Numerical solution of weakly singular Volterra integro-differential equations with change of variables

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Abstract: - We discuss a possibility to construct high order methods on uniform or mildly graded grids for the numerical solution of linear Volterra integro-differential equations with weakly singular or other nonsmooth kernels. Using an integral equation reformulation of the initial value problem, we apply to it a smoothing transformation so that the exact solution of the resulting equation does not contain any singularities in its derivatives up to a certain order. After that the regularized equation is solved by a piecewise polynomial collocation method on a mildly graded or uniform grid. In particular, a numerical method based on the Haar wavelets can be constructed.

Key–Words: - Weakly singular integro-differential equation, Smoothing, Collocation method, Haar wavelet

1 Introduction

We consider an initial-value problem of the form

$$y'(t) = a(t)y(t) + b(t) + \int_0^t K(t,s)y(s)ds, \quad t \in [0,T],\ T > 0, \tag{1}$$

$$y(0) = y_0, \quad y_0 \in \mathbb{R} = (-\infty, \infty). \tag{2}$$

Such problems arise naturally in many mathematical models of various physical and biological phenomena, see, e.g. [2, 3]. In what follows, we assume that

$$a, b \in C^{m,\nu}(0,T],\ K \in \mathcal{W}^{m,\nu}(\Delta_T),$$

$$m \in \mathbb{N} = \{1, 2, \ldots\},\ \nu \in \mathbb{R},\ \nu < 1.$$

Here, $C^{m,\nu}(0,T]$, $m \in \mathbb{N}$, $-\infty < \nu < 1$, is defined as the set of all m times continuously differentiable functions $u : (0,T] \to \mathbb{R}$ such that the estimation

$$|u^{(k)}(t)| \le c \begin{cases} 1 & \text{if} \quad k < 1-\nu, \\ 1 + |\log t| & \text{if} \quad k = 1-\nu, \\ t^{1-\nu-k} & \text{if} \quad k > 1-\nu \end{cases} \tag{3}$$

holds with a constant $c = c(u)$ for all $t \in (0,T]$ and $k = 0, 1, \ldots, m$.

Note that $C^m[0,T]$, the set of m times continuously differentiable functions on $[0,T]$, belongs to $C^{m,\nu}(0,T]$ for arbitrary $\nu < 1$. On the other hand, a function $u \in C^{m,\nu}(0,T]$ ($m \in \mathbb{N}$, $\nu < 1$) is uniformly continuous on $(0,T]$ and therefore has a continuous extension to the closed interval $[0,T]$; below such an extension will be denoted again by u.

The set $\mathcal{W}^{m,\nu}(\Delta_T)$, $m \in \mathbb{N}$, $-\infty < \nu < 1$, with

$$\Delta_T = \{(t,s) \in \mathbb{R}^2 : \ 0 \le t \le T,\ 0 \le s < t\},$$

consists of all m times continuously differentiable functions $K\colon\ \Delta_T \to \mathbb{R}$ satisfying

$$\left| \left(\frac{\partial}{\partial t}\right)^i \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial s}\right)^j K(t,s) \right| \le c \begin{cases} 1 & \text{if} \quad \nu + i < 0, \\ 1 + |\log(t-s)| & \text{if} \quad \nu + i = 0, \\ (t-s)^{-\nu-i} & \text{if} \quad \nu + i > 0, \end{cases} \tag{4}$$

with a constant $c = c(K)$ for all $(t,s) \in \Delta_T$ and all non-negative integers i and j such that $i + j \le m$.

It follows from (4) (with $i = j = 0$, $0 \le \nu < 1$) that the kernel $K(t,s)$ of equation (1) may possess a weak singularity as $s \to t$. In the case $\nu < 0$, the kernel $K(t,s)$ is bounded on Δ_T, but its derivatives may be singular as $s \to t$. Often the kernel K of equation (1) has the form $K(t,s) = K_\nu(t,s)$, with

$$K_\nu(t,s) = \kappa(t,s)(t-s)^{-\nu}, \quad 0 < \nu < 1,$$

or

$$K_0(t,s) = \kappa(t,s)\log(t-s),$$

where κ is a m times continuously differentiable function on $\overline{\Delta}_T = \{(t,s) \in \mathbb{R}^2 : \ 0 \le s \le t \le T\}$. Clearly, $K_\nu \in \mathcal{W}^{m,\nu}(\Delta_T)$ and $K_0 \in \mathcal{W}^{m,0}(\Delta_T)$.

The regularity of the solution of equation (1) can be characterized by the following lemma.

Lemma 1 [4] *Let $a, b \in C^{m,\nu}(0,T]$, $K \in \mathcal{W}^{m,\nu}(\Delta_T)$, $m \in \mathbb{N}$, $-\infty < \nu < 1$. Then the Cauchy problem $\{(1),(2)\}$ has a unique solution $y \in C^{m+1,\nu-1}(0,T]$.*

Thus, under the conditions of Lemma 1, the solution $y(t)$ of equation (1) and its first derivative $y'(t)$ are continuous for $t \in [0,T]$ but $y''(t),\ldots,y^{(m)}(t)$ may be singular as $t \to 0$. If one wants to construct for $\{(1),(2)\}$ a numerical algorithm possessing a high order convergence on the whole interval $[0,T]$, one has take into account, in some way, the possible singular behaviour of the exact solution. In collocation methods this behaviour can be taken into account by using special graded grids

$$\Pi_N = \{t_0,\ldots,t_N : 0 = t_0 < t_1 < \ldots < t_N = T\} \tag{5}$$

with the nodes

$$t_j = T(j/N)^r, \quad j = 0,\ldots,N. \tag{6}$$

Here the real number $r \in [1,\infty)$ characterizes the non-uniformity of the grid Π_N. If $r = 1$ then the grid points (6) are distributed uniformly; if $r > 1$, then the grid points (6) are more densely clustered near the left endpoint of the interval $[0,T]$ where the derivatives of the solution of equation (1) may be singular.

By using a collocation method based on {(5),(6)} and piecewise polynomials of degree $m-1$, one can reach a convergence of order N^{-m} for sufficiently large values of r. For instance, in case $0 < \nu < 1$ the convergence behaviour of order N^{-m} is available for $r \geq m/(1-\nu)$, see [4, 5]. A problem which may arise with the use of graded grids for large values of r is that it can sometimes create significant round-off errors in calculations over a long interval of integration, since a number of these calculations is performed with a very small step size in the neighbourhood of the left endpoint of the interval $[0,T]$.

The purpose of the present paper is to construct such high order algorithms for the numerical solution of problem $\{(1),(2)\}$ which do not need strongly graded grids. To this end, we first introduce an equivalent integral equation reformulation of $\{(1),(2)\}$. Then we apply to it a smoothing transformation so that the singularities of the derivatives of the exact solution of the resulting equation will be milder or disappear. After that we solve the transformed equation by a piecewise polynomial collocation method on a mildly graded or uniform grid. In particular, a Haar wavelet solution can be constructed.

Our approach is based on the ideas of [9, 10, 11, 12] (see also [1, 6, 7, 8]) and the smoothness properties of the exact solution of problem $\{(1),(2)\}$ given by Lemma 1.

2 Piecewise polynomial interpolation

For given integers $m \geq 0$ and $-1 \leq n \leq m-1$, let $S_m^{(n)}(\Pi_N)$ be the spline space of piecewise polynomial functions on the grid (5):

$$\begin{gathered} S_m^{(n)}(\Pi_N) = \{u\colon u|_{[t_{j-1},t_j]} =: u_j \in \pi_m, \\ j = 1,\ldots,N;\quad u_j^{(k)}(t_j) = u_{j+1}^{(k)}(t_j), \\ k = 0,\ldots,n;\ j = 1,\ldots,N-1\}. \end{gathered}$$

Here π_m denotes the set of polynomials of degree not exceeding m and $u|_{[t_{j-1},t_j]}$ is the restriction of u to the subinterval $[t_{j-1},t_j]$, $j = 1,\ldots,N$. Note that elements of $S_m^{(-1)}(\Pi_N) = \{u\colon u|_{[t_{j-1},t_j]} \in \pi_m$, $j = 1,\ldots,N\}$ may have jump discontinuities at the interior grid points $t_1,\ldots,t_{N-1}$.

In every subinterval $[t_{j-1},t_j]$, we introduce m interpolation points $t_{j1},\ldots,t_{jm}$:

$$\begin{aligned} &t_{jk} = t_{j-1} + \eta_k(t_j - t_{j-1}), \\ &k = 1,\ldots,m; j = 1,\ldots,N, \end{aligned} \tag{7}$$

where the parameters $\eta_1,\ldots,\eta_m$ do not depend on j and N and satisfy

$$0 \leq \eta_1 < \ldots < \eta_m \leq 1. \tag{8}$$

To a given continuous function $z : [0,T] \to \mathbb{R}$ we assign a piecewise polynomial interpolation function $P_N z = P_N^{(m)} z \in S_{m-1}^{(-1)}(\Pi_N)$ which interpolates z at the points (7): $(P_N z)(t_{jk}) = z(t_{jk})$, $k = 1,\ldots,m$; $j = 1,\ldots,N$. Thus, $(P_N z)(t)$ is independently defined in every subinterval $[t_{j-1},t_j]$, $j = 1,\ldots,N$, and may be discontinuous at the interior grid points $t = t_j$, $j = 1,\ldots,N-1$. Note that in the case $\eta_1 = 0$, $\eta_m = 1$ (see (8)), $P_N z$ is a continuous function on $[0,T]$.

We introduce also an interpolation operator $P_N = P_N^{(m)}$ which assigns to every continuous function $z : [0,T] \to \mathbb{R}$ its piecewise polynomial interpolation function $P_N z$.

Lemma 2 [4, 13] *Let $z \in C^{m,\nu}(0,T]$, $m \in \mathbb{N}$, $-\infty < \nu < 1$, and let the interpolation nodes (7) with grid points (6) and parameters (8) be used. Then*

$$\max_{x\in[t_{j-1},t_j]} \left|z(x) - (P_N z)(x)\right| \leq c(t_j - t_{j-1})^m \begin{cases} 1 & \text{if m} < 1-\nu, \\ 1 + |\log t_j| & \text{if m} = 1-\nu, \\ t_j^{1-\nu-m} & \text{if m} > 1-\nu, \end{cases}$$

where $j = 1, \dots, N$. Moreover, we have

$$\sup_{x\in[0,T]} \left|z(x) - (P_N z)(x)\right| \le c\varepsilon_N^{(m,\nu,r)} . \qquad (9)$$

Here c is a positive constant which do not depend on N and

$$\varepsilon_N^{(m,\nu,r)} = \begin{cases} N^{-m}, & m < 1-\nu, r \ge 1\,, \\ N^{-m}(1+\log N), & m = 1-\nu, r = 1\,, \\ N^{-m}, & m = 1-\nu, r > 1\,, \\ N^{-r(1-\nu)}, & m > 1-\nu, \\ & 1 \le r < m/(1-\nu)\,, \\ N^{-m}, & m > 1-\nu, \\ & r \ge m/(1-\nu). \end{cases} \qquad (10)$$

In what follows, for given Banach spaces E and F we denote by $\mathcal{L}(E,F)$ the Banach space of linear bounded operators $A\colon\ E \to F$ with the norm $||A|| = \sup\{||Az|| :\ z \in E,\ ||z|| \le 1\}$. By $C(\Omega)$ we denote the set of continuous functions on Ω. In particular, by $C[0,T]$ we denote the Banach space of continuous functions $z : [0,T] \to \mathbb{R}$ with the norm $\|z\|_{C[0,T]} = \max\{|z(x)| : 0 \le x \le T\}$. By $c, c_1, c_2, \dots$ we will denote positive constants which may be different in different occurrences.

Lemma 3 [4] *Let $S\colon\ L^\infty(0,T) \to C[0,T]$ be a linear compact operator. Then*

$$||S - P_N S||_{\mathcal{L}(L^\infty(0,T),L^\infty(0,T))} \to 0 \quad \text{as} \quad N \to \infty.$$

3 Smoothing transformation

Following [10], [12], for given $m \in \mathbb{N}$ and $1 \le d \in \mathbb{R}$, we denote by $\Phi_{m,d}$ the set of all transformations of the form $\varphi : [0,T] \to [0,T]$ that map $[0,T]$ onto $[0,T]$ and satisfy the following conditions:

$$\varphi \in C^m[0,T]\,, \qquad (11)$$

$$c_1 x^{d-1} \le \varphi'(x) \le c_2 x^{d-1},\ 0 \le x \le T, \qquad (12)$$

$$\left|\varphi^{(j)}(x)\right| \le c x^{d-j},\ 0 \le x \le T,\ 0 \le j \le \min\{d,m\}, \qquad (13)$$

where $c_2 \ge c_1 > 0$ and $c > 0$ are some constants.

It follows from (11) and (12) that a transformation $\varphi \in \Phi_{m,d}$ has a continuous inverse $\varphi^{-1} : [0,T] \to [0,T]$, $\varphi^{-1}(0) = 0$. Moreover, we have (see [10])

$$\left|\varphi(x_1) - \varphi(x_2)\right| \ge c_0|x_1 - x_2|\left(x_1^{d-1} + x_2^{d-1}\right),$$
$$x_1, x_2 \in [0,T],$$

with a constant $c_0 > 0$ which is independent of $x_1, x_2 \in [0,T]$.

A simplest example of a transformation $\varphi \in \Phi_{m,d}$ is given by

$$\varphi(x) = T^{1-d}x^d, \quad 0 \le x \le T, \quad d \in \mathbb{N}. \qquad (14)$$

We are interested in transformations $\varphi \in \Phi_{m,d}$ with $d > 1$ since they possess a smoothing property for $z(\varphi(x))$ with singularities of $z(t)$ at $t = 0$, see Lemma 4 below.

Lemma 4 [10] *Assume that:*
1) $z \in C^{m,\nu}(0,T]$, $m \in \mathbb{N}$, $\nu \in \mathbb{R}$, $\nu < 1$;
2) $\varphi \in \Phi_{m,d}$, where $d \in \mathbb{N}$ in case $d \le m$ and $d \in \mathbb{R}$ in case $d > m$;
3) $u(x) = z(\varphi(x))$, $x \in [0,T]$.
Then $u \in C^{m,\nu_d}(0,T]$ with $\nu_d = 1 - d(1-\nu)$.

4 Numerical method

First we consider a reformulation of problem $\{(1),(2)\}$. It is based on introducing a new unknown function

$$z = y'.$$

Using $y' = z$, $\{(1),(2)\}$ may be rewritten as a linear Volterra integral equation of the second kind with respect to z:

$$\begin{aligned} z(t) &= f(t) + a(t)\int_0^t z(s)ds \\ &+\int_0^t K(t,s)\left(\int_0^s z(\tau)d\tau\right) ds, \quad t \in [0,T], \end{aligned} \qquad (15)$$

where

$$f(t) = b(t) + y_0 a(t) + y_0 \int_0^t K(t,s)ds,\ t \in [0,T]. \qquad (16)$$

Next we introduce in (15) a change of variables. The aim of the change of variables is to obtain a new integral equation whose solution does not involve any more singularities in its derivatives up to a certain order.

Let $\varphi \in \Phi_{m,d}$, $m \in \mathbb{N}$, $d \ge 1$. Introducing in (15) the change of variables

$$t = \varphi(x),\ s = \varphi(\mu),\ \tau = \varphi(\sigma),\ x, \mu, \sigma \in [0,T],$$

we obtain an integral equation of the form

$$\begin{aligned} z_\varphi(x) &= f_\varphi(x) + a_\varphi(x)\int_0^x z_\varphi(\mu)\varphi'(\mu)d\mu \\ &+ \int_0^x K_\varphi(x,\mu)\Big(\int_0^\mu z_\varphi(\sigma)\varphi'(\sigma)d\sigma\Big)d\mu,\ 0 \le x \le T, \end{aligned} \qquad (17)$$

where

$$f_\varphi(x) = f(\varphi(x)), \quad 0 \le x \le T,$$
$$a_\varphi(x) = a(\varphi(x)), \quad 0 \le x \le T,$$
$$K_\varphi(x,\mu) = K(\varphi(x),\varphi(\mu))\varphi'(\mu), \quad 0 \le \mu < x \le T,$$

and

$$z_\varphi(x) = z(\varphi(x)) \quad (0 \le x \le T) \tag{18}$$

is a function which we have to find. Changing the order of integration in the double integral of (17), we present (17) in the form

$$z_\varphi = f_\varphi + T_\varphi z_\varphi, \tag{19}$$

where

$$(T_\varphi z_\varphi)(x) = \int_0^x L_\varphi(x,\mu) z_\varphi(\mu)d\mu, \; x \in [0,T], \tag{20}$$

with

$$\begin{aligned} L_\varphi(x,\mu) &= a_\varphi(x)\varphi'(\mu) \\ &+\varphi'(\mu)\int_\mu^x K_\varphi(x,\sigma)d\sigma, \quad 0 \le \mu \le x \le T. \end{aligned} \tag{21}$$

Since $a, b \in C^{m,\nu}(0,T]$, $K \in \mathcal{W}^{m,\nu}(\Delta_T)$, $m \in \mathbb{N}$, $\nu < 1$, $\varphi \in \Phi_{m,d}$, we obtain that $K_\varphi \in C(\Delta_T)$ and

$$|K_\varphi(x,\mu)| = |K(\varphi(x),\varphi(\mu))|\varphi'(\mu)$$
$$\le c \begin{cases} 1 & \text{if } \nu < 0, \\ 1 + |\log(x-\mu)| & \text{if } \nu = 0, \\ (x-\mu)^{-\nu} & \text{if } \nu > 0, \end{cases}$$

where $0 \le \mu < x \le T$. Now it follows from (21) and (20) that $L_\varphi \in C(\overline{\Delta}_T)$, $T_\varphi \in \mathcal{L}(L^\infty(0,T), C[0,T])$ and

$$T_\varphi : L^\infty(0,T) \to C[0,T] \quad \text{is compact.} \tag{22}$$

This together with $f_\varphi \in C[0,T]$ yields that equation (19) (equation (17)) has a unique solution $z_\varphi \in C[0,T]$.

We look for an approximation $v = v_{N,m,r,\varphi}$ to the solution z_φ of equation (17) in $S_{m-1}^{(-1)}(\Pi_N)$, $m, N \in \mathbb{N}$. We determine

$$v \in S_{m-1}^{(-1)}(\Pi_N) \; (m \ge 1) \tag{23}$$

by the collocation method from the following conditions:

$$\begin{aligned} v_j(t_{jk}) &= f_\varphi(t_{jk}) + a_\varphi(t_{jk}) \int_0^{t_{jk}} v(\mu)\varphi'(\mu)d\mu \\ &+ \int_0^{t_{jk}} K_\varphi(t_{jk},\mu)\Big(\int_0^\mu v(\sigma)\varphi'(\sigma)d\sigma\Big)d\mu, \\ & k = 1,\dots,m; \; j = 1,\dots,N, \end{aligned} \tag{24}$$

where $v_j = v|_{[t_{j-1},t_j]}$ is the restriction of v to the interval $[t_{j-1},t_j]$, $j = 1,\dots,N$, and the set of collocation points $\{t_{jk}\}$ is given by (7).

Having determined the approximation v for z_φ, we can determine an approximation $u = u_{N,m,r,\varphi}$ for y, the solution of the Cauchy problem $\{(1),(2)\}$, setting

$$u(t) = y_0 + \int_0^t v(\varphi^{-1}(s))ds, \quad 0 \le t \le T. \tag{25}$$

Remark 5 *The choice of parameters* (8) *with* $\eta_1 = 0$, $\eta_m = 1$ *in* $\{(23),(24)\}$ *actually implies that the resulting collocation approximation* v *belongs to the smoother polynomial spline space* $S_{m-1}^{(0)}(\Pi_N)$.

Remark 6 *Conditions* (23) *and* (24) *determine a system of linear equations whose exact form is specified by the choice of a basis in the space* $S_{m-1}^{(-1)}(\Pi_N)$ *(or in* $S_{m-1}^{(0)}(\Pi_N)$ *if* $\eta_1 = 0$, $\eta_m = 1$*). For example, one can use on each subinterval* $[t_{j-1},t_j] \subset [0,T]$ *(*$j = 1,\dots,N$*) a representation of the form*

$$v(x) = \sum_{l=1}^m c_{jl} \prod_{\substack{k=1\\k\ne l}}^m \Big(\frac{x - t_{jk}}{t_{jl} - t_{jk}}\Big), \; t_{j-1} \le x \le t_j,$$

where $\{t_{jk}\}$ *are the points* (7) *and* $\{c_{jl}\}$ *are unknown coefficients. Now the above-mentioned conditions acquire the form of linear equations for the coefficients* $\{c_{jl}\}$.

Theorem 7 *Let* $a, b \in C^{m,\nu}(0,T]$, $K \in \mathcal{W}^{m,\nu}(\Delta_T)$, $m \in \mathbb{N}$, $-\infty < \nu < 1$. *Let* $\varphi \in \Phi_{m,d}$, *with* $d \in \mathbb{N}$ *in case* $d \le m$ *and* $d \in \mathbb{R}$ *in case* $d > m$. *Finally, assume that the collocation points* (7)*, with grid points* (6) *and parameters* (8)*, are used.*

Then, for all sufficiently large $N \in \mathbb{N}$*, say* $N \ge N_0$*, and every choice of parameters* (8) *with* $\eta_1 > 0$ *or* $\eta_m < 1$*, the settings* (25) *and* $\{(23),(24)\}$ *determine unique approximations* $u \in S_m^{(0)}(\Pi_N)$ *and* $v \in S_{m-1}^{(-1)}(\Pi_N)$ *to the solution* y *of the Cauchy problem* $\{(1),\ (2)\}$ *and its derivative* y'*, respectively. If in* (8) $\eta_1 = 0$, $\eta_m = 1$*, then* $u \in S_m^{(1)}(\Pi_N)$ *and* $v \in S_{m-1}^{(0)}(\Pi_N)$*. Moreover, for* $N \ge N_0$ *the following error estimates hold:*

$$\max_{0 \le t \le T} |u(t) - y(t)| \le c\varepsilon_N^{(m,\nu_d,r)}, \tag{26}$$

$$\sup_{0 \le t \le T} |v(\varphi^{-1}(t)) - y'(t)| \le c\varepsilon_N^{(m,\nu_d,r)}. \tag{27}$$

Here $\nu_d = 1 - d(1-\nu)$, $\varepsilon_N^{(m,\nu_d,r)}$ *is defined by* (10) *and* c *is a positive constant not depending on* N.

Proof: We already know that equation (19) (equation (17)) has a unique solution $z_\varphi = (I - T_\varphi)^{-1} f_\varphi \in C[0,T]$. Here I is the identity mapping and $(I - T_\varphi)^{-1} \in \mathcal{L}(C[0,T], C[0,T])$. It follows from [4] that equation (15) has a unique solution $z \in C^{m,\nu}(0,T]$. This together with (18) and Lemma 4 yields that $z_\varphi \in C^{m,\nu_d}(0,T]$, with $\nu_d = 1 - d(1-\nu)$.

Further, the conditions (23) and (24) have the operator equation representation

$$v - P_N T_\varphi v = P_N f_\varphi, \tag{28}$$

with T_φ, given by (20), and P_N, defined in Section 2. From (22), Lemma 3 and from the boundedness of $(I - T_\varphi)^{-1}$ in $L^\infty(0,T)$ we obtain that $I - P_N T_\varphi$ is invertible in $L^\infty(0,T)$ for all sufficiently large N, say $N \geq N_0$, and the norms of $(I - P_N T_\varphi)^{-1}$ are uniformly bounded in N,

$$\|(I - P_N T_\varphi)^{-1}\|_{\mathcal{L}(L^\infty(0,T), L^\infty(0,T))} \leq c, \; N \geq N_0, \tag{29}$$

with a constant c which is independent of N. Thus, equation (28) has a unique solution $v \in S_{m-1}^{(-1)}(\Pi_N)$ for $N \geq N_0$. We have for it and z_φ, the solution of equation (19) that

$$v - z_\varphi = (I - P_N T_\varphi)^{-1}(P_N z_\varphi - z_\varphi). \tag{30}$$

Therefore, by (29),

$$\|v - z_\varphi\|_{L^\infty(0,T)} \leq c\|P_N z_\varphi - z_\varphi\|_{L^\infty(0,T)}, \tag{31}$$

where $N \geq N_0$ and c is a positive constant not depending on N. Further, we have

$$\begin{aligned} \|v - z_\varphi\|_{L^\infty(0,T)} &= \sup_{x\in[0,T]} |v(x) - z_\varphi(x)| \\ &= \sup_{t\in[0,T]} |v(\varphi^{-1}(t)) - y'(t)|. \end{aligned} \tag{32}$$

Taking into account that $z_\varphi \in C^{m,\nu_d}(0,T]$, $\nu_d = 1 - d(1-\nu)$, and applying Lemma 2, we obtain from (31) and (32) the estimate (27). Since

$$\begin{aligned} &|u(t) - y(t)| \\ &\leq \int_0^t |v(\varphi^{-1}(s)) - y'(s)| ds, \; 0 \leq t \leq T, \end{aligned} \tag{33}$$

the estimate (26) is a consequence of (27). □

Remark 8 *According to* (26)*, in the case $m > d(1-\nu)$, the estimate*

$$\max_{0\leq t\leq T} |u(t) - y(t)| \leq cN^{-m} \tag{34}$$

is guaranteed for $r \geq m/d(1-\nu)$. For ν close to 1 ($\nu < 1$), this condition to r may be too restrictive. Actually, to obtain the estimate (34)*, the condition on r can be considerable relaxed, as shown in the following theorem.*

Theorem 9 *Let the conditions of Theorem 7 be fulfilled. Then, with the notation of Theorem 7, we have the following estimates for $N \geq N_0$:*

1) if $m \leq 2 - \nu_d = 1 + d(1-\nu)$, then

$$\max_{0\leq t\leq T} |u(t) - y(t)| \leq c\varepsilon_N^{(m,\nu_d - 1, r)};$$

2) if $m > 2 - \nu_d$, then

$$\max_{0\leq t\leq T} |u(t) - y(t)|$$

$$\leq c \begin{cases} N^{-r(2-\nu_d)} & \text{for } 1 \leq \mathrm{r} < \mathrm{m}/(2-\nu_\mathrm{d}), \\ N^{-m}(1 + \log N) & \text{for } \quad \mathrm{r} = \mathrm{m}/(2-\nu_\mathrm{d}), \\ N^{-m} & \text{for } \quad \mathrm{r} > \mathrm{m}/(2-\nu_\mathrm{d}). \end{cases} \tag{35}$$

Proof: We consider only the case $m > 2 - \nu_d$. For $m \leq 2 - \nu_d$ the argument is similar. Using the equality

$$(I - P_N T_\varphi)^{-1} = I + (I - P_N T_\varphi)^{-1} P_N T_\varphi, \; N \geq N_0,$$

we rewrite the error (30) in the form

$$\begin{aligned} v - z_\varphi &= P_N z_\varphi - z_\varphi \\ &+ (I - P_N T_\varphi)^{-1} P_N T_\varphi (P_N z_\varphi - z_\varphi). \end{aligned} \tag{36}$$

Due to continuity of $L_\varphi(x,\mu)$ on $\overline{\Delta}_T$, the operator T_φ is bounded as an operator from $L^1(0,T)$ into $C[0,T]$. An observation shows that

$$\|P_N\|_{\mathcal{L}(C[0,T], L^\infty(0,T))} \leq c, \; N \in \mathbb{N}. \tag{37}$$

This together with $y' = z$, $\varphi \in \Phi_{m,d}$, (25) and (36) yields

$$\begin{aligned} |u(\varphi(x)) - y(\varphi(x))| &= |\int_0^x [v(\mu) - z_\varphi(\mu)]\, \varphi'(\mu) d\mu| \\ &\leq \int_0^x |v(\mu) - z_\varphi(\mu)| \varphi'(\mu) d\mu \\ &\leq c \int_0^T |(P_N z_\varphi)(s) - z_\varphi(s)| ds, \end{aligned}$$

where $0 \leq x \leq T$ and c is a constant not depending on N. Then, by Lemma 2,

$$\begin{aligned} &|u(\varphi(x)) - y(\varphi(x))| \\ &\leq c \sum_{l=1}^{N} \int_{t_{l-1}}^{t_l} |(P_N z_\varphi)(s) - z_\varphi(s)| ds \\ &\leq c_1 \sum_{l=1}^{N} (t_l - t_{l-1})^{m+1} t_l^{1-\nu_d - m}, \end{aligned}$$

where $0 \le x \le T$ and the constant c_1 in independent of N. It follows from (6) that

$$(t_l - t_{l-1})^{m+1} t_l^{1-\nu_d - m} \le T^{2-\nu_d} r^{m+1} N^{-r(2-\nu_d)} l^{r(2-\nu_d)-m-1},$$

where $l = 1, \dots, N$. Therefore

$$\max_{t\in[0,T]} |u(t) - y(t)| = \max_{x\in[a,b]} |u(\varphi(x)) - y(\varphi(x))| \le cN^{-r(2-\nu_d)} \sum_{l=1}^{N} l^{r(2-\nu_d)-m-1}, \tag{38}$$

with a constant c not depending on N. Furthermore, for a fixed $\alpha \in \mathbb{R}$ we have

$$\sum_{l=1}^{N} l^\alpha \le c \begin{cases} N^{\alpha+1} & \text{if} \quad \alpha > -1, \\ 1 + \log N & \text{if} \quad \alpha = -1, \\ 1 & \text{if} \quad \alpha < -1, \end{cases} \tag{39}$$

with a constant c which is independent of N. Applying (39) with $\alpha = r(2 - \nu_d) - m - 1$ to (38) it is easy to see that the estimate (35) holds. □

5 Concluding remarks

This paper has been concerned with the numerical solution of linear weakly singular Volterra integro-differential equations. The solutions of such equations are typically nonsmooth at the left endpoint of the interval of integration $[0, T]$, where their higher order derivatives become unbounded. If one wants to construct a high order numerical method for such equations one has take into account, in some way, the singular behaviour of the exact solution. It can be done using polynomial splines on special non-uniform grids. A problem which may arise with the use of strongly non-uniform grids is that it can sometimes create significant round-off errors in calculations. The approach discussed in this paper allows us to construct such high order algorithms for solving weakly singular integro-differential equations which do not need strongly non-uniform grids. In particular, numerical schemes of arbitrary high order on the uniform grid can be constructed.

Acknowledgements: This work was supported by Estonian Science Foundation, Grant 5859.

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