A Dynamic Allocation Scheme for a Multi–agent Nash Equilibrium

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Abstract: A multi–agent game, in which each one of the N agents allocates its fixed resource against others to achieve dominance, results in a Nash equilibrium in a static game under perfect information. When an agent has knowledge only of his own allocations and allocations of the others against itself, then the only way to achieve a Nash equilibrium is to dynamically update its own allocations in time. This paper provides a formal scheme which is guaranteed to converge to a Nash equilibrium under the aforementioned information structure. This result has applications in the theory of balance of power in an international political systems, as well as in the analysis of piece–wise linear multi–model dynamical system.

Key–Words: Game theory, Multimodel systems, Nash equilibrium, Dynamic resource allocation, Convergence to equilibrium

1 Introduction and Formulation

Consider a system $\mathcal{S}$ that consists of $N \geq 2$ subsystems (agents) $\mathcal{S}_i$, $i \in \mathbf{N} = \{1, 2, \ldots, N\}$. Each $\mathcal{S}_i$ has a resource $r_i > 0$ which it uses to allocate against the remaining subsystems. Without loss of generality, we assume that

$$r_1 \geq r_2 \geq \cdots \geq r_N$$

The allocation of resources can be described by a nonnegative allocation matrix $A = [\, a_{ij} \,]$, where a_{ij} denotes the allocation of $\mathcal{S}_i$ against $\mathcal{S}_j$. Thus the ith row of A

$$\mathbf{a}_i^T = [\, a_{i1} \;\; a_{i2} \;\; \cdots \;\; a_{iN} \,]$$

describes the allocation of the resource of $\mathcal{S}_i$, and the ith column of A

$$\mathbf{c}_i = [\, a_{1i} \;\; a_{2i} \;\; \cdots \;\; a_{Ni} \,]^T$$

describes the allocations against $\mathcal{S}_i$. The whole resource r_i of $\mathcal{S}_i$ is distributed among the remaining $\mathcal{S}_j$'s so that a_{ij}'s satisfy

$$a_{ii} = 0 \,, \; 0 \leq a_{ij} \leq r_i \,, \; \forall \, i \neq j \in \mathbf{N} \qquad (1)$$

and

$$\sum_{j \in \mathbf{N}_i} a_{ij} = r_i \,, \; \forall \, i \in \mathbf{N} \qquad (2)$$

where $\mathbf{N}_i = \mathbf{N} - \{i\}$.

The system $\mathcal{S}$ may be interpreted as the world, with $\mathcal{S}_i$ representing the states, and r_i their economical or military resources or as a swarm, with $\mathcal{S}_i$ representing the individuals, and r_i their forces or powers, etc. In any case, the subsystems $\mathcal{S}_i$ allocate their available resources against each other to achieve dominance or balance. In the rest of this section, we summarize the results obtained in [1], which have been applied to the theory of international political systems of Waltz, [2].

A set of allocations A is called a *balanced equilibrium*, or a B-equilibrium in short, if

$$a_{ij} = a_{ji} \,, \; \forall \, i \neq j \in \mathbf{N} \qquad (3)$$

Clearly, a set of allocations is a B-equilibrium if an only if the corresponding allocation matrix A is symmetric. If S has a B-equilibrium, then it is said to be *balanced*. We are concerned with the following questions:

Q1. When is $\mathcal{S}$ balanced?

Q2. What allocations yield a B-equilibrium when $\mathcal{S}$ is balanced?

Q3. If $\mathcal{S}$ is not balanced, is there an optimal set of allocations that is close to a B-equilibrium with respect to a meaningful measure?

Q4. How can a B-equilibrium (an optimal allocation) be achieved when $\mathcal{S}$ is balanced (unbalanced)?

The following result by [1] provides conditions for existence and uniqueness of a B-equilibrium:

Lemma 1 *$\mathcal{S}$ is balanced if and only if*

$$r_1 \le R = \sum_{j=2}^{N} r_j \tag{4}$$

in which case there is a unique B-equilibrium if and only if either $N \le 3$ or equality holds in (4).

The condition in (4) simply requires that no subsystem controls more than half of the total resources, that is, there is no hegemon.

Let $s_{ij} = a_{ij} - a_{ji}$ denote the excess allocation of $\mathcal{S}_i$ over $\mathcal{S}_j$, which can be interpreted as the "security" of $\mathcal{S}_i$ against $\mathcal{S}_j$. Thus

$$\mathbf{s}_i = \mathbf{a}_i - \mathbf{c}_i = [\, s_{i1} \;\; s_{i2} \;\; \cdots \;\; s_{iN} \,]^T$$

describes the securities of $\mathcal{S}_i$ against other subsystems. Clearly, if $\mathcal{S}$ is balanced and A^* is a B-equilibrium, then $s^*_{ij} = 0$ for all (i,j). If $\mathcal{S}$ is not balanced, then we can conveniently use

$$\min_A J(A) \tag{5}$$

as a measure of imbalance, where

$$J(A) = \max_{i,j} \{\, |\, s_{ij} \,| \,\} \tag{6}$$

Let

$$D = \max \{\, 0 \,, \frac{r_1 - R}{N-1} \,\}$$

Clearly, $D = 0$ when $\mathcal{S}$ is balanced, and $D > 0$, otherwise. The significance of d is that it measures the degree of imbalance as stated by the following result of [1]:

Lemma 2

$$J^* = \min_A J(A) = d$$

The minimum is achieved at $A = A^*$ where

- *if $D = 0$, then A^* is any B-equilibrium*
- *if $D > 0$, then A^* is uniquely given as*

$$\begin{array}{rcll} a^*_{1j} & = & r_j + d\,, & j \ne 1 \\ a^*_{j1} & = & r_j\,, & j \ne 1 \\ a^*_{ij} & = & 0\,, & i \ne 1 \ne j \end{array}$$

An interesting property of the solutions of the optimization problem in (5) is that each solution (if not unique) is a Nash equilibrium (N-equilibrium),[3], of a game among $\mathcal{S}_i$, in which each $\mathcal{S}_i$ tries to maximize its minimum security

$$J_i(\mathbf{s}_i) = \min_{j \in \mathbf{N}_i} \{\, s_{ij} \,\}\,,\; i \in \mathbf{N} \tag{7}$$

by choosing its own allocations $\mathbf{a}_i$ based on the locally available information $\mathbf{s}_i$.

We illustrate these concepts with an example.

Example 1 We consider three systems:

(a) A system with $N = 3$ subsystems having the resources

$$(r_1, r_2, r_3) = (40, 30, 20)$$

is balanced. The unique B-equilibrium is given by the allocation matrix

$$A^* = \begin{bmatrix} 0 & 25 & 15 \\ 25 & 0 & 5 \\ 15 & 5 & 0 \end{bmatrix}$$

(b) A system with $N = 4$ subsystems having the resources

$$(r_1, r_2, r_3, r_4) = (50, 40, 30, 10)$$

is also balanced. B-equilibria are characterized parametrically as

$$\begin{array}{rcccl} a^*_{12} & = & a^*_{21} & = & 25 + \alpha \\ a^*_{13} & = & a^*_{31} & = & 15 + \beta \\ a^*_{14} & = & a^*_{41} & = & 10 - \alpha - \beta \\ a^*_{23} & = & a^*_{32} & = & 15 - \alpha - \beta \\ a^*_{24} & = & a^*_{42} & = & \beta \\ a^*_{34} & = & a^*_{43} & = & \alpha \end{array}$$

where $\alpha, \beta \ge 0$ and $\alpha + \beta \le 10$.

(c) On the other hand, a system with $N = 3$ subsystems having the resources

$$(r_1, r_2, r_3) = (50, 30, 10)$$

is not balanced. Optimum allocation of resources is unique, and is given by

$$A^* = \begin{bmatrix} 0 & 35 & 15 \\ 30 & 0 & 0 \\ 10 & 0 & 0 \end{bmatrix}$$

which results in

$$J^* = J^*_1 = 5\,,\; J^*_2 = J^*_3 = -5$$

Note that, in this case, the hegemon distributes its excess resources evenly against the remaining subsystems. Note also that, A^* is an N-equilibrium: With $\mathbf{a}_j = \mathbf{a}^*_j$ and $\mathbf{a}_k = \mathbf{a}^*_k$,

$$\max_{\mathbf{a}_i} J_i(A)$$

is achieved at $\mathbf{a}_i = \mathbf{a}^*_i$ for all $\{i, j, k\} = \{1, 2, 3\}$.

2 Dynamic Resource Allocation

We now turn to the question of achieving the optimal allocation starting from an arbitrary initial allocation of resources, in which the main contribution of this study lies. The constraints are that

- (1) and (2) must hold at all times,
- $\mathcal{S}_i$ has access only to $\mathbf{a}_i$ and $\mathbf{s}_i$, and therefore, $\mathbf{c}_i$ in the process of updating its own allocations $\mathbf{a}_i$

We note that there are many attempts of setting up dynamic games which converge to Nash equilibria whenever one exists (see e.g., [5], [6], [7]. Our approach differs from those in the sense that we search among all possible dynamic update schemes those that yield a Nash equilibrium in the long run, rather than setting up a "dynamic game" by imposing further motives and preferences on agents.

2.1 A Discrete Update Scheme

The simplest approach is to update the allocations $\mathbf{a}_i$ of each $\mathcal{S}_i$ iteratively by making use of the N-equilibrium property of the optimal allocation: With $\mathbf{a}_i(k)$ and $\mathbf{s}_i(k)$ denoting the allocation and security vectors of $\mathcal{S}_i$ at the kth step of iterations, we simply choose

$$\mathbf{a}_i(k+1) = \mathbf{a}_i^*(k) = \arg\max_{\mathbf{a}_i(k)} J_i(\mathbf{s}_i(k)) \qquad (8)$$

subject to (1) and (2). Unfortunately, it is shown (see e.g. [4]) that this approach may fail to converge to an optimal allocation due to possible oscillations. However, we have observed that a modified iterative scheme

$$\mathbf{a}_i(k+1) = \alpha\mathbf{a}_i(k) + (1-\alpha)\mathbf{a}_i^*(k)\,,\ 0 < \alpha < 1, \quad (9)$$

which updates the allocations more slowly, often converges to an optimal allocation.[1]

2.2 A Continuous Update Scheme

We look for an update scheme of the form

$$\dot{\mathbf{a}}_i = \mathbf{f}_i(\mathbf{a}_i, \mathbf{c}_i)\,,\ i \in \mathbf{N} \qquad (10)$$

that defines the dynamics of $\mathcal{S}_i$.

If $0 < a_{ij} < r_i$, then a suitable update scheme is

$$\dot{a}_{ij} = c_i d_{ij}\,,\ j \in \mathbf{N}_i \qquad (11)$$

[1]Although we have not proved the convergence of the modified iterative scheme, we strongly believe that it does. Because, the modified scheme is closely related to a discrete version of the continuous update scheme given in the next sub-section.

where $c_i > 0$, and

$$d_{ij} = -(N-2)s_{ij} + \sum_{q \neq i,j} s_{iq}$$

This scheme

- updates the allocation a_{ij} of $\mathcal{S}_i$ against $\mathcal{S}_j$ at a rate that is inversely proportional to its security against $\mathcal{S}_j$ but directly proportional to its securities against other $\mathcal{S}_q$,
- treats all $\mathcal{S}_q$, $q \neq j$, equally in updating a_{ij},
- guarantees that

$$\sum_{j \in \mathbf{N}_i} \dot{a}_{ij} = 0$$

as required by (2).

If $a_{ij} = 0$, then as long as $d_{ij} \geq 0$ we can use (11) to update a_{ij}. However, if $a_{ij} = 0$ and $d_{ij} < 0$, then (11) can no longer be used as it would force a_{ij} to go negative. In that case, we force $\dot{a}_{ij} = 0$, and omit s_{ij} from d_{ip} in computing $\dot{a}_{ip}$ for $p \neq j$. Repeating the process, we end up with a maximal subset $\mathbf{J}_i \subset \mathbf{N}_i$ such that with

$$d_{ij} = -(J_i - 1)s_{ij} + \sum_{\substack{q \in \mathbf{J}_i \\ q \neq j}} s_{iq}$$

where $J_i = |\,\mathbf{J}_i\,|$, we have

$$\begin{array}{lll} a_{ij} = 0\,, \quad j \in \mathbf{J}_i & \Rightarrow & d_{ij} \geq 0 \\ a_{ij} = r_i\,, \quad j \in \mathbf{J}_i & \Rightarrow & d_{ij} \leq 0 \end{array}$$

We then set

$$\dot{a}_{ij} = \begin{cases} c_i d_{ij}, & j \in \mathbf{J}_i \\ 0, & j \notin \mathbf{J}_i \end{cases} \qquad (12)$$

Construction of $\mathbf{J}_i$ and (12) guarantees that whenever a particular allocation a_{ij} hits the boundary of the allowable allocation interval (i.e., 0 or r_i), then it either stays at the boundary or it is pushed back into the open interval $(0, r_i)$. (12) defines a piece–wise linear multimodel system (see, e.g., [8]) that has different linear dynamics in different parts of the state space.

We now state:

Theorem 1 *Solution $A(t, A_0)$ of the allocation dynamics (12) starting from an initial allocation $A(0) = A_0$ converges to a constant equilibrium $A^*(A_0)$, which is*

- *a B-equilibrium if $\mathcal{S}_1$ is not a hegemon,*
- *the unique N-equilibrium (independent of A_0) if $\mathcal{S}_1$ is a hegemon*

The rather lengthy proof of Theorem 1 can be found on the website www.ee.bilkent.edu.tr/∼sezer. It suffices to point out that the proof is based on construction of a common Liapunov function [9] for all linear regimes of (12).

3 Simulation Results

We simulated the discrete and continuous update mechanisms for the three systems considered in Example 1(a-c).

(a) The discrete update scheme (8) for the system in Example 1(a), starting with

$$A(0) = \begin{bmatrix} 0 & 0 & 40 \\ 25 & 0 & 5 \\ 15 & 5 & 0 \end{bmatrix}$$

resulted in oscillations in $a_{13}(k)$ and $a_{31}(k)$ and did not converge. Whereas, the modified update scheme (9) with $\alpha = 0.5$ starting with the same initial allocations converged to the unique B-Equilibrium given in Example 1 as shown in Fig. 1.

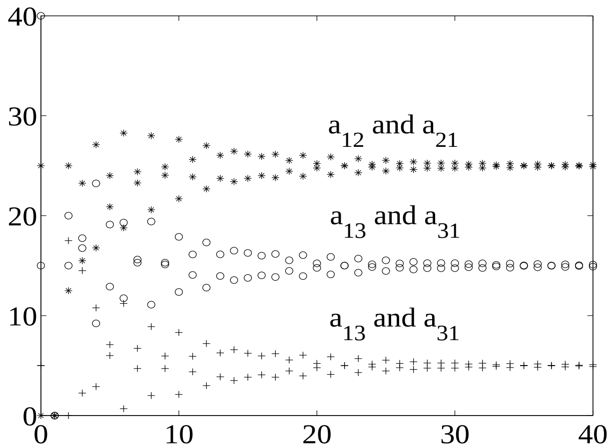

Figure 1: Discrete Allocation Dynamics $\mathcal{S}_a$

The continuous update scheme with $c_1 = 1, c_2 = 2, c_3 = 3$ also converged to the unique B-Equilibrium as shown in Fig. 2. Note that a_{32} was already at the equilibrium value initially. However, combined dynamics of $\mathcal{S}$ resulted first in a decrease in a_{32} until it hit the boundary, and later in a gradual increase to the final equilibrium value. The piece-wise linear nature of the allocation dynamics is apparent from the figure.

(b) The continuous allocation scheme with $c_1 = 1, c_2 = 5, c_3 = 3, c_4 = 1$ for the system in Example 1(b) starting from

$$A(0) = \begin{bmatrix} 0 & 10 & 20 & 20 \\ 0 & 0 & 30 & 10 \\ 5 & 5 & 0 & 20 \\ 0 & 0 & 10 & 0 \end{bmatrix}$$

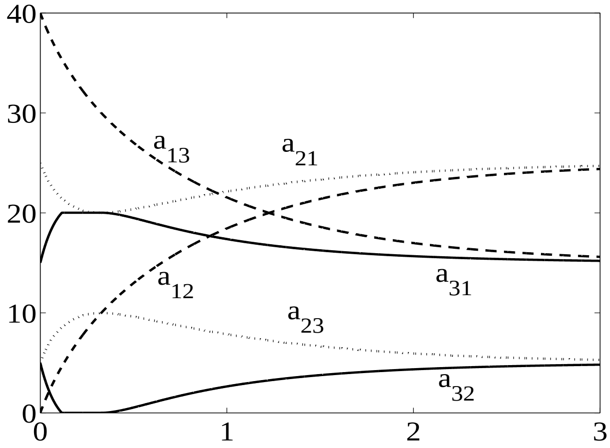

Figure 2: Continuous Allocation Dynamics for $\mathcal{S}_a$

converged to the N-equilibrium

$$A^* = \begin{bmatrix} 0.00 & 25.00 & 16.34 & 8.66 \\ 25.00 & 0.00 & 13.66 & 1.34 \\ 16.34 & 13.66 & 0.00 & 0.00 \\ 8.66 & 1.34 & 0.00 & 0.00 \end{bmatrix}$$

as shown in Fig. 3.

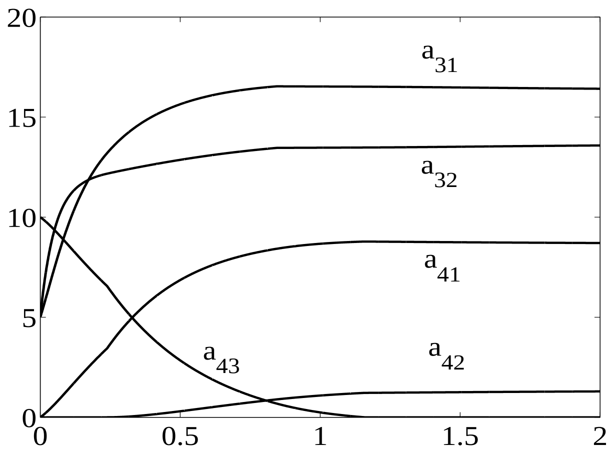

Figure 3: Continuous Allocation Dynamics for $\mathcal{S}_b$

A different initial allocation

$$A(0) = \begin{bmatrix} 0 & 20 & 0 & 30 \\ 10 & 0 & 20 & 10 \\ 0 & 20 & 0 & 10 \\ 5 & 5 & 0 & 0 \end{bmatrix}$$

resulted in a different N-equilibrium

$$A^* = \begin{bmatrix} 0 & 25 & 15 & 10 \\ 25 & 0 & 15 & 0 \\ 15 & 15 & 0 & 0 \\ 10 & 0 & 0 & 0 \end{bmatrix}$$

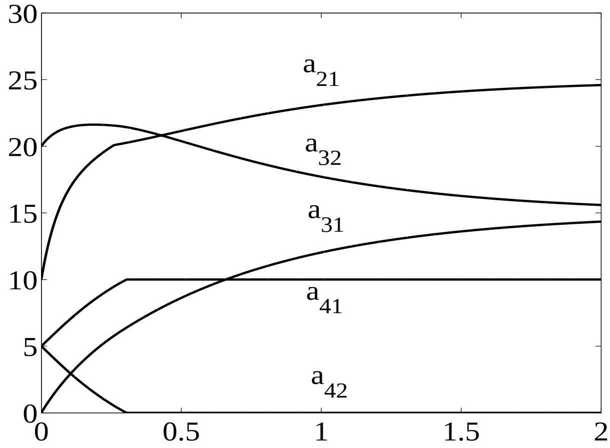

Figure 4: Continuous Allocation Dynamics for $\mathcal{S}_b$

as shown in Fig. 4.

(c) Finally, the continuous update scheme with $c_1 = 1, c_2 = 2, c_3 = 1$ for the unbalanced system in Example 1(c) starting from

$$A(0) = \begin{bmatrix} 0 & 25 & 25 \\ 15 & 0 & 15 \\ 5 & 5 & 0 \end{bmatrix}$$

resulted in the unique N-Equlibrium given in Example 1(c) as shown in Fig. 5. Other initial allocations also converged to the same N-equilibrium.

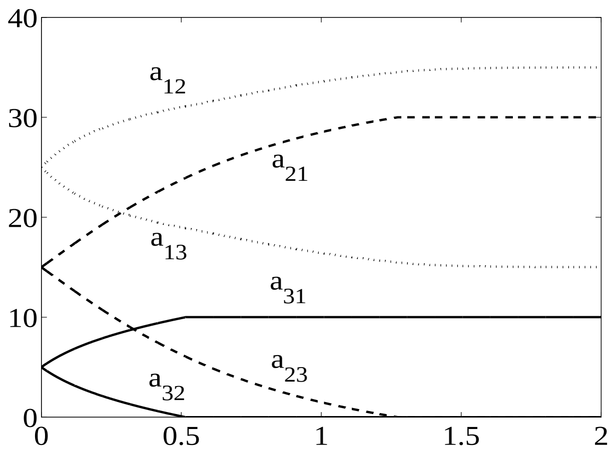

Figure 5: Continuous Allocation Dynamics for $\mathcal{S}_c$

References:

[1] A. B Özgüler, S. Ş. Güner, and N. M. Alemdar, Structure of multipolar international systems, *Preprint* Bilkent University, Ankara, Turkey, 1998.

[2] K. Waltz, *Theory of International Politics* Addison-Wesley, Reading, Massachussetts 1979.

[3] M. J. Osborne and A. Rubinstein, *A Course in Game Theory* The MIT Press, Cambridge, Massachusetts, 1994.

[4] O. Kesten, A dynamic allocation model for multipolar international systems, *Preprint*, Bilkent University, Ankara, Turkey, 2000

[5] M. Kandori, G. J. Mailath and R. Rob. Learning, mutation, and long run equilibria in games, *Econometrica* 61, pp. 29–56, 1993.

[6] J. M. Smith, *Evolution and The Theory of Games* Cambridge University Press, Cambridge, 1982.

[7] D. Friedman, Evolutionary games in economics, *Econometrica* 59: pp. 637–666, 1991.

[8] E. Sontag, Nonlinear regulation: The piece–wise linear approach, *IEEE Trans. Autom. Contr.* 26, pp. 346–358, 1981.

[9] M. Vidyasagar, *Nonlinear System Analysis* Prentice–Hall,Englewood Cliffs, New Jersey, 1978.