot$$

$$\cdot\begin{pmatrix} P & 0 & 0 & 0 & & & & D_{2n} & 0 \\ F_E & R & 0 & 0 & & & & D_{2n-1} & 0 \\ 0 & 0 & P & 0 & & & & D_{2n-2} & 0 \\ -\lambda F_E+F_A & 0 & F_E & R & & & & D_{2n-3} & 0 \\ & & & & \ddots & & & & \\ & & & & & P & 0 & D_3 & 0 \\ & & & & & F_E & R & D_2 & 0 \\ & & & & & 0 & 0 & P & 0 \\ & & & & & -\lambda F_E+F_A & 0 & D_1 & R \end{pmatrix}$$

where $D_j = \begin{cases} 0 & \text{if } j \neq 2i-1 \\ -\lambda F_E + A & \text{if } j = 2i-1 \end{cases}$
and i is in such a way that $C_i \neq 0$.
Observe that, $C'_i = \begin{cases} -\lambda E' + A' & \text{if } i = l \\ 0 & \text{if } i \neq l, \end{cases}$
for $l = 1, 2, \ldots$ □

Suppose now, the triple (E, A, B) is standardizable, then we can compute τ_l^{n-1}, for all ℓ, using the equivalent triple (I_n, A_1, B_1) with (A_1, B_1) in the Kronecker reduced form.

Making row and columns block elementary transformations in the matrix

$$\begin{pmatrix} I_n & B_1 & & & & C'_n & 0 \\ -\lambda I_n+A_1 & 0 & & & & C'_{n-1} & 0 \\ & & \ddots & & & & \\ & & & I_n & B_1 & C'_2 & 0 \\ & & & -\lambda I_n+A_1 & 0 & C'_1 & B_1 \end{pmatrix}$$

we have that ,

$$\tau_l^{n-1} = (n-1)n + \\ +\operatorname{rank}\left((-\lambda I_n+A_1)^l \; B_1 \; A_1B_1 \; \ldots \; A_1^{n-1}B_1 \right),$$

$l = 1, 2, \ldots$

Definition 4 *Let (E, A, B) be an standardizable but non controllable system in M, and $\lambda_1, \ldots, \lambda_s$ its eigenvalues.*

For each eigenvalue λ_i we will call Segre characteristic of this eigenvalue to the conjugate partition of a $[n - \nu_1, \nu_1 - \nu_2, ...]$, where

$$\nu_l = \tau_l^{n-1} - (n-1)n = \\ \operatorname{rank}\left((-\lambda_i I_n+A')^l \; B' \; A'B' \; \ldots \; A'^{n-1}B' \right)$$

Remark 2 Let (E, A, B) be a triple of matrices equivalent to (I_n, A_1, B_1). The eigenvalues of the triple (E, A, B), correspond with eigenvalues of (A_1, B_1) and the Segre characteristic of each eigenvalue correspond to the Segre Characteristic of the corresponding eigenvalue in (A_1, B_1).

Using the collection of invariants presented, we can deduce the canonical reduced form.

Proposition 6 *Let (E, A, B) be a triple in M, $[k_1, ..., k_r]$ the controllability indices of the triple, $\lambda_1, \ldots, \lambda_s$ the eigenvalues, and for each λ_i the Segre characteristic. Then, the triple can be reduced to (I_n, A_1, B_1) with*

$$A_1 = \begin{pmatrix} N_1 & & & \\ & \ddots & & \\ & & N_p & \\ & & & J \end{pmatrix}, B_1 = \begin{pmatrix} B_1^1 & & & \\ & \ddots & & \\ & & B_p^1 & \\ & & & 0 \end{pmatrix}$$

con $N_i = \begin{pmatrix} 0 & 1 & & 0 \\ & \ddots & \ddots & \\ & & & 1 \\ & & & 0 \end{pmatrix} \in M_{k_i}(\mathbb{C}), B_i^1 = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} \in M_{k_i \times 1}(\mathbb{C})$.

Example 3 Let (E, A, B) a triple with
$E = \begin{pmatrix} 3&2&1&0 \\ 0&3&2&1 \\ 0&0&3&2 \\ 0&0&0&3 \end{pmatrix}$, $A = \begin{pmatrix} 0&3&2&1 \\ 0&0&4&4 \\ 0&0&6&7 \\ 0&0&0&6 \end{pmatrix}$, $B = \begin{pmatrix} 2 \\ 3 \\ 0 \\ 0 \end{pmatrix}$
$\rho_0 = 1$, $\rho_1 = 1$, so controllability indices are $k_1 = 2$. The eigenvalues of (E, A, B) are $\lambda = 2$. $\nu_1 = 3$, $\nu_2 = \nu_3 = \ldots = 2$, then, the Segre characteristic is the conjugate partition of $[4 - 3 = 1, 3 - 2 = 1]$.

The canonical reduced form is (E', A', B') with

$$E' = \begin{pmatrix} 1&0&0&0 \\ 0&1&0&0 \\ 0&0&1&0 \\ 0&0&0&1 \end{pmatrix}, A = \begin{pmatrix} 0&1&0&0 \\ 0&0&0&0 \\ 0&0&2&1 \\ 0&0&0&2 \end{pmatrix}, B = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}.$$

6 Conclusions

In this paper a simple method to analyze the controllability as well the poles of a singular system in the form $E\dot{x} = Ax + Bu$, is presented. This method is based in the computation of the rank of a certain matrices depending only in the matrices E, A, B defining the system.

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