COMPUTATION OF INVERSE D-TRANSFORM VIA NLI ALGORITHM

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Abstract: The D-transform (©J. Hekrdla, [1]) converts a continuous-time signal into a sequence in such a way that – in contrast to the classical sampling – the relations between the derivative of continuous-time signal and the difference of discrete-time signal are strictly preserved. This fact enables, among other things, changing the problem of numerical solution of differential equations into the simpler problem of solving difference equations. The inverse D-transform represents a difficult problem, which has been solved numerically by means of Laguerr's finite series. This paper describes a novel method of D-transform inversion, which is based on mutual correspondence between the D-, z-, and Laplace transforms. Numerical accuracy is provided by the algorithm of the inverse Laplace transform, which works reliably also in the case of periodical or divergent signals.

Keywords: - D-transform, s-z correspondence, continuous-time signal, discrete-time signal.

1 Introduction

A new integral transformation is published in [1] which transfers a function $x_a(t)$ of real variable t into a sequence $x(k)$ according to the formula

$$x(k) = \int_0^\infty x_a(t) e^{-\frac{t}{T}} \frac{\left(\frac{t}{T}\right)^k}{k!} dt, \tag{1}$$

where k is a non-negative integer, and T is a real constant. Eq. (1) defines the so-called D-transform of function $x_a(t)$. This transformation can be denoted by means of the D-operator:

$$x(k) = \mathrm{D}\{x_a(t)\}. \tag{2}$$

The constant T plays the role of sampling period. In contrast to classical sampling, the discrete-time (DT) signal $x(k)$, acquired from continuous-time (CT) signal $x_a(t)$, preserves a number of its features. Applying the D-transform to the impulse response of a CT system, we get the impulse response of the equivalent DT system and a difference equation, describing its behavior. It is proven in [1] that this difference equation can be unambiguously determined from the differential equation of CT system after replacing derivatives by differences. This basic feature of the D- transform can be described as follows:

The D-transform of a derivative is the difference of a D-transform,

or

$$\mathrm{D}\left\{\frac{d}{dt} x_a(t)\right\} = \Delta \mathrm{D}\{x_a\} = \Delta x(k) = \frac{x(k) - x(k-1)}{T}. \tag{3}$$

The correspondence between the differential and difference equations of CT and DT systems, which is provided by the D-transform, is mutually explicit. This feature is not true for any correspondence based on the well-known s-z transformations, when the derivative-difference replacement represents only an approximation [2, 3, 4]. It really means that the D-transform enables **precise** modeling of CT systems via DT systems that are equivalent in terms of the D-transform. However, such modeling also requires backward transition from the DT signal into the CT signal, i.e. the inverse D-transform.

It is shown in [5] that no such transformation kernel $g(k,t)$ exists that would enable writing the inverse D-transform in a common form

$$x_a(t) = \mathrm{D}^{-1}\{x(k)\} = \sum_{k=0}^{\infty} x(k) g(k,t).$$

That is why the problem of the inversion of the D-transform is solved indirectly via the well-known problem of inverting another transformation. In [5], the D-transform is converted into the Laguerre transform with subsequent time-domain inversion.

Another procedure is described in this paper. This method uses the relationships between the D-, z-, and Laplace transforms. As a result, the computation is faster and more accurate than in the case of the method published in [5].

2 Inversion of D-transform by numerical inversion of Laplace transform

The following relationships between the D-, z-, and Laplace transforms can be derived from Eq. (1), which defines the D-transform [1]:

$$x(k) = \mathrm{D}\{x_a(t)\} \Rightarrow \mathrm{Z}\{x(k)\} = \mathrm{L}\{x_a(t)\}\Big|_{s=\frac{1-z^{-1}}{T}} . \qquad (4)$$

The symbols D, Z, and L denote the operators of the D-, z- and Laplace transforms.

It is obvious from (4) that the problem of finding the signal $x_a(t)$ from the sequence $x(k)$ can be solved in the following steps:

1 Finding the z-transform of the sequence $x(k)$.
2 Transposing the above z-transform to the Laplace transform the of the signal $x_a(t)$, applying the substitution $s = (1\text{-}z^{-1})/T$ or $z^{-1} = 1\text{-}sT$ according to (4).
3 Computing the signal $x_a(t)$ from its Laplace transform by means of the algorithm of the numerical Laplace inversion (NLI).

Finding z-transform of sequence *x*(*k*)

The implementation of this step depends on the type of the problem being solved. Let us omit the trivial case, when we know an algorithm of $x(k)$ generation in the form of difference equation. If only numerical values of this sequence are available, such a rational fraction function of the z-operator must be found that forms the z-transform pair with this sequence. This classical problem of identifying the DT system from its impulse or another response is solved, for instance, in [6, 7].

Transposing z-transform to Laplace transform

Consider the z-transform of original $x(k)$ in the form of rational fraction function:

$$K(z^{-1}) = \frac{\sum_{i=0}^{m} a_i z^{-i}}{\sum_{i=0}^{n} b_i z^{-i}}, \; m \le n \cdot \qquad (5)$$

As evident from (4), applying the substitution $s=(1\text{-}z^{-1})/T$ yields the Laplace transform of the resulting CT signal. Tedious derivation leads to the closed-form solution:

$$K(s) = \frac{\sum_{k=0}^{m} c_k s^k}{\sum_{k=0}^{n} d_k s^k}, \; m \le n,$$

$$c_k = (-T)^k \sum_{l=k}^{m} a_l \binom{l}{k}, \; d_k = (-T)^k \sum_{l=k}^{n} b_l \binom{l}{k}. \qquad (6)$$

From the point of view of s-z transformations, the transposition of transfer function (5) to (6) represents the so-called BD (Backward-Difference) transformation [2], [3]. Eq. (6) is in agreement with the results from [2], relating to the BD transformation.

Computing $x_a(t)$ by NLI algorithm

An algorithm of a precise inversion of the Laplace transform is described in [8-10]. The Laplace transform is regarded as the rational fraction function of the *s*-operator. Consider the Laplace transform *K*(s) according to Eq. (6) in order to find its time-domain representation $x_a(t)$. *K*(s) can be considered the transfer function of an *n*-th order linear system with an input $w(t)$ and an output $y(t)$. Then the signal $x_a(t)$ can be found as the impulse response of this system, i.e. its forced response to the Dirac impulse $w(t) = \delta(t)$. Then K(s) is the ratio of the Laplace transforms of output and input signals:

$$K(s) = \frac{\mathrm{L}\{y(t)\}}{\mathrm{L}\{w(t)\}} = \frac{Y(s)}{W(s)} = \frac{\sum_{k=0}^{m} c_k s^k}{\sum_{k=0}^{n} d_k s^k}, \; m \le n. \qquad (7)$$

On the assumption of normalized polynomial in the denominator, i.e. $d_n = 1$, the following state equations can be assigned to transfer function (7) [10, 12]

$$(s\mathbf{I} - \mathbf{A})\mathbf{X}(s) = \mathbf{B}W(s) + \mathbf{x}(0), \qquad (8)$$

where **X**(s) and *W*(s) are the Laplace transforms of state vector $\mathbf{x}(t)$ ($nx1$) and input signal $w(t)$; $\mathbf{x}(0)$ is the state vector of initial conditions at t=0, and **I** is an (nxn) unity matrix. The state matrix **A** and vector **B** have the following structures:

$$\mathbf{A} = \begin{vmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -d_0 & -d_1 & -d_2 & \cdots & -d_{n-1} \end{vmatrix}, \; \mathbf{B} = \begin{vmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{vmatrix}.$$

The time-domain output equation, corresponding to transfer function (6), is as follows:

$$y(t) = \sum_{i=0}^{n-1} (c_i - c_n d_i) x_{i+1}(t) + c_n w(t), \qquad (9)$$

where x_i , i=1,2,..,n are elements of the state vector.

The time-domain representation of transfer function (6) is found in two steps:

1 The numerical Laplace inversion (NLI) of matrix equation (8),

2 Computing the final response from output Eq. (9).

As demonstrated in [9], this procedure results in a very precise inversion of the Laplace transform, even if the time-domain signal has an oscillating or unstable behavior.

The transformation of Eq. (8) into the time domain utilizes the so-called "resetting principle" [9-11]. The time axis is divided into consecutive intervals. The NLI algorithm, described in [9], is applied to Eq. (8) over the first interval on the assumption of zero initial state $\mathbf{x}(0) = \mathbf{0}$ and the Dirac impulse as input signal with the Laplace transform $W(s)$ = 1. The computational step must be sufficiently small to avoid error in the NLI algorithm, which grows with longer simulation times. After finishing the NLI, the state vector is saved in the memory, and it is used in the next step as a vector of initial conditions. Starting from the second step, the natural response to these initial conditions is solved. The algorithm is repeated over all the intervals of the time axis. Every time the state vector is computed, the output signal is also determined by evaluating Eq. (9).

4 Algorithm testing

Consider the DT signal in Fig. 1, which represents the input signal $x(k)$ before the inverse D-transform. The sampling period is T = 0.1ms. The signal is described by the equation

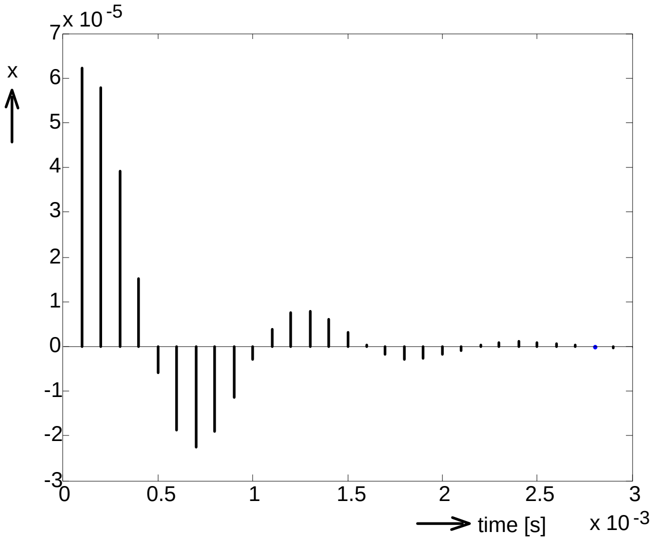

Fig. 1: DT signal for inverse D-transformation.

$$x(k) = X_{\max} a^k \sin(k.w_d + w_d), \qquad (10)$$

where

X_{max} = 8.347e-005,
a = 8.347e-001,
w_d = 5.521e-001 rads.

Note that in the time domain this signal represents a harmonic signal with a frequency of 878.7Hz, a phase shift of 31.6°, and a damping time constant of 533.6μs.

The proposed algorithm for the inversion of the D-transform has been programmed in MATLAB. The resulting CT signal is in Fig. 2. It has been identified as CT harmonic signal with the following parameters:

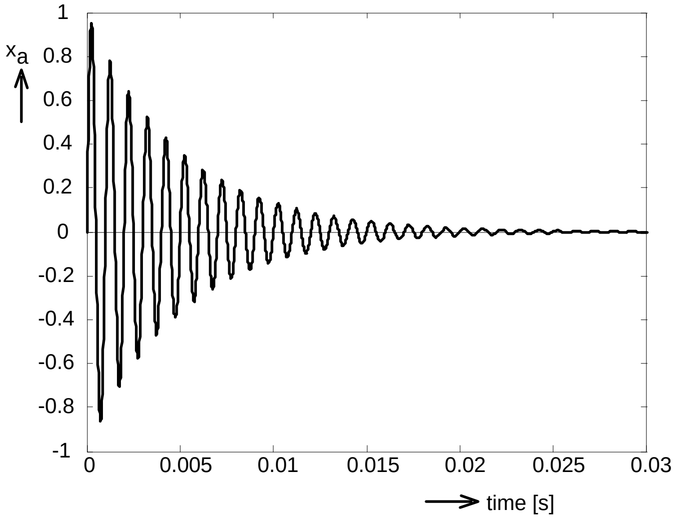

Fig. 2: CT signal after inverse D-transform.

$$x_a(t) = X_{a\max} e^{-\frac{t}{t}} \sin(w_a t + j), \qquad (11)$$

where

X_{amax}=0.99994,
τ = 5.003ms,
φ = -0.0034°,
w_a = 6.2832e+003 rads/s.

Note that the D-transform preserves the character of the signal, which is in an exponentially damped harmonic form in both cases. However, the parameters of signals are totally different.

To verify this result, setting (11) into definition equation (1) of the D-transform is not a preferred method, because it leads to a brain-teaser integral. That is why we utilize Eq. (4): we compute the z-transform of DT signal (10), and perform the substitution z^{-1}=1-sT. In this way we obtain the Laplace transform of the signal $x_a(t)$. After the Laplace inversion we would get Eq. (11).

The z-transform of (10) leads to the following result:

$$\mathrm{Z}\{x(k)\} = X_{\max} \sin w_d \frac{z}{z^2 - 2az\cos w_d + a^2}.$$

Applying the above substitution and arrangement yields the Laplace transform of CT signal:

$$\mathrm{L}\{x_a(t)\} = \frac{\dfrac{X_{\max}\sin w_d}{a^2T^2}}{s^2 + \dfrac{2}{aT}(\cos w_d - a)s + \dfrac{1 - 2a\cos w_d + a^2}{a^2T^2}} =$$

$$= \frac{\dfrac{X_{\max}\sin w_d}{a^2T^2}}{\left(s + \dfrac{\cos w_d - a}{aT}\right)^2 + \left(\dfrac{\sin w_d}{aT}\right)^2}.$$

After the inverse Laplace transform we obtain $x_a(t)$ exactly in the form of (11), where

$$X_{a\max} = \frac{X_{\max}}{a} = 1,$$

$$t = \frac{aT}{\cos w_d - a} \doteq 4.991\text{ms},$$

$$w_a = \frac{\sin w_d}{aT} \doteq 6.2834\text{krad/s} \Rightarrow f_a \doteq 1\text{kHz},$$

$$j = 0.$$

These results are in good agreement with data from our numerical algorithm.

5 Conclusions

An effective method of numerical inversion of the D-transform into continuous-time signal is described in the paper. The problem of the inverse D-transform is translated into a formerly resolved problem of numerical inversion of the Laplace transform. The algorithm is relatively simple and its practical testing has confirmed a high precision of the resulting CT signal.

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