System of Models for Transport Processes in Layered Strata

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Abstract: - In this paper the normalized form of the generalized integral parabolic spline is described, which interpolates the integral averaged values of piecewise-smooth function. The three-dimensional system of partial differential equations as model of transport processes for porous layered stratum with semi-permeable interlayer is proposed. The generalized integral parabolic spline is used for the approximate transformation of the 3-D problem into 2-D system by the original method of conservative averaging. The order of the 2-D system is equal to the number of productive layers. This system of 2-D partial differential equations with continuous coefficients fulfills all conservation laws of initial problem in averaged sense.

Key-Words: - integral spline, layered media, transport processes, three-dimensional, conservative averaging.

1 Introduction

Many real processes take place in layered systems, consisting of separate layers with different thickness and different physical properties [1]-[4]. Very often such systems are composed from two types of layers which are located alternatively, e.g. aquifers and aquitards (aquiludes) [1] in groundwater flows. In these cases, in the mathematical model of such processes we have jump in coefficients of differential equations on the surfaces between two layers. Discontinuity of coefficients of PDE (partial differential equations) brings out additional difficulties by standard use of traditional mathematical methods.

One of the authors has developed the conservative averaging method and he introduced a special new type of splines for wide class of PDE problems with discontinuous coefficients [5]-[9].

In this paper we give a description of generalized integral parabolic spline (GIPS) in so called normalized form [10], [11] and employ this spline for some groundwater (or other fluids) flows and pollution problems in layered stratum, which consists of two different types of layers: main (productive) and interlayers (semi-permeable layers). This type of models automatically reduces (if the thickness of interlayers trends to zero) to models, which have only productive layers - considered in our previous works [10], [11]. Proposed method differs from methods traditionally used in various transport processes in natural or artificial porous media. Our method as outcome gives little bit more complicated mathematical model, but it allows analyzing broader spectrum of physical phenomena and wider variety of geometrical and physical parameters.

2 Integral Splines

In this section we introduce the integral averaged values interpolating special splines (parabolic and generalized parabolic).

2.1 Interpolation Problem for Piecewise-Smooth Function

Let it be given a continuous, piecewise-smooth function $U(x), x \in [a,b]$. Further, let it be given, that in the different inner points $x_i, i = 1,...,N$ the first derivative $U'(x)$ of the function has a finite jump:

$$k_{i-1}U'(x_i - 0) = k_i U'(x_i + 0). \tag{1}$$

Here coefficients $k_i > 0,\ i = \overline{0,N}$ are known for all $x \in (x_i, x_{i+1}),\quad x_0 = a, x_{N+1} = b$. Since the function $U(x)$ is continuous on the closed interval $[a,b]$, we additionally have following continuity equalities in these points:

$$U(x_i - 0) = U(x_i + 0), i = 1,...,N. \tag{2}$$

Let additionally be given the integral averaged values u_i of the function $U(x)$ over the all sub-segments $[x_i, x_{i+1}]$:

$$u_i = H_i^{-1} \int_{x_i}^{x_{i+1}} U(x)dx, H_i = x_{i+1} - x_i, i = 0, ..., N. \quad (3)$$

The interpolation problem consists in approximate reconstruction of the function $U(x)$, and it is based on conditions (1)-(3) and following general boundary conditions (BC) on end points $x = a$ and $x = b$:

$$-\nu_0 k_0 U'(a) + \lambda_0 U(a) = \Phi_0, \quad (4)$$

$$\nu_1 k_N U'(b) + \lambda_1 U(b) = \Phi_1. \quad (5)$$

This interpolation problem can be solved by polynomial spline IPS

$$S(x) = u_i + m_i(x - \overline{x}_i) + e_i \left[\frac{(x - \overline{x}_i)^2}{k_i H_i} - \frac{G_i}{12} \right],$$
$$\overline{x}_i = \frac{x_i + x_{i+1}}{2}, G_i = \frac{H_i}{k_i} > 0. \quad (6)$$

For the determination of $2(N+1)$ free coefficients, we have exactly the same number of equations (1),(2),(4) and (5). In [5], [6], [10] and [11] it was proved, that all coefficients m_i can be represented through coefficients e_i in two forms:

a) for $i = 0, ..., N-1$

$$k_i m_i (G_i + G_{i+1}) = 2(u_{i+1} - u_i) -$$
$$e_i (G_i / 3 + G_{i+1}) - 2/3 e_{i+1} G_{i+1}; \quad (7_a)$$

b) for $i = 1, ..., N$

$$k_i m_i (G_i + G_{i-1}) = 2(u_i - u_{i-1}) +$$
$$e_i (G_i / 3 + G_{i-1}) + 2/3 e_{i-1} G_{i-1}. \quad (7_b)$$

The Elimination of the coefficients m_i from these expressions gives us for $i = 1, ..., N-1$ following system regarding e_i (see [10] and [11]):

$$a_i e_{i-1} + (1 + a_i + b_i) e_i + b_i e_{i+1} =$$
$$f_i^- u_{i-1} - f_i u_i + f_i^+ u_{i+1}, i = 1, ..., N-1. \quad (8)$$

Here $f_i = f_i^- + f_i^+$ and

$$a_i = G_{i-1} / \left(G_i + G_{i-1}\right), b_i = G_{i+1} / \left(G_i + G_{i+1}\right),$$
$$f_i^- = 3 / \left(G_i + G_{i-1}\right), f_i^+ = 3 / \left(G_i + G_{i+1}\right). \quad (9)$$

For the transformation of the BC (4), (5) some additional notations must be used and two different cases are distinguished:

1) $\lambda_0 \neq 0$ (and $\lambda_1 \neq 0$). Then

$$G_{-1} = 2\nu_0 / \lambda_0, G_{N+1} = 2\nu_1 / \lambda_1,$$
$$u_{-1} = \Phi_0 / \lambda_0, u_{N+1} = \Phi_1 / \lambda_1; \quad (10_0)$$

2) $\lambda_0 = 0$ (and $\lambda_1 = 0$). Then

$$G_{-1} = 2\nu_0 - G_0, G_{N+1} = 2\nu_1 - G_N,$$
$$u_{-1} = \Phi_0 + u_0, u_{N+1} = \Phi_1 + u_N. \quad (10_1)$$

We have obtained from BC (4), (5) in papers [10], [11] following equations for the first and last equations:

$$(1 + a_0 + b_0) e_0 + b_0 e_1 = f_0^- u_{-1} - f_0 u_0 + f_0^+ u_1,$$
$$a_N e_{N-1} + (1 + a_N + b_N) e_N = \quad (11)$$
$$f_N^- u_{N-1} - f_N u_N + f_N^+ u_{N+1}.$$

In the case $\lambda_0 = 0$ ($\lambda_1 = 0$) we have special formulas for coefficients a_0 and b_N: $a_0 = b_N = 1$. We proposed in our papers [10], [11] the representation for coefficients e_i of IPS through all averaged integral values. This representation shows in explicit form also the impact of the BC type and its right hand side on the spline:

$$e_i = \gamma_i^{(0)} f_0^- u_{-1} + \gamma_i^{(1)} f_N^+ u_{N+1} + \sum_{j=0}^{N} \beta_{ij} u_j, \; i = \overline{0, N}. \quad (12)$$

The coefficients in the representation (12) are determinate from three systems of linear algebraic equations.

a) The system for $\gamma_i^{(0)}$ (shows the impact of BC (4)):

$$(1 + a_0 + b_0)\gamma_0^{(0)} + b_0 \gamma_1^{(0)} = 1,$$
$$a_i \gamma_{i-1}^{(0)} + (1 + a_i + b_i)\gamma_i^{(0)} + b_i \gamma_{i+1}^{(0)} = 0,$$
$$i = 1, ..., N-1, \quad (13_0)$$
$$a_N \gamma_{N-1}^{(0)} + (1 + a_N + b_N)\gamma_N^{(0)} = 0.$$

b) The system for $\gamma_i^{(1)}$ (shows the impact of BC (5)):

$$(1 + a_0 + b_0)\gamma_0^{(1)} + b_0 \gamma_1^{(1)} = 0,$$
$$a_i \gamma_{i-1}^{(1)} + (1 + a_i + b_i)\gamma_i^{(1)} + b_i \gamma_{i+1}^{(1)} = 0,$$
$$i = 1, ..., N-1, \quad (13_1)$$
$$a_N \gamma_{N-1}^{(1)} + (1 + a_N + b_N)\gamma_N^{(1)} = 1.$$

c) The $N+1$ systems ($j = 0, ..., N$) for β_{ij}:

$$(1 + a_0 + b_0)\beta_{0,j} + b_0 \beta_{1,j} = 0,$$
$$a_i \beta_{i-1,j} + (1 + a_i + b_i)\beta_{ij} + b_i \beta_{i+1,j} =$$
$$f_j^- \delta_{i-1,j} - f_j \delta_{i,j} + f_j^+ \delta_{i+1,j}, i = 1, ..., N-1, \quad (14)$$
$$a_N \beta_{N-1,j} + (1 + a_N + b_N)\beta_{N,j} = 0.$$

2.2 Model of Heat Conduction in Multi-layered Bar

Now, by easiest one-dimensional multi-layer temperature conduction model, we will show how IPS can be used to simplify the initial problem. We consider following heat conduction model with piecewise-constant coefficients:

$$c_i \frac{\partial U_i}{\partial t} = \frac{\partial}{\partial x}(k_i \frac{\partial^2 U_i}{\partial x}) + F_i(x,t), x_i < x < x_{i+1}, \quad (15)$$
$$i = \overline{0,N}, x_0 = a, x_{N+1} = b, t \in (0,T],$$

$$U_{i-1}(x_i - 0) = U_i(x_i + 0),$$
$$k_{i-1}\frac{\partial U_{i-1}(x_i - 0)}{\partial x} = k_i \frac{\partial U_i(x_i + 0)}{\partial x}, \quad (16)$$

$$-\nu_0 k_0 \frac{\partial U_0(a)}{\partial x} + \lambda_0 U_0(a) = \Phi_0(t),$$
$$\nu_1 k_N \frac{\partial U_N(b)}{\partial x} + \lambda_N U_N(b) = \Phi_1(t), \quad (17)$$

$$U_i(x,0) = U_i^0(x). \quad (18)$$

We introduce in conformity with the method of conservative averaging (see [8]-[11]) the integral averaged values $u_i(t)$:

$$u_i(t) = \frac{1}{H_i}\int_{x_i}^{x_{i+1}} U_i(x,t)dx, H_i = x_{i+1} - x_i, i = 0,...,N.$$

The integration of main equation (15) gives exact consequences:

$$c_i \frac{du_i}{dt} = \frac{k_i}{H_i}\frac{\partial U_i}{\partial x}\bigg|_{x=x_i}^{x=x_{i+1}} + f_i(t), i = \overline{0,N}, \quad (19)$$

$$f_i(t) = \frac{1}{H_i}\int_{x_i}^{x_{i+1}} F_i(x,t)dx.$$

It remains to replace the first term in the right hand side of the equation (19) by the first derivative difference of IPS (6):

$$\frac{k_i}{H_i}\frac{\partial U_i}{\partial x}\bigg|_{x=x_i}^{x=x_{i+1}} \approx \frac{k_i}{H_i}\frac{dS}{dx}\bigg|_{x=x_i}^{x=x_{i+1}} = \frac{2e_i}{H_i}.$$

As the last step, we use the representation (12) for the spline coefficient e_i. This finally gives:

$$c_i \frac{du_i}{dt} = \frac{2}{H_i}\left[\sum_{j=0}^{N} \beta_{ij}u_j + \gamma_i^{(0)} f_0^- u_{-1} + \gamma_i^{(1)} f_N^+ u_{N+1}\right]$$
$$+ f_i(t), i = \overline{0,N}, t \in (0,T]. \quad (20)$$

To this system of ordinary differential equations (ODE) averaged over sub-segments $[x_i, x_{i+1}]$, initial conditions (18) must be added (the conjugations conditions (16) and BC (17) were already used by construction of IPS):

$$u_i(0) = u_i^0 := \frac{1}{H_i}\int_{x_i}^{x_{i+1}} U_i^0(x)dx. \quad (21)$$

So, we have reduced the one-dimensional problem with discontinuous coefficients for PDE to the system of ODE with continuous coefficients. After solving problem (20), (21), we can approximately reconstruct the solution of original problem (15)-(18) by IPS (6).

2.3 Generalized Integral Parabolic Spline

Let it again be given a continuous, piecewise-smooth function $U(x), x \in [a,b]$, for which the first derivative $U'(x)$ has first kind discontinuities in $2N$ different inner points $x_i, x_{i-1/2}, i = 1,...,N$:

$$k_{i-1}U'(x_{i-1/2} - 0) = k_{i-1/2}U'(x_{i-1/2} + 0),$$
$$k_{i-1/2}U'(x_i - 0) = k_i U'(x_i + 0). \quad (22)$$

The continuity property of the function $U(x)$ gives following $2N$ equalities:

$$U(x_{i-1/2} - 0) = U(x_{i-1/2} + 0),$$
$$U(x_i - 0) = U(x_i + 0). \quad (23)$$

Here the function (coefficients) $k(x)$ is piecewise-constant function:

$$k(x) = \begin{cases} k_{i-1/2}, if \ x \in (x_{i-1/2}, x_i), \\ k_i, \quad if \ x \in (x_i, x_{i+1/2}) \end{cases}$$

with property $k_{i-1/2} \ll k_{i-1}$, $k_{i-1/2} \ll k_i$. Here must be mentioned that the $x_{N+1/2} = x_{N+1} = b$, it means the function $U(x)$ can neither end nor begin with a linear part of the segment $[a,b]$.)

The $N+1$ integral averaged values u_i of the function $U(x)$ are given additionally over sub-segments $[x_i, x_{i+1/2}]$:

$$\tilde{u}_i = \frac{1}{\tilde{H}_i}\int_{x_i}^{x_{i+1/2}} U(x)dx, \ \tilde{H}_i = x_{i+1/2} - x_i, i = 0,N. \quad (24)$$

Further, it is known, that on the sub-segments $\left[x_{i+1/2}, x_{i+1}\right], i=\overline{0, N-1}$ the function can be approximated by linear function.

Finally, the BC (4), (5) must be fulfilled.

In our paper [7] it was shown that there exists exactly one spline fulfilling all mentioned conditions. We will seek this spline in the form:

$$\tilde{S}(x)=\begin{cases}\tilde{u}_i+\tilde{m}_i(x-\tilde{x}_i)+\tilde{e}_i\left[\dfrac{(x-\tilde{x}_i)^2}{k_i\tilde{H}_i}-\dfrac{\tilde{G}_i}{12}\right],\\ x\in[x_i, x_{i+1/2}], \tilde{x}_i=\dfrac{x_i+x_{i+1/2}}{2}, i=\overline{0,N};\\ u_{i-1/2}+m_{i-1/2}(x-\overline{x}_{i-1/2}), x\in[x_{i-1/2}, x_i],\\ \overline{x}_{i-1/2}=\dfrac{x_{i-1/2}+x_i}{2}, i=\overline{1,N}.\end{cases} \tag{25}$$

Here $\tilde{G}_i=\tilde{H}_i/k_i$ are lengths parameters reduced by conductions coefficient, which can be called as "characteristic conduction lengths". Similarly we introduce N additional lengths parameters for second type of layers $G_{i-1/2}=H_{i-1/2}/k_{i-1/2}$, $i=\overline{1,N}$, which will be used immediately.

The continuity (23) of generalized IPS (GIPS) at discontinuity points $x_{i+1/2}, i=\overline{0,N-1}$ gives following equalities:

$$\tilde{u}_i+k_i\tilde{m}_i\frac{\tilde{G}_i}{2}+\tilde{e}_i\frac{\tilde{G}_i}{6}= u_{i+1/2}-k_{i+1/2}m_{i+1/2}\frac{G_{i+1/2}}{2}. \tag{26}$$

The same property at points $x_i, i=\overline{0,N-1}$ gives similar equalities:

$$\tilde{u}_{i+1}-k_{i+1}\tilde{m}_{i+1}\frac{\tilde{G}_{i+1}}{2}+\tilde{e}_{i+1}\frac{\tilde{G}_{i+1}}{6}= u_{i+1/2}+k_{i+1/2}m_{i+1/2}\frac{G_{i+1/2}}{2}. \tag{27}$$

The conjugation conditions (22) lead to the equalities for $i=\overline{0,N-1}$:

$$k_i\tilde{m}_i+\tilde{e}_i=k_{i+1/2}m_{i+1/2}=k_{i+1}\tilde{m}_{i+1}-\tilde{e}_{i+1}. \tag{28}$$

Equalities (28) allow us to exclude $k_{i+1/2}m_{i+1/2}$ and $k_{i+1}\tilde{m}_{i+1}$ from (26), (27). Instead of (7_a) we obtain following generalized chain of equalities for $i=\overline{0,N-1}$:

$$\begin{aligned}&3k_i\tilde{m}_i(\tilde{G}_i+2G_{i+1/2}+\tilde{G}_{i+1})+\tilde{e}_i(\tilde{G}_i+6G_{i+1/2}\\&+3\tilde{G}_{i+1})+2\tilde{e}_{i+1}\tilde{G}_{i+1}=6(\tilde{u}_{i+1}-\tilde{u}_i).\end{aligned} \tag{29_0}$$

In its turn, from (7_b) we obtain for $i=\overline{1,N}$:

$$\begin{aligned}&3k_i\tilde{m}_i(\tilde{G}_i+2G_{i-1/2}+\tilde{G}_{i-1})+\tilde{e}_i(\tilde{G}_i+6G_{i-1/2}\\&+3\tilde{G}_{i-1})-2\tilde{e}_{i-1}\tilde{G}_{i-1}=6(\tilde{u}_i-\tilde{u}_{i-1}).\end{aligned} \tag{29_1}$$

The last step consists of excluding the term $k_i\tilde{m}_i$ from last two equalities. We obtain finally:

$$\begin{aligned}&\tilde{a}_i\tilde{e}_{i-1}+(1+\tilde{a}_i+\tilde{b}_i)\tilde{e}_i+\tilde{b}_i\tilde{e}_{i+1}=\\&\tilde{f}_i^-\tilde{u}_{i-1}-\tilde{f}_i\tilde{u}_i+\tilde{f}_i^+\tilde{u}_{i+1}, i=1,\dots,N-1,\\&(1+\tilde{a}_0+\tilde{b}_0)\tilde{e}_0+\tilde{b}_0\tilde{e}_1=\\&\tilde{f}_0^-\tilde{u}_{-1}-\tilde{f}_0\tilde{u}_0+\tilde{f}_0^+\tilde{u}_1,\\&\tilde{a}_N\tilde{e}_{N-1}+(1+\tilde{a}_N+\tilde{b}_N)\tilde{e}_N=\\&\tilde{f}_N^-\tilde{u}_{N-1}-\tilde{f}_N\tilde{u}_N+\tilde{f}_N^+\tilde{u}_{N+1}.\end{aligned} \tag{30}$$

Here the coefficients of linear algebraic system have following expressions:

$$\begin{aligned}&\tilde{a}_i=\tilde{G}_{i-1}/\left(\tilde{G}_i+G_{i-1/2}+\tilde{G}_{i-1}\right),\\&\tilde{b}_i=\tilde{G}_{i+1}/\left(\tilde{G}_i+G_{i+1/2}+\tilde{G}_{i+1}\right),\\&\tilde{f}_i^-=3/\left(\tilde{G}_i+G_{i-1/2}+\tilde{G}_{i-1}\right),\\&\tilde{f}_i^+=3/\left(\tilde{G}_i+G_{i+1/2}+\tilde{G}_{i+1}\right),\\&\tilde{f}_i=\tilde{f}_i^-+\tilde{f}_i^+, G_{-1/2}=G_{N+1/2}=0.\end{aligned} \tag{31}$$

The expressions for $\tilde{G}_{-1}, \tilde{G}_{N+1}, \tilde{u}_{-1}, \tilde{u}_{N+1}$ are identical with the expressions $(10_k), k=0,1$.

We would like to underline an interesting moment regarding GIPS: the system of linear algebraic equations for the calculations of spline's coefficients contains only the "parabolic part" coefficients. Linear part characteristics are represented trough coefficients (31). Second important aspect: GIPS naturally transforms to IPS when all $x_{i-1/2}=x_i$. The same property holds for the explicit representation for coefficients $\tilde{e}_i$ of GIPS:

$$\tilde{e}_i=\tilde{\gamma}_i^{(0)}\tilde{f}_0^-\tilde{u}_{-1}+\tilde{\gamma}_i^{(1)}\tilde{f}_N^+\tilde{u}_{N+1}+\sum_{j=0}^{N}\tilde{\beta}_{ij}\tilde{u}_j. \tag{32}$$

As it was mentioned in our papers [10], [11] we would like to draw one's attention to some important aspects of this new type of (integral) splines. As one can see, the components of the vector $\gamma^{(k)}=(\gamma_i^{(k)})_{i=0}^N, k=\{0,1\}$ and of the matrix

$\beta = (\beta_{ij})_{i,j=0}^{N}$ depend on the location of grid points x_i, coefficients k_i and type of BC (4), (5), but *they are independent* from averaged integral values u_i and right hand sides' values Φ_0, Φ_1 of BC. This property implies that for fixed grid points and coefficients k_i (and $k_{i-1/2}$) we need to calculate the components of the two vectors $\gamma^{(k)}$ (or $\tilde{\gamma}^{(k)}$) and the matrix β (or $\tilde{\beta}$) only once. After this calculation, for the construction of the integral parabolic spline we need only to compute the finite sum (12) (or (32) in case of GIPS).

3 Transforming the 3-D Formulation to 2-D System

3.1 3-D Transport Process in Orthotropic Layered Media with Interlayers

Let it be given the domain $G \subset R^3$, where the domain G is multilayered cylinder with base $D \in R^2$ and with the height $H = b - a$: $G = D \otimes \{z \in [a = x_0, b = x_{N+1}]\}$.

The equation for medium characteristic $U_i(x, y, z, t)$ (concentration, temperature etc.) in the $i-th$ productive (type aquifer) layer (for $(x, y) \in D$ and $z_i < z < z_{i+1/2}, i = \overline{0, N}$) we assume in following form:

$$c_1^i \frac{\partial \tilde{U}_i}{\partial t} = \frac{\partial}{\partial x}\left(k_{11}^i \frac{\partial \tilde{U}_i}{\partial x}\right) + \frac{\partial}{\partial y}\left(k_{22}^i \frac{\partial \tilde{U}_i}{\partial y}\right) + \frac{\partial}{\partial z}\left(k_{33}^i \frac{\partial \tilde{U}_i}{\partial z}\right) + F_i(x, y, z, t). \tag{33}$$

We assume that the source term consists of the advection and of the internal source $\Phi_i(x, y, z, t)$:

$$F_i(x, y, z, t) = \Phi_i(x, y, z, t) - \frac{\partial}{\partial x}(c_1^i v_{i,1} \tilde{U}_i) - \frac{\partial}{\partial y}(c_1^i v_{i,2} \tilde{U}_i).$$

We have excluded the possible motion in the $z-th$ direction: orthogonal to the layer plane and we allow the dependence only on the velocity components $v_{i,1}, v_{i,2}$ and coefficients k_{jj}^i, κ_{jj}^i of the arguments x, y, t .

The BC in the $z-$direction for the first and the last layer we assume in general form including all three classical types of BC:

$$-\nu_0 k_{33}^0 \left.\frac{\partial \tilde{U}_0}{\partial z}\right|_{z=z_0} + \lambda_0 \tilde{U}_0 = \Phi_0(x, y, t),$$

$$\nu_1 k_{33}^N \left.\frac{\partial \tilde{U}_N}{\partial z}\right|_{z=z_N} + \lambda_1 \tilde{U}_N = \Phi_1(x, y, t).$$

The equation for interlayer (aquitard type) we write out in substantially simpler form:

$$c_{i-1/2} \frac{\partial U_{i-1/2}}{\partial t} = \frac{\partial}{\partial z}\left(k_{i-1/2} \frac{\partial U_{i-1/2}}{\partial z}\right), \quad z_{i-1/2} < z < z_i, i = \overline{1, N}. \tag{34}$$

The initial conditions are in the traditional form:

$$\tilde{U}_i\Big|_{t=0} = \tilde{U}_i^0(x, y, z), U_{i-1/2}\Big|_{t=0} = U_{i-1/2}^0(x, y, z). \tag{35}$$

We assume fulfilling one of all traditional BC on the lateral boundary $\partial D \otimes \{z \in [a = x_0, b = x_{N+1}]\}$ of the cylinder. Specific its properties don't play important role for proposed method of conservative averaging.

3.2 Transformation to 2-D Problem

We will use our original method of conservative averaging and for this goal we introduce averaged integral values:

$$\tilde{u}_i(x, y, t) = \tilde{H}_i^{-1} \int_{z_i}^{z_{i+1/2}} \tilde{U}_i(x, y, z, t) dz. \tag{36}$$

Now we integrate the differential equation (33) in the $z-$ direction:

$$c_1^i \frac{\partial \tilde{u}_i}{\partial t} = \frac{\partial}{\partial x}\left(k_{11}^i \frac{\partial \tilde{u}_i}{\partial x}\right) + \frac{\partial}{\partial y}\left(k_{22}^i \frac{\partial \tilde{u}_i}{\partial y}\right) + H_i^{-1} k_{33}^i \left.\frac{\partial \tilde{U}_i}{\partial z}\right|_{z=z_i}^{z=z_{i+1/2}} + f_i(x, y, t),$$

(37)

$$f_i(x, y, t) = \varphi_i(x, y, t) - \frac{\partial}{\partial x}(c_1^i v_{i,1} \tilde{u}_i) - \frac{\partial}{\partial y}(c_1^i v_{i,2} \tilde{u}_i),$$

$$\varphi_i(x, y, t) = H_i^{-1} \int_{z_i}^{z_{i+1}} \Phi_i(x, y, z, t) dz.$$

Next step of our conservative averaging method is the approximation of the function $\tilde{U}_i(x, y, z, t)$ by the spline (25) in the $z-$ direction. As it was shown in the section 2.3 the construction of spline reduces to the calculation of its coefficients $\tilde{e}_i, i = \overline{0, N}$. The following important step is the approximation of the fluxes difference by spline's derivative in the differential equation (37):

$$k_{33}^i \frac{\partial \tilde{U}_i}{\partial z}\bigg|_{z=z_i}^{z=z_{i+1/2}} \approx \frac{d\tilde{S}}{dz} = 2\tilde{e}_i. \quad (38)$$

It again must be underlined that this is exclusive step in which the approximate substitution in the conservative averaging method is made.

The next and in the same time the last step of the conservative averaging method is to use the representation (32) in (38) and to substitute this approximate equality in the integrated differential equation (37). We obtain:

$$c_1^i \frac{\partial \tilde{u}_i}{\partial t} = \frac{\partial}{\partial x}\left(k_{11}^i \frac{\partial \tilde{u}_i}{\partial x}\right) + \frac{\partial}{\partial y}\left(k_{22}^i \frac{\partial \tilde{u}_i}{\partial y}\right) + f_i(x, y, t)$$

$$+\frac{2}{\tilde{H}_i}\left[\sum_{j=0}^{N} \tilde{\beta}_{ij}\tilde{u}_j + \tilde{\gamma}_i^{(0)}\tilde{f}_0^{-}\tilde{u}_{-1} + \tilde{\gamma}_i^{(1)}\tilde{f}_N^{+}\tilde{u}_{N+1}\right]. \quad (39)$$

The initial conditions transform in following form:

$$\tilde{u}_i\big|_{t=0} = \tilde{u}_i^0(x, y), u_{i-1/2}\big|_{t=0} = u_{i-1/2}^0(x, y). \quad (40)$$

Here the right hands sides of both equations are got by averaging procedure (36), respectively by formula

$$u_{i-1/2}(x, y, t) = H_{i-1/2}^{-1} \int_{z_{i-1/2}}^{z_i} U_{i-1/2}(x, y, z, t)dz.$$

The same procedure must be done for the BC on the lateral surface the finite cylinder.

After solving the problem for system of PDE (39) (together with initial and boundary conditions) we can reconstruct the approximate solution $\tilde{U}_i(x, y, z, t)$ for all productive layers. Then, it is the right time to find the solution for interlayers. Here we have two possible ways to write out it. The assumed linearity in $z-$direction allows us to give following simple explicit formula immediately:

$$u_{i-1/2}^{(1)}(x, y, t) = 0.5[\tilde{U}_{i-1}(x, y, z_{i-1/2}, t) + \tilde{U}_i(x, y, z_i, t)].$$

The second way consists of solving the integrate PDE (34):

$$c_{i-1/2}\frac{\partial u_{i-1/2}^{(2)}}{\partial t} = \frac{1}{2}[\tilde{U}_{i-1}(x, y, z_{i-1/2}, t) + \tilde{U}_i(x, y, z_i, t)].$$

The difference in both approaches gives a posteriori evaluation of errors in the proposed method.

4 Conclusion

The generalized integral parabolic spline allows transforming 3-D problem for layered stratum with interlayers (for partial differential equations with discontinuous coefficients) to 2-D system of PDE with continuous coefficients with number of equations equal to the number of productive layers.

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