System of Models for Transport Processes in Layered Strata

MARGARITA BUIKE, ANDRIS BUIKIS
Institute of Mathematics and Computer Science
University of Latvia
Raina bulv. 29, Riga, LV1459
LATVIA
http://www.lza.lv/scientists/buikis.htm

Abstract: - In this paper the normalized form of the generalized integral parabolic spline is described, which interpolates the integral averaged values of piecewise-smooth function. The three-dimensional system of partial differential equations as model of transport processes for porous layered stratum with semi-permeable interlayer is proposed. The generalized integral parabolic spline is used for the approximate transformation of the 3-D problem into 2-D system by the original method of conservative averaging. The order of the 2-D system is equal to the number of productive layers. This system of 2-D partial differential equations with continuous coefficients fulfills all conservation laws of initial problem in averaged sense.

Key-Words: - integral spline, layered media, transport processes, three-dimensional, conservative averaging.

1 Introduction
Many real processes take place in layered systems, consisting of separate layers with different thickness and different physical properties [1]-[4]. Very often such systems are composed from two types of layers which are located alternatively, e.g. aquifers and aquitards (aquiludes) [1] in groundwater flows. In these cases, in the mathematical model of such processes we have jump in coefficients of differential equations on the surfaces between two layers. Discontinuity of coefficients of PDE (partial differential equations) brings out additional difficulties by standard use of traditional mathematical methods.

One of the authors has developed the conservative averaging method and he introduced a special new type of splines for wide class of PDE problems with discontinuous coefficients [5]-[9].

In this paper we give a description of generalized integral parabolic spline (GIPS) in so called normalized form [10], [11] and employ this spline for some groundwater (or other fluids) flows and pollution problems in layered stratum, which consists of two different types of layers: main (productive) and interlayers (semi-permeable layers). This type of models automatically reduces (if the thickness of interlayers trends to zero) to models, which have only productive layers - considered in our previous works [10], [11]. Proposed method differs from methods traditionally used in various transport processes in natural or artificial porous media. Our method as outcome gives little bit more complicated mathematical model, but it allows analyzing broader spectrum of physical phenomena and wider variety of geometrical and physical parameters.

2 Integral Splines
In this section we introduce the integral averaged values interpolating special splines (parabolic and generalized parabolic).

2.1 Interpolation Problem for Piecewise-Smooth Function
Let it be given a continuous, piecewise-smooth function \( U(x), x \in [a,b] \). Further, let it be given, that in the different inner points \( x_i, i = 1, ..., N \) the first derivative \( U'(x) \) of the function has a finite jump:

\[
k_{i-1} U'(x_i-0) = k_i U'(x_i+0).
\]  

Here coefficients \( k_i > 0, i = 0, N \) are known for all \( x \in (x_i, x_{i+1}) \), \( x_0 = a, x_{N+1} = b \). Since the function \( U(x) \) is continuous on the closed interval \([a,b] \), we additionally have following continuity equalities in these points:

\[
U(x_i-0) = U(x_i + 0), i = 1, ..., N.
\]  

Let additionally be given the integral averaged values \( u_i \) of the function \( U(x) \) over the all subsegments \([x_i, x_{i+1}]\):

Proceedings of the 5th WSEAS Int. Conf. on System Science and Simulation in Engineering, Tenerife, Canary Islands, Spain, December 16-18, 2006
The interpolation problem consists in approximate reconstruction of the function \( U(x) \), and it is based on conditions (1)-(3) and following general boundary conditions (BC) on end points \( x = a \) and \( x = b \):

\[
-v_k \lambda U'(a) + \lambda U(a) = \Phi_0, \\
\nu_k \lambda U'(b) + \lambda U(b) = \Phi_1.
\]

(4)

(5)

This interpolation problem can be solved by polynomial spline IPS

\[
S(x) = u_i + m_i(x - x_i) + e_i \left[ \frac{(x-x_i)^2}{k_i H_i} \right],
\]

(6)

\[
x_i = \frac{x + x_{i+1}}{2}, G_i = \frac{H_i}{k_i} > 0.
\]

For the determination of \( 2(N + 1) \) free coefficients, we have exactly the same number of equations (1),(2),(4) and (5). In [5], [6], [10] and [11] it was proved, that all coefficients \( m_i \) can be represented through coefficients \( e_i \) in two forms:

a) for \( i = 0, ..., N - 1 \)

\[
k_m(G_i + G_{i+1}) = 2(u_{i+1} - u_i) - e_i(G_i/3 + G_{i+1}) - 2/3 e_i G_{i+1};
\]

(7a)

b) for \( i = 1, ..., N \)

\[
k_m(G_i + G_{i-1}) = 2(u_i - u_{i-1}) + e_i(G_i/3 + G_{i-1}) + 2/3 e_i G_{i-1}.
\]

(7b)

The Elimination of the coefficients \( m_i \) from these expressions gives us for \( i = 1, ..., N - 1 \) following system regarding \( e_i \) (see [10] and [11]):

\[
a_i e_{i+1} + (1 + a_i + b_i) e_i + b_i e_{i-1} = f_i^* u_{i+1} - f_i u_i + f_i^* u_{i-1}, i = 1, ..., N - 1.
\]

(8)

Here \( f_i = f_i^* + f_i^+ \) and

\[
a_i = G_{i-1}/(G_i + G_{i-1}), b_i = G_{i+1}/(G_i + G_{i+1}),
\]

\[
f_i^+ = 3/(G_i + G_{i+1}), f_i^* = 3/(G_i + G_{i-1}).
\]

(9)

For the transformation of the BC (4), (5) some additional notations must be used and two different cases are distinguished:

1) \( \lambda_0 \neq 0 \) (and \( \lambda_i \neq 0 \)). Then

\[
G_{i-1} = 2 \nu / \lambda_0, G_{N+1} = 2 \nu / \lambda_1,
\]

\[
u_i = \Phi_i / \lambda_0, u_{N+1} = \Phi_1 / \lambda_1;
\]

(10a)

2) \( \lambda_0 = 0 \) (and \( \lambda_i = 0 \)). Then

\[
G_{i-1} = 2 \nu - G_0, G_{N+1} = 2 \nu - G_N,
\]

\[
u_i = \Phi_i + u_0, u_{N+1} = \Phi_1 + u_N.
\]

(10b)

We have obtained from BC (4), (5) in papers [10], [11] following equations for the first and last equations:

(1) \( a_0 e_0 + b_0 e_0 = f_0 u_0 - f_0 u_0 + f_0^* u_1, \)

(11)

\[
a_0 e_{N-1} + (1 + a_N + b_N) e_N = f_N^* u_{N-1} - f_N u_N + f_N^* u_{N+1}.
\]

In the case \( \lambda_0 = 0 \) (\( \lambda_i = 0 \)) we have special formulas for coefficients \( a_0 \) and \( b_N \)

\[
a_0 = b_N = 1. \]

We proposed in our papers [10], [11] the representation for coefficients \( e_i \) of IPS through all averaged integral values. This representation shows in explicit form also the impact of the BC type and its right hand side on the spline:

\[
e_i = \gamma_i^{(0)} f_0 u_{i-1} + \gamma_i^{(1)} f_N u_{N+1} + \sum_{j=0}^{N} \beta_j u_j, i = 0, N.\]

(12)

The coefficients in the representation (12) are determinate from three systems of linear algebraic equations.

a) The system for \( \gamma_i^{(0)} \) (shows the impact of BC (4)):

\[
(1 + a_0 + b_0) \gamma_i^{(0)} + b_0 \gamma_{i+1}^{(0)} = 1,
\]

\[
a_0^{(0)} + (1 + a_i + b_i) \gamma_i^{(0)} + b_i \gamma_{i+1}^{(0)} = 0, \]

\[
i = 1, ..., N - 1,
\]

(13a)

b) The system for \( \gamma_i^{(1)} \) (shows the impact of BC (5)):

\[
(1 + a_0 + b_0) \gamma_i^{(1)} + b_0 \gamma_{i+1}^{(1)} = 0,
\]

\[
a_0^{(1)} + (1 + a_i + b_i) \gamma_i^{(1)} + b_i \gamma_{i+1}^{(1)} = 0, \]

\[
i = 1, ..., N - 1,
\]

(13b)

c) The \( N + 1 \) systems ( \( j = 0, ..., N \) ) for \( \beta_j \):

\[
(1 + a_0 + b_0) \beta_{ij} + a_i \beta_{i-1,j} + b_i \beta_{i+1,j} = f_j \delta_{i-1,j} - f_j \delta_{i,j} + f_j^* \delta_{i+1,j}, i = 1, ..., N - 1,
\]

\[
a_0 \beta_N + (1 + a_N + b_N) \beta_{N,j} = 0.
\]

(14)
2.2 Model of Heat Conduction in Multilayered Bar

Now, by easiest one-dimensional multi-layer temperature conduction model, we will show how IPS can be used to simplify the initial problem. We consider following heat conduction model with piecewise-constant coefficients:

\[
\frac{c_i}{\partial t} = \frac{\partial}{\partial x} \left( k_i \frac{\partial^2 U}{\partial x^2} \right) + F_i(x,t), x_i < x < x_{i+1}, \quad i = 0, N,
\]

\[
U_{i-1}(x - 0) = U_i(x + 0),
\]

\[
k_i \frac{\partial U_i}{\partial x}(x - 0) = k_j \frac{\partial U_j}{\partial x}(x + 0), \quad i = 0, N, \quad j = 0, N.
\]

For the spline coefficient \(e_i\), this finally gives:

\[
c_i \frac{du_i}{dt} = \frac{2}{H_i} \left[ \sum_{j=0}^{N} \beta_j u_j + \gamma_i^{(0)} f_0^i u_{i-1} + \gamma_i^{(1)} f_N^i u_{N+1} \right] + f_i(t), i = 0, N, t \in (0, T).
\]

To this system of ordinary differential equations (ODE) averaged over sub-segments \([x_i, x_{i+1}]\), initial conditions (18) must be added (the conjugations conditions (16) and BC (17) were already used by construction of IPS):

\[
u_i(x, 0) = u_i^0 := \frac{1}{H_i} \int_{x_i}^{x_{i+1}} U_i^0(x) dx.
\]

So, we have reduced the one-dimensional problem with discontinuous coefficients for PDE to the system of ODE with continuous coefficients. After solving problem (20), (21), we can approximately reconstruct the solution of original problem (15)-(18) by IPS (6).

2.3 Generalized Integral Parabolic Spline

Let it again be given a continuous, piecewise-smooth function \(U(x), x \in [a, b]\), for which the first derivative \(U'(x)\) has first kind discontinuities in \(2N\) different inner points \(x_i, x_{i+1/2}, i = 1, ..., N\):

\[
k_{i-1/2} U'(x_{i-1/2} - 0) = k_{i+1/2} U'(x_{i+1/2} + 0),
\]

\[
k_{i-1} U'(x_i - 0) = k_U U'(x_i + 0),
\]

The continuity property of the function \(U(x)\) gives following \(2N\) equalities:

\[
U(x_{i+1/2} - 0) = U(x_{i+1/2} + 0),
\]

\[
U(x_i - 0) = U(x_i + 0).
\]

Here the function (coefficients) \(k(x)\) is piecewise-constant function:

\[
k(x) = \begin{cases} k_{i-1/2}, & \text{if } x \in (x_{i-1/2}, x_i), \\ k_i, & \text{if } x \in (x_i, x_{i+1/2}) \end{cases}
\]

with property \(k_{i-1/2} \ll k_{i-1}, \quad k_{i+1/2} \ll k_i\). Here must be mentioned that the \(x_{N+1/2} = x_{N+1} = b\), it means the function \(U(x)\) can neither end nor begin with a linear part of the segment \([a, b]\).

The \(N + 1\) integral averaged values \(\bar{u}_i\) of the function \(U(x)\) are given additionally over sub-segments \([x_i, x_{i+1/2}]\):

\[
\bar{u}_i = \frac{1}{H_i} \int_{x_i}^{x_{i+1/2}} U(x) dx, \quad \bar{H}_i = x_{i+1/2} - x_i, i = 0, N.
\]
Further, it is known, that on the sub-segments \([x_{i+1/2}, x_{i+1}],[x_{i+1}, x_{i+1/2}]\), \(i = 0, N - 1\) the function can be approximated by linear function.

Finally, the BC (4), (5) must be fulfilled.

In our paper [7] it was shown that there exists exactly one spline fulfilling all mentioned conditions. We will seek this spline in the form:

\[
\tilde{u}_i + \tilde{m}_i (x - \tilde{x}_i) + \tilde{e}_i \left[ \frac{(x - \tilde{x}_i)^2}{k_i H_i} - \tilde{G}_i \right],
\]

\[
\tilde{S}(x) = \begin{cases} 
\begin{align*}
\tilde{u}_i + \tilde{m}_i (x - \tilde{x}_i) + \tilde{e}_i \left[ \frac{(x - \tilde{x}_i)^2}{k_i H_i} - \tilde{G}_i \right], \\
\tilde{x}_i = \frac{x_j + x_{j+1}}{2}, i = 0, N; \\
u_{i-1/2} + m_{i-1/2} (x - x_{i-1/2}), x \in [x_{i-1/2}, x_i], \\
\tilde{x}_{i-1/2} = (x_{i-1/2} + x_{i})/2, i = 1, N.
\end{align*}
\end{cases}
\]

Here \(\tilde{G}_i = \tilde{H}_i / k_i\) are lengths parameters reduced by conductions coefficient, which can be called as “characteristic conduction lengths”. Similarly we introduce \(N\) additional lengths parameters for second type of layers \(G_{i-1/2} = H_{i-1/2} / k_{i-1/2}\), \(i = 1, N\), which will be used immediately.

The continuity (23) of generalized IPS (GIPS) at discontinuity points \(x_{i+1/2}, i = 0, N - 1\) gives following equalities:

\[
u_{i+1/2} - k_{i+1/2} m_{i+1/2} \frac{G_{i+1/2}}{2}.
\]

The same property at points \(x_i, i = 0, N - 1\) gives similar equalities:

\[
u_{i+1} - k_{i+1} m_{i+1} \frac{G_{i+1}}{2}.
\]

The conjugation conditions (22) lead to the equalities for \(i = 0, N - 1:\)

\[
k_i \tilde{m}_i + \tilde{e}_i = k_{i+1} \tilde{m}_{i+1} = k_{i+1/2} \tilde{m}_{i+1/2} - \tilde{e}_{i+1}.
\]

Equalities (28) allow us to exclude \(k_{i+1/2} m_{i+1/2}\) and \(k_{i+1} \tilde{m}_{i+1}\) from (26), (27). Instead of (7) we obtain following generalized chain of equalities for \(i = 0, N - 1:\)

\[
3k_i \tilde{m}_i (\tilde{G}_i + 2G_{i+1/2} + \tilde{G}_{i+1}) + \tilde{e}_i (\tilde{G}_i + 6G_{i+1/2} + 3\tilde{G}_{i+1})) + 2\tilde{e}_{i-1} \tilde{G}_{i+1} = 6(\tilde{u}_{i+1} - \tilde{u}_i).
\]

In its turn, from (7b) we obtain for \(i = 1, N:\)

\[
3k_i \tilde{m}_i (\tilde{G}_i + 2G_{i+1/2} + \tilde{G}_{i+1}) + \tilde{e}_i (\tilde{G}_i + 6G_{i+1/2} + 3\tilde{G}_{i+1}) - 2\tilde{e}_{i-1} \tilde{G}_{i+1} = 6(\tilde{u}_{i+1} - \tilde{u}_i).
\]

The last step consists of excluding the term \(k_i \tilde{m}_i\) from last two equalities. We obtain finally:

\[
\tilde{a}_i \tilde{e}_{i-1} + (1 + \tilde{a}_i + \tilde{b}_i) \tilde{e}_i + \tilde{b}_i \tilde{e}_{i+1} = \tilde{f}_i \tilde{u}_{i-1} - \tilde{f}_i \tilde{u}_i + \tilde{f}_i^+ \tilde{u}_{i+1}, i = 1, ..., N - 1,
\]

\[
(1 + \tilde{a}_i + \tilde{b}_i) \tilde{e}_i + \tilde{b}_i \tilde{e}_{i+1} = \tilde{f}_i \tilde{u}_{i-1} - \tilde{f}_i \tilde{u}_i + \tilde{f}_i^+ \tilde{u}_{i+1}.
\]

Here the coefficients of linear algebraic system have following expressions:

\[
\tilde{a}_i = \tilde{G}_{i-1} / (\tilde{G}_i + G_{i+1/2} + \tilde{G}_{i+1}),
\]

\[
\tilde{b}_i = \tilde{G}_{i+1/2} / (\tilde{G}_i + G_{i+1/2} + \tilde{G}_{i+1}),
\]

\[
\tilde{f}_i = 3 / (\tilde{G}_i + G_{i+1/2} + \tilde{G}_{i+1}),
\]

\[
\tilde{f}_i^+ = 3 / (\tilde{G}_i + G_{i+1/2} + \tilde{G}_{i+1}),
\]

\[
\tilde{f}_i = \tilde{f}_i^+ + \tilde{G}_{i+1/2} = G_{i+1/2} = 0.
\]

The expressions for \(\tilde{G}_{i-1}, \tilde{G}_{i+1}, \tilde{u}_i, \tilde{u}_{i+1}\) are identical with the expressions (10, k), \(k = 0, 1\).

We would like to underline an interesting moment regarding GIPS: the system of linear algebraic equations for the calculations of spline’s coefficients contains only the “parabolic part” coefficients. Linear part characteristics are represented trough coefficients (31). Second important aspect: GIPS naturally transforms to IPS when all \(x_{i-1/2} = x_i\). The same property holds for the explicit representation for coefficients \(\tilde{e}_i\) of GIPS:

\[
\tilde{e}_i = \gamma_i^{(0)} \tilde{f}_0 \tilde{u}_i + \gamma_i^{(1)} \tilde{f}_i^+ \tilde{u}_{i+1} + \sum_{j=0}^{N} \tilde{\beta}_j \tilde{u}_j.
\]
\[ \beta = (\beta_0)_{i,j=0}^N \] depend on the location of grid points \( x_i \) , coefficients \( k_i \) and type of BC (4), (5), but they are independent from averaged integral values \( u_i \) and right hand sides’ values \( \Phi_0, \Phi_1 \) of BC. This property implies that for fixed grid points and coefficients \( k_i \) and \( \kappa_j \) we need to calculate the components of the two vectors \( \gamma^{(k)} \) (or \( \tilde{\gamma}^{(k)} \)) and the matrix \( \beta \) (or \( \tilde{\beta} \)) only once. After this calculation, for the construction of the integral parabolic spline we need only to compute the finite sum (12) (or (32) in case of GIPS).

3 Transforming the 3-D Formulation to 2-D System

3.1 3-D Transport Process in Orthotropic Layered Media with Interlayers

Let it be given the domain \( G \subset \mathbb{R}^3 \), where the domain \( G \) is multilayered cylinder with base \( D \subset \mathbb{R}^2 \) and with the height \( H = b - a \) :

\[ G = D \otimes \{ z \in [a=x_0, b=x_{N+1}] \} . \]

The equation for medium characteristic (concentration, temperature etc.) in the \( i-th \) productive (type aquifer) layer (for \( (x,y) \in D \) and \( z_i < z < z_{i+1/2}, i = 0, N \) ) we assume in following form:

\[ c_i \frac{\partial U_i}{\partial t} = \frac{\partial}{\partial x} \left( k_{i1}^{(i)} \frac{\partial U_i}{\partial x} \right) + \frac{\partial}{\partial y} \left( k_{i2}^{(i)} \frac{\partial U_i}{\partial y} \right) + \frac{\partial}{\partial z} \left( k_{i3}^{(i)} \frac{\partial U_i}{\partial z} \right) + F_i(x,y,z,t). \]  (33)

We assume that the source term consists of the advection and of the internal source \( \Phi_i(x,y,z,t) : \)

\[ F_i(x,y,z,t) = \Phi_i(x,y,z,t) - \frac{\partial}{\partial x} (c_{i1}^{(i)} \frac{\partial U_i}{\partial x}) - \frac{\partial}{\partial y} (c_{i2}^{(i)} \frac{\partial U_i}{\partial y}) . \]

We have excluded the possible motion in the \( z-th \) direction: orthogonal to the layer plane and we allow the dependence only on the velocity components \( v_{i1}, v_{i2} \) and coefficients \( k_{j1}^{(i)}, \kappa_{j2}^{(i)} \) of the arguments \( x, y, t \).

The BC in the \( z \) – direction for the first and the last layer we assume in general form including all three classical types of BC:

\[ -v_0^{(i)} k_{33}^{(i)} \frac{\partial U_i}{\partial z} \bigg|_{z=z_0} + \lambda_0^{(i)} U_i = \Phi_0^{(i)}(x,y,t), \]

\[ v_1^{(i)} k_{33}^{(i)} \frac{\partial U_i}{\partial z} \bigg|_{z=z_N} + \lambda_1^{(i)} U_i = \Phi_1^{(i)}(x,y,t). \]

The equation for interlayer (aquitard type) we write out in substantially simpler form:

\[ c_{i-1/2}^{(i)} \frac{\partial U_{i-1/2}}{\partial t} = \frac{\partial}{\partial z} \left( k_{i-1/2}^{(i)} \frac{\partial U_{i-1/2}}{\partial z} \right) . \]  (34)

The initial conditions are in the traditional form:

\[ U_i^{(i)}(x,y,z,t_{0}) = \Phi_i^{(i)}(x,y,z,t_{0}) . \]  (35)

We assume fulfilling one of all traditional BC on the lateral boundary \( \partial D \otimes \{ z \in [a=x_0, b=x_{N+1}] \} \) of the cylinder. Specific its properties don’t play important role for proposed method of conservative averaging.

3.2 Transformation to 2-D Problem

We will use our original method of conservative averaging and for this goal we introduce averaged integral values:

\[ \bar{U}_i^{(i)}(x,y,z,t) = \bar{H}_i^{(i)} \int_{z_i}^{z_{i+1/2}} \bar{U}_i(x,y,z,t)dz. \]  (36)

Now we integrate the differential equation (33) in the \( z \) – direction:

\[ c_i^{(i)} \frac{\partial \bar{U}_i}{\partial t} = \frac{\partial}{\partial x} \left( k_{i1}^{(i)} \frac{\partial \bar{U}_i}{\partial x} \right) + \frac{\partial}{\partial y} \left( k_{i2}^{(i)} \frac{\partial \bar{U}_i}{\partial y} \right) + \frac{\partial}{\partial z} \left( H_i^{(i)} k_{i3}^{(i)} \frac{\partial \bar{U}_i}{\partial z} \right) + f_i(x,y,t), \]

\[ f_i(x,y,t) = \phi_i(x,y,t) - \frac{\partial}{\partial x} (c_{i1}^{(i)} \frac{\partial \bar{U}_i}{\partial x}) - \frac{\partial}{\partial y} (c_{i2}^{(i)} \frac{\partial \bar{U}_i}{\partial y}) + \tilde{\phi}_i(x,y,z,t), \]

\[ \phi_i(x,y,t) = H_i^{(i)} \int_{z_i}^{z_{i+1}} \Phi_i(x,y,z,t)dz. \]

Next step of our conservative averaging method is the approximation of the function \( \bar{U}_i(x,y,z,t) \) by the spline (25) in the \( z \) – direction. As it was shown in the section 2.3 the construction of spline reduces to the calculation of its
coefficients $\tilde{e}_i, i = 0, N$. The following important step is the approximation of the fluxes difference by spline’s derivative in the differential equation (37):

$$k_{33}^{-1} \frac{\partial \tilde{U}}{\partial z} \bigg|_{z_{i-1/2}} \approx \frac{dS}{dz} = 2\tilde{e}_i.$$  \hspace{1cm} (38)

It again must be underlined that this is exclusive step in which the approximate substitution in the conservative averaging method is made. The next and in the same time the last step of the conservative averaging method is to use the representation (32) in (38) and to substitute this approximate equality in the integrated differential equation (37). We obtain:

$$c_i \left( \frac{\partial \tilde{u}}{\partial x} \left( k_{11}^{-1} \frac{\partial \tilde{u}}{\partial x} \right) + \frac{\partial}{\partial y} \left( k_{22}^{-1} \frac{\partial \tilde{u}}{\partial y} \right) + f_i(x, y, t) \right) + \frac{2}{H_i} \sum_{j=0}^{N} \tilde{z}_j \tilde{u}_j + \tilde{z}_i \tilde{u}_{i-1} + \tilde{z}_i \tilde{f}_i \tilde{u}_{N+i+1} \right].$$  \hspace{1cm} (39)

The initial conditions transform in following form:

$$\tilde{u}_{i,0} = \tilde{u}_i^0(x, y), u_{i-1/2,0} = u_{i-1/2}^0(x, y).$$  \hspace{1cm} (40)

Here the right hands sides of both equations are got by averaging procedure (36), respectively by formula

$$u_{i-1/2}(x, y, t) = H_{i-1/2}^{-1} \int_{z_{i-1/2}}^{z} U_{i-1/2}(x, y, z, t) dz.$$  \hspace{1cm} (34)

The same procedure must be done for the BC on the lateral surface the finite cylinder.

After solving the problem for system of PDE (39) (together with initial and boundary conditions) we can reconstruct the approximate solution $\tilde{U}_i(x, y, z, t)$ for all productive layers. Then, it is the right time to find the solution for interlayers. Here we have two possible ways to write out it. The assumed linearity in $z-$direction allows us to give following simple explicit formula immediately:

$$u_{i-1/2}^{(1)}(x, y, t) = 0.5[\tilde{U}_{i-1}(x, y, z_{i-1/2}, t) + \tilde{U}_{i}(x, y, z_{i}, t)].$$

The second way consists of solving the integrate PDE (34):

$$c_i^{-1} \frac{\partial \tilde{u}_{i-1/2}}{\partial t} = \frac{1}{2}[\tilde{U}_{i-1}(x, y, z_{i-1/2}, t) + \tilde{U}_{i}(x, y, z_{i}, t)].$$

The difference in both approaches gives a posteriori evaluation of errors in the proposed method.

4 Conclusion

The generalized integral parabolic spline allows transforming 3-D problem for layered stratum with interlayers (for partial differential equations with discontinuous coefficients) to 2-D system of PDE with continuous coefficients with number of equations equal to the number of productive layers.

Acknowledgements:

Research was supported by University of Latvia (project Y2-22AP06-100) and Council of Sciences of Latvia (grant 05.1525).

References: