On the best approximation of function classes from values on a uniform grid in the real line

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Abstract: Given the values of function f on a uniform grid of a step size h in the real line, we construct the spline interpolant of order m, defect 1 using the B-spline basis obtained by wavelet-type dilation and shifts from the father B-spline which can be considered as a scaling function for a special (nonorthogonal) spline-wavelet system. We establish an unimprovable error estimate on the class of functions with bounded mth derivative, and we show that in the sense of worst case, other approximations to f using the same information about f are more coarse than the spline interpolant. Some further results of this type are established.

Key–Words: splines, nonorthogonal wavelets, interpolation, error estimates, exact constants

1 Introduction

This paper is devoted to unimprovable error estimates for spline interpolation and to the optimality of spline interpolants compared with other approximations. The spline interpolation has been widely examined during the last 40–50 years, see in particular the monographs [1, 3, 5, 6, 10] and the references there. Nevertheless, there are open problems concerning exact (unimprovable) constants in error estimates for interpolants and quasi-interpolants. Such results are of great general interest; our special interest is caused by examining fast solvers for integral equations (see, e.g., [7, 9]).

In Section 2 of the present paper we discuss the construction of the spline interpolants of functions on the real line. Given the values of a function f on a uniform grid $\Delta_h \subset \mathbb{R}$ of a step size h, we construct the spline interpolant $Q_{h,m}f$ of order m, defect 1 using the B-spline basis obtained by wavelet-type dilation and shifts from the father B-spline B_m which can be considered as a scaling function for a special (nonorthogonal) wavelet system, see [2]. Technically, our approach is equivalent to that in [6] but we use another start idea.

Our main results are presented in Section 3. We establish an unimprovable estimate for $||f - Q_{h,m}f||_\infty$ on the class $V^{m,\infty}(\mathbb{R})$ of functions with bounded mth derivative. This estimate essentially extends a result of [3] concerning 1-periodic functions f and the step size of the form $h = 1/n$ with even $n \in \mathbb{N}$. Further, we show that in the sense of the worst case in $V^{m,\infty}(\mathbb{R})$, other approximations to f using the same information about f as $Q_{h,m}f$ are more coarse than $Q_{h,m}f$. We also present some further estimates for $Q_{h,m}f$, in particular, error estimates for derivatives and error estimates in the case of modestly smooth f.

We use the following standard notations:

$$\mathbb{R} = (-\infty, \infty), \quad \mathbb{N} = \{1, 2, ...\},$$
$$\mathbb{Z} = \{..., -2, -1, 0, 1, 2, ...\}.$$

Let us characterise more precisely the spaces of functions on $\mathbb{R}$ used in the sequel. As usual, $C(\mathbb{R})$ is the space of continuous functions on $\mathbb{R}$, and $C^m(\mathbb{R})$ is the space of functions on $\mathbb{R}$ that have continuous derivatives up to the order m. By $BC(\mathbb{R})$ we mean the Banach space of bounded continuous functions f on $\mathbb{R}$ equipped with the norm

$$||f||_\infty = \sup_{x \in \mathbb{R}} |f(x)|;$$

$BUC(\mathbb{R})$ is the (closed) subspace of $BC(\mathbb{R})$ consisting of bounded uniformly continuous functions on $\mathbb{R}$. The Sobolev space $W^{m,\infty}(\mathbb{R})$, $m \in \mathbb{N}$, consists of functions f such that f itself and its derivatives up to the order m are measurable, bounded functions on $\mathbb{R}$ (actually then $f, f', ..., f^{(m-1)}$ are continuous in $\mathbb{R}$; the derivatives are understood in the sense of distributions). Finally, the Sobolev space $V^{m,\infty}(\mathbb{R})$ consists of functions f such that $f^{(m)}$ is measurable and bounded in $\mathbb{R}$; then $f, f', ..., f^{(m-1)}$ are continuous but not necessarily bounded in $\mathbb{R}$. With the help of the Taylor formula

$$f(x)=\sum_{l=0}^{m-1}\frac{f^{(l)}(0)}{l!}x^l+\frac{1}{(m-1)!}\int_0^x(x-t)^{m-1}f^{(m)}(t)dt$$

we observe that for $f \in V^{m,\infty}(\mathbb{R})$, $|x| \to \infty$ it holds

$$|f(x)| \leq \frac{1}{m!}\,||f^{(m)}||_\infty\,|x|^m + O(x^{m-1}).$$

There is an equivalent way to define $V^{m,\infty}(\mathbb{R})$ as the space of functions $f \in C^{(m-1)}(\mathbb{R})$ such that $f^{(m-1)}$ is uniformly Lipschitz continuous:

$$|f^{(m-1)}(x_1)-f^{(m-1)}(x_2)|\leq L_f|x_1-x_2|,\ x_1,x_2\in\mathbb{R}.$$

A Lipschitz continuous function, $f^{(m-1)}$in our case, is differentiable almost everywhere as well as differentiable in the sense of distributions, and

$$\inf L_f = ||f^{(m)}||_\infty = \text{vraisup}_{x\in\mathbb{R}}|f^{(m)}(x)|$$

where the infimum is taken over all Lipschitz constants L_f for $f^{(m-1)}$.

We do not need norms in $W^{m,\infty}(\mathbb{R})$ and $V^{m,\infty}(\mathbb{R})$.

Clearly, $W^{m,\infty}(\mathbb{R}) + \mathcal{P}_m \subset V^{m,\infty}(\mathbb{R})$; this inclusion is strict.

2 Construction of the interpolant

The formula for the father B-spline B_m is given by

$$B_m(x) = \frac{1}{(m-1)!}\sum_{i=0}^{m}(-1)^i\begin{pmatrix} m \\ i \end{pmatrix}(x-i)_+^{m-1},$$
$$x \in \mathbb{R}, \quad m \in \mathbb{N},$$

where, as usual, $0! = 1$, $0^0 := \lim_{x\downarrow 0}x^x = 1$,

$$(x-i)_+^{m-1} := \left\{\begin{array}{ll}(x-i)^{m-1}, & x-i\geq 0\\ 0, & x-i<0\end{array}\right\}.$$

Some properties of B_m, $m \geq 2$, are as follows:

$B_m|_{[i,i+1]} \in \mathcal{P}_{m-1}$ (polynomials of degree $m-1$),

$i \in \mathbb{Z}$, $B_m \in C^{(m-2)}(\mathbb{R})$, i.e., B_m is a spline of degree $m-1$, defect 1 on the "cardinal" knot set $\mathbb{Z}$,

$$\text{supp}\,B_m = [0,m], \quad B_m(x) > 0 \quad \text{for } 0 < x < m,$$
$$B_m(x) = B_m(m-x),\ 0 \leq x \leq m,$$
$$B_m\left(\tfrac{m}{2}\right) = \max_{x\in\mathbb{R}} B_m(x),$$
$$\int_{\mathbb{R}} B_m(x)dx = 1, \quad \sum_{j\in\mathbb{Z}} B_m(x-j) = 1, \quad x \in \mathbb{R}.$$

Introduce in $\mathbb{R}$ the uniform grid $h\mathbb{Z} = \{ih\colon i\in\mathbb{Z}\}$ of a step size $h > 0$. Denote by $S_{h,m}$, $m \in \mathbb{N}$, the space of splines of order m (or, of degree $m-1$) and defect 1 with the knot set $h\mathbb{Z}$. Clearly the dilated and shifted B-splines $B_m(h^{-1}x - j)$, $j \in \mathbb{Z}$, belong to $S_{h,m}$, and the same is true for $\sum_{j\in\mathbb{Z}} d_j B_m(h^{-1}x - j)$ with arbitrary coefficients d_j; there are no problems with the convergence of the series since it is locally finite: for $x \in [ih,(i+1)h))$, $i \in \mathbb{Z}$, it holds

$$\sum_{j\in\mathbb{Z}} d_j B_m(h^{-1}x-j) = \sum_{j=i-m+1}^{i} d_j B_m(h^{-1}x-j).$$

Given a function $f \in C(\mathbb{R})$ of possibly polynomial growth as $|x| \to \infty$, we look for the interpolant $Q_{h,m}f\in S_{h,m}$ in the form

$$(Q_{h,m}f)(x) = \sum_{j\in\mathbb{Z}} d_j B_m(h^{-1}x-j), \quad x \in \mathbb{R}, \quad (1)$$

and determine the coefficients d_j from the interpolation conditions

$$(Q_{h,m}f)((k+\frac{m}{2})h) = f((k+\frac{m}{2})h), \quad k \in \mathbb{Z}. \quad (2)$$

This leads to the bi-infinite system of linear equations

$$\sum_{j\in\mathbb{Z}} B_m(k+\frac{m}{2}-j)d_j = f((k+\frac{m}{2})h), \quad k \in \mathbb{Z},$$

or

$$\sum_{j\in\mathbb{Z}} b_{k-j}d_j = f_k, \quad k \in \mathbb{Z}, \quad (3)$$

where for $k \in \mathbb{Z}$

$$b_k = b_{k,m} = B_m(k+\frac{m}{2}),\ f_k = f_{k,h,m} = f((k+\frac{m}{2})h), \quad (4)$$

$$b_k = b_{-k} > 0 \quad \text{for } |k| \leq \mu,$$
$$b_k = 0 \quad \text{for } |k| > \mu, \quad \sum_{|k|\leq\mu} b_k = 1, \quad (5)$$

$$\mu := \text{int}\,((m-1)/2) = \left\{\begin{array}{ll}(m-2)/2, & m \text{ even}\\ (m-1)/2, & m \text{ odd}\end{array}\right\}.$$

Thus (3) is a bi-infinite system with the symmetric Toeplitz band matrix $\mathfrak{B} = (b_{k-j})_{k,j\in\mathbb{Z}}$ of the band width $2\mu+1$. For $m = 2$, system (3) reduces to relations $d_k = f((k+1)h))$, $k \in \mathbb{Z}$, and $(Q_{h,2}f)(x) = \sum_{j\in\mathbb{Z}} f((j+1)h))B_2(nx-j)$ is the usual piecewise linear interpolant which can be constructed on every subinterval $[ih,(i+1)h]$ independently from other subintervals. All is clear in the cases $m = 1, 2$

and we focuse our attention to the case $m \geq 3$. A delicate problem appears that the solution of system (3) always exists but is nonunique for $m \geq 3$ if we allow an exponential growth of $|d_j|$ as $|j| \to \infty$. Only one of the solutions of system (3) is reasonable. We call it the Wiener solution since it is related to the Wiener theorem (see [11]) about trigonometric (or equivalent Laurent) series. The following construction of the Wiener interpolant is in more details elaborated in [8] and it is equivalent to that in [6].

With $b_k = b_{k,m} = B_m(k + \frac{m}{2})$ defined in (4), introduce the following functions:

$$b(z) = b^m(z) := \sum_{|k| \leq \mu} b_k z^k = b_0 + \sum_{k=1}^{\mu} b_k(z^k + z^{-k}), \quad (6)$$

$$P_{2\mu}(z) = P_{2\mu}^m(z) = z^\mu b(z) \quad (7)$$

(the characteristic polynomial of B_m). Denote by $z_\nu = z_{\nu,m}$, $\nu = 1, ..., 2\mu$, the roots of the characteristic polynomial $P_{2\mu}^m \in \mathcal{P}_{2\mu}$ (we call them the *characteristic roots*). From (6) we observe that together with z_ν also $1/z_\nu$ is a characteristic root. The polynomials (7) were introduced in [6] starting from different considerations. As proved in [6], all characteristic roots are real and simple; then clearly $z_\nu < 0, \nu = 1, ..., 2\mu$, and $z_\nu \neq -1$, $\nu = 1, ..., 2\mu$, thus there are exactly μ characteristic roots z_ν, $\nu = 1, ..., \mu$, in the interval $(-1, 0)$ and μ characteristic roots $z_{\mu+\nu} = 1/z_\nu$, $\nu = 1, ..., \mu$ in the interval $(-\infty, -1)$. It is not complicated to show that the function

$$a(z) = a^m(z) := 1/b^m(z) = z^\mu / P_{2\mu}^m(z) \quad (8)$$

has the Laurent expansion $a^m(z) = \sum_{k \in \mathbb{Z}} a_{k,m} z^k$ with

$$a_{k,m} = \sum_{\nu=1}^{\mu} \frac{z_{\nu,m}^{\mu-1}}{P'_{2\mu}(z_{\nu,m})} z_{\nu,m}^{|k|} = a_{-k,m}, \quad k \in \mathbb{Z}, \quad (9)$$

and the Wiener solution of system (3) is given by

$$d_k = \sum_{j \in \mathbb{Z}} a_{k-j,m} f((j + \frac{m}{2})h), \quad k \in \mathbb{Z}. \quad (10)$$

An important observation from (9) is that $a_{k,m}$ decays exponentially as $|k| \to \infty$, thus the series in (10) converges provided that f is bounded or of a polynomial growth as $|x| \to \infty$. Thus we have the following result.

Theorem 1 *For a bounded or polynomially growing $f \in C(\mathbb{R})$, the Wiener interpolant $Q_{h,m}f$ is well defined by the formulae (1), (9), (10).*

Further properties of $a_{k,m}$ is that

$$\sum_{k \in \mathbb{Z}} a_{k,m} = 1, \quad \sum_{k \in \mathbb{Z}} |a_{k,m}| = \frac{(-1)^\mu}{P_{2\mu}(-1)},$$
$$a_{k,m} = (-1)^k |a_{k,m}| \neq 0, \ k \in \mathbb{Z}.$$

In the vector space X of all bisequences $(d_j)_{j \in \mathbb{Z}}$, the null space $\mathcal{N}(\mathfrak{B})$ of the matrix $\mathfrak{B} = (b_{k-j})_{k,j \in \mathbb{Z}}$ of system (3) is of dimension 2μ being spanned by bisequences $(z_\nu^j)_{j \in \mathbb{Z}}$, $\nu = 1, ..., 2\mu$. Hence for any nontrivial $(d_j^{(0)}) \in \mathcal{N}(\mathfrak{B})$, $d_j^{(0)}$ grows exponentially as $j \to \infty$ or as $j \to -\infty$.

Clearly, $||Q_{h,m}||_{BC(\mathbb{R}) \to BC(\mathbb{R})} \leq \sum_{k \in \mathbb{Z}} |a_{k,m}|$ but this estimate is coarse. To present an exact formula for $||Q_{h,m}||_{BC(\mathbb{R}) \to BC(\mathbb{R})}$, introduce the *fundamental spline* $F_m(x) := \sum_{k \in \mathbb{Z}} a_{k,m} B_m(x - k)$; it decays exponentially as $|x| \to \infty$ and satisfies $F_m(j + \frac{m}{2}) = \delta_{j,0}$, $j \in \mathbb{Z}$, where $\delta_{j,k}$ is the Kronecker symbol. Clearly,

$$(Q_{h,m}f)(x) = \sum_{j \in \mathbb{Z}} f((j + \frac{m}{2})h) F_m(h^{-1}x + j),$$

and this implies

$$\begin{aligned} &||Q_{h,m}||_{BC(\mathbb{R}) \to BC(\mathbb{R})} \\ &= \sup_{x \in \mathbb{R}} \sum_{j \in \mathbb{Z}} |F_m(x + j)| \\ &= \max_{x \in [\frac{m}{2}, \frac{m+1}{2}]} \sum_{j \in \mathbb{Z}} |F_m(x + j)|. \end{aligned}$$

Numerical values of $q_m := ||Q_{h,m}||_{BC(\mathbb{R}) \to BC(\mathbb{R})}$ and $\alpha_m := \sum_{k \in \mathbb{Z}} |a_{k,m}|$ for some m are presented in the following table taken from [4]:

m	3	4	5	9	10	20
q_m	1.414	1.549	1.706	2.075	2.142	2.583
α_m	2.000	3.000	4.800	29.11	45.73	4182

For $4 \leq m \leq 20$, the computed values of q_m fit into the model $q_m \leq \frac{e}{4} + \frac{2}{\pi}\log m$, and it seems that $q_m - (\frac{e}{4} + \frac{2}{\pi}\log m) \to 0$ as $m \to \infty$; for $m = 20$ this difference is of order 0.001. We can also observe that $\alpha_{m+1}/\alpha_m \to \pi/2 = 1.5707963268...$ as $m \to \infty$; for $m = 20$ this ratio is 1.570796327, already very close to $\pi/2$. It is challenging to confirm these empiric guesses analytically.

Finally, it is easily seen that for any bisequence d_j, $j \in \mathbb{Z}$, with $\sup_j |d_j| < \infty$, it holds

$$\frac{\sup_j |d_j|}{\alpha_m} \leq \sup_{x \in \mathbb{R}} |\sum_{j \in \mathbb{Z}} d_j B(h^{-1}x - j)| \leq \sup_j |d_j|.$$

3 Error estimates of the spline interpolant

To formulate the main results of the paper, we need some information concerning the Euler splines [3]. A spline $E \in S_{h,m}$ satisfying

$$E^{(m-1)}(x) = (-1)^i \quad \text{for} \quad ih < x < (i+1)h, \ i \in \mathbb{Z},$$

is called *perfect*. If $E \in S_{h,m}$ is perfect then so is $E + g$ with any $g \in \mathcal{P}_{m-2}$.

For $m = 1$, the *Euler perfect spline* $E_{h,1} \in S_{h,1}$ is defined by the formula

$$E_{h,1}(x) = \text{sign } \sin(h^{-1}\pi x) = \frac{4}{\pi}\sum_{k=0}^{\infty} \frac{\sin(2k+1)h^{-1}\pi x}{2k+1}. \tag{11}$$

For $m \geq 2$, the *Euler perfect spline* $E_{h,m} \in S_{h,m}$ is determined recursively as a special integral function of $E_{h,m-1}$, namely,

$$E_{h,m}(x) = \left\{ \begin{array}{ll} \int_{h/2}^{x} E_{h,m-1}(y)dy, & m = 2l \\ \int_{0}^{x} E_{h,m-1}(y)dy, & m = 2l+1 \end{array} \right\};$$

the lower bounds of integration are chosen so that the $2h$-periodicity and the zero mean value of $E_{h,m-1}$ over a period is inherited to $E_{h,m}$. Starting from (11) we recursively find that

$$E_{h,m}(x) = \left\{ \begin{array}{ll} \frac{4}{\pi}\frac{(-1)^l h^{m-1}}{\pi^{m-1}} \sum_{k=0}^{\infty} \frac{\cos(2k+1)h^{-1}\pi x}{(2k+1)^m}, & m = 2l \\ \frac{4}{\pi}\frac{(-1)^l h^{m-1}}{\pi^{m-1}} \sum_{k=0}^{\infty} \frac{\sin(2k+1)h^{-1}\pi x}{(2k+1)^m}, & m = 2l+1 \end{array} \right\}. \tag{12}$$

By construction, $E'_{h,m} = E_{h,m-1}$ for $m \geq 2$. Further, we observe that $x = (i + \frac{1}{2})h$, $i \in \mathbb{Z}$, are the zeroes of $E_{h,m}$ for even m, and $x = ih$, $i \in \mathbb{Z}$, are the zeroes of $E_{h,m}$ for odd m. A unified formulation is that $x = (i + \frac{m-1}{2})h$, $i \in \mathbb{Z}$, are the zeroes of $E_{h,m}$ and $x = (i + \frac{m}{2})h$, $i \in \mathbb{Z}$, are the local extrema of $E_{h,m}$ (the zeroes of $E'_{h,m} = E_{h,m-1}$). There are no other zeroes and extrema of $E_{h,m}$ – this can be easily seen recursively, since by Rolle's theorem an additional zero of $E_{h,m}$ involves an additional zero of $E'_{h,m} = E_{h,m-1}$. It is clear also that the zeroes of $E_{h,m}$ are simple. Further, for $m = 2l$,

$$||E_{h,m}||_\infty = |E_{h,m}(0)| = \frac{4}{\pi}\frac{h^{m-1}}{\pi^{m-1}}\sum_{k=0}^{\infty}\frac{1}{(2k+1)^m}$$

(the absolute value of $E_{h,m}$ at other local extremum points $x = ih$ is same). Similarly, for $m = 2l+1$,

$$||E_{h,m}||_\infty = |E_{h,m}(\frac{h}{2})| = \frac{4}{\pi}\frac{h^{m-1}}{\pi^{m-1}}\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)^m}.$$

Unifying these two formulae, we can write

$$||E_{h,m}||_\infty = \Phi_m \pi^{-(m-1)} h^{m-1}, \quad m \in \mathbb{N}, \tag{13}$$

with

$$\Phi_m = \frac{4}{\pi}\left\{ \begin{array}{ll} \sum_{k=0}^{\infty} \frac{1}{(2k+1)^m}, & m = 2l \\ \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^m}, & m = 2l+1 \end{array} \right\}, \ m \in \mathbb{N}, \tag{14}$$

known as the *Favard constant*. In particular,

$$\Phi_1 = 1, \quad \Phi_2 = \pi/2, \quad \Phi_3 = \pi^2/8, \quad \Phi_4 = \pi^3/24,$$

and it holds $\lim_{m\to\infty} \Phi_m = \frac{4}{\pi}$,

$$\Phi_1 < \Phi_3 < \Phi_5 < \ldots < \frac{4}{\pi} < \ldots < \Phi_6 < \Phi_4 < \Phi_2.$$

We are ready to formulate our main result.

Theorem 2 *For $f \in V^{m,\infty}(\mathbb{R})$, $m \in \mathbb{N}$, there hold the pointwise estimate*

$$|f(x) - (Q_{h,m}f)(x)| \leq ||f^{(m)}||_\infty |E_{h,m+1}(x)|, \ x \in \mathbb{R}, \tag{15}$$

and the uniform estimate

$$||f - Q_{h,m}f||_\infty \leq \Phi_{m+1}\pi^{-m}h^m ||f^{(m)}||_\infty. \tag{16}$$

For $f = E_{h,m+1} \in W^{m,\infty}(\mathbb{R}) \subset V^{m,\infty}(\mathbb{R})$, inequalities (15) and (16) turn into equalities.

This theorem is known [3] in the case of 1-periodic f and $h = 1/n$ with an even $n \in \mathbb{N}$ (then also $Q_{h,m}f$ and $E_{h,m+1}$ are 1-periodic). Actually, only (15) is a complicated assertion for the proof, whereas (16) is an obvious consequence of (15), (13), and the assertion about the sharpness of estimates for $f = E_{h,m+1}$ is elementary. We step after step extend the theorem first for functions f with compact support, then for $f \in V^{m,\infty}(\mathbb{R})$ of slow growth as $|x| \to \infty$, and finally for arbitrary $f \in V^{m,\infty}(\mathbb{R})$. Technically, the extensions are based on the following lemma; for details, see [8].

Lemma 3 *Suppose that functions $g_\delta \in C(\mathbb{R})$, $\delta > 0$, satisfy*

$$g_\delta(x) \to 0 \quad as \quad \delta \to 0 \quad \forall x \in \mathbb{R},$$
$$|g_\delta(x)| \leq c(1 + |x|^r) \quad \forall x \in \mathbb{R}$$

where $r \geq 0$. Then

$$(Q_{h,m}g_\delta)(x) \to 0 \quad as \quad \delta \to 0 \quad \forall x \in \mathbb{R}.$$

Remark 4 *Using Banach–Steinhaus theorem and Theorem 2, it is easily seen that for* $f \in BUC(\mathbb{R})$, $||f - Q_{h,m}f||_\infty \to 0$ *as* $h \to 0$.

Let us discuss **optimality properties of the spline interpolation** compared with other methods that use the same information about the values of f on the uniform grid $\Delta_h = \{(j + \frac{m}{2})h : j \in \mathbb{Z}\}$. Such a method can be identified with a mapping $M_h : C(\Delta_h) \to C(\mathbb{R})$ where $C(\Delta_h)$ is the vector space of grid functions defined on Δ_h and having values in $\mathbb{R}$ or $\mathbb{C}$.

Remark 5 *For given* $\gamma > 0$, *we have in accordance to Theorem 2*

$$\sup_{f \in V^{m,\infty}(\mathbb{R}),\, ||f^{(m)}||_\infty \le \gamma} ||f - Q_{h,m}f||_\infty = \Phi_{m+1}\pi^{-m}h^m\gamma$$

whereas for any mapping $M_h : C(\Delta_h) \to C(\mathbb{R})$ *(linear or nonlinear, continuous or discontinuous), it holds*

$$\sup_{f \in V^{m,\infty}(\mathbb{R}),\, ||f^{(m)}||_\infty \le \gamma} ||f - M_h(f|_{\Delta_h})||_\infty \ge \Phi_{m+1}\pi^{-m}h^m\gamma. \quad (17)$$

Indeed, (17) is trivially fulfilled if $M_h(\mathbf{0}) \notin BC(\mathbb{R})$, so we may assume that $M_h(\mathbf{0}) \in BC(\mathbb{R})$. Consider two functions $f_\pm = \pm\gamma E_{h,m+1}$. Clearly, $f_\pm \in W^{m,\infty}(\mathbb{R}) \subset V^{m,\infty}(\mathbb{R})$, $||f_\pm^{(m)}||_\infty = \gamma$, and since $E_{h,m+1}|_{\Delta_h} = \mathbf{0}$, we obtain (17) by the following argument:

$$\begin{aligned}
&\sup_{f \in V^{m,\infty}(\mathbb{R}),||f^{(m)}||_\infty \le \gamma} ||f - M_h(f|_{\Delta_h})||_\infty \\
&\ge \max\{||f_+ - M_h(f_+|_{\Delta_h})||_\infty, ||f_- - M_h(f_-|_{\Delta_h})||_\infty\} \\
&= \max\{||f_+ - M_h(\mathbf{0})||_\infty, ||f_- - M_h(\mathbf{0})||_\infty\} \\
&\ge \tfrac{1}{2}(||f_+ - M_h(\mathbf{0})||_\infty + ||f_- - M_h(\mathbf{0})||_\infty) \\
&\ge \tfrac{1}{2}||f_+ - f_-||_\infty = ||E_{h,m+1}||_\infty\gamma = \Phi_{m+1}\pi^{-m}h^m\gamma.
\end{aligned}$$

Remark 6 *Let* $h = 1/n$ *with an even* $n \in \mathbb{N}$. *Consider the subspace* $C_{per}(\mathbb{R})$ *of* $C(\mathbb{R})$ *consisting of 1-periodic continuous functions on* $\mathbb{R}$, *and denote* $W^{m,\infty}_{per}(\mathbb{R}) = C_{per}(\mathbb{R}) \cap W^{m,\infty}(\mathbb{R})$; *denote by* $C_{per}(\Delta_h)$ *the space of* 1*-periodic (grid) functions on the grid* Δ_h, *i.e.,* $f_h(ih) = f_h(1 + ih)$, $i \in \mathbb{Z}$, *for* $f_h \in C_{per}(\Delta_h)$. *Then for any mapping* $M_h : C_{per}(\Delta_h) \to C_{per}(\mathbb{R})$, *it holds*

$$\sup_{f \in W^{m,\infty}_{per}(\mathbb{R}),||f^{(m)}||_\infty \le \gamma} ||f - M_h(f|_{\Delta_h})||_\infty \ge \Phi_{m+1}\pi^{-m}h^m\gamma\,.$$

The proof is same as in the case of Remark 5, we only need to observe that $E_{h,m+1} \in W^{m,\infty}_{\text{per}}(\mathbb{R})$ for $h = 1/n$ with an even $n \in \mathbb{N}$.

Remark 7 *For functions with compact supports, similar result as Remark 5 holds asymptotically as* $h \to 0$. *Denote by* $W^{m,\infty}_0(\mathbb{R})$ *the subspace of* $W^{m,\infty}(\mathbb{R})$ *consisting of functions* $f \in W^{m,\infty}(\mathbb{R})$ *with support in* $[0, 1]$. *For any mapping* $M_h : C(\Delta_h) \to C(\mathbb{R})$, *it holds*

$$\liminf_{h\to 0} \sup_{f \in W^{m,\infty}_0(\mathbb{R}),||f^{(m)}||_\infty \le \gamma} \frac{||f - M_h(f|_{\Delta_h})||_\infty}{\Phi_{m+1}\pi^{-m}h^m\gamma} \ge 1\,.$$

This follows by a slight modification of the argument in Remark 5. Namely, instead of $f_\pm = \pm\gamma E_{h,m+1}$, use $f_\pm = \pm\gamma e E_{h,m+1}$ where $e \in C^m(\mathbb{R})$ is supported in $(0, 1)$, $0 \le e(x) \le 1$ for all $x \in \mathbb{R}$ and $e(x) = 1$ for $\frac{1}{3} \le x \le \frac{2}{3}$. Then for sufficiently small h, it still holds $||eE_{h,m+1}||_\infty = ||E_{h,m+1}||_\infty = \Phi_{m+1}\pi^{-m}h^m$, and the Leibniz differentiation rule yields $||(eE_{h,m+1})^{(m)}||_\infty \to 1$ as $h \to 0$.

Next we formulate some further error estimates.

Theorem 8 *For* $f \in V^{m,\infty}(\mathbb{R})$, $l = 1, ..., m - 1$, *it holds*

$$||f^{(l)} - (Q_{h,m}f)^{(l)}||_\infty \le \Phi_{m-l+1}\pi^{-(m-l)}h^{m-l}\,(1 + \alpha_m)\,||f^{(m)}||_\infty, \quad (18)$$

$$||f^{(l)} - (Q_{h,m}f)^{(l)}||_\infty \le \Phi_{m-l+1}\pi^{-(m-l)}h^{m-l}\,(1 + q_{m-l}\alpha_{m,l})\,||f^{(m)}||_\infty \quad (19)$$

where $\alpha_m = \sum_{k\in\mathbb{Z}} |a_{k,m}|$,

$$\alpha_{m,l} = \sum_{k\in\mathbb{Z}} \left| \sum_{|j|\le int\{(m-l-1)/2\}} a_{k-j,m}b_{j,m-l} \right| < \alpha_m, \quad (20)$$

$q_{m-l} = ||Q_{h,m-l}||_{BC(\mathbb{R})\to BC(\mathbb{R})}$,
$b_{j,m-l} = B_{m-l}(j + \frac{m-l}{2})$ *(cf. (4)), and* $a_{k,m}$ *are defined in (9).*

Remark 9 *For* $f \in V^{l,\infty}(\mathbb{R})$ *with* $f^{(l)} \in BUC(\mathbb{R})$, $0 < l < m$, *it holds* $||f^{(l)} - (Q_{h,m}f)^{(l)}||_\infty \to 0$ *as* $h \to 0$.

Theorem 10 *For* $f \in V^{l,\infty}(\mathbb{R})$, $0 < l < m$, *it holds*

$$||f - Q_{h,m}f||_\infty \le \Phi_{l+1}\pi^{-l}h^l\,(1 + \alpha_m)\,||f^{(l)}||_\infty, \quad (21)$$

$$||f - Q_{h,m}f||_\infty \le \Phi_{l+1}\pi^{-l}h^l\,(1 + q_{m-l}\alpha_{m,l})\,||f^{(l)}||_\infty \quad (22)$$

with constants α_m and $\alpha_{m,l}$ defined in (20). If, in addition, $f^{(l)} \in BUC(\mathbb{R})$ then

$$||f - Q_{h,m}f||_\infty = o(h^l) \quad \textit{as } h \to 0.$$

The proof of Theorems 8 and 10 is based on the following lemma.

Lemma 11 *For $f \in V^{l,\infty}(\mathbb{R})$, $\mathbb{N} \ni l < m$, it holds*

$$||(Q_{h,m}f)^{(l)}||_\infty \le \alpha_m ||f^{(l)}||_\infty$$
$$||(Q_{h,m}f)^{(l)}||_\infty \le q_{m-l}\alpha_{m,l}||f^{(l)}||_\infty$$

with constants defined in Theorem 8.

Introduce the space $V_h^{m,\infty}(\mathbb{R})$, $m \ge 2$, of functions $g \in C^{m-2}(\mathbb{R})$ such that the derivatives $g^{(m-1)}$, $g^{(m)}$ exist on every interval $(ih, (i+1)h)$ and

$$g^{(m-1)}|_{(ih,(i+1)h)} \in C((ih, (i+1)h)),$$
$$g^{(m)}|_{(ih,(i+1)h)} \in L^\infty((ih, (i+1)h)),\ i \in \mathbb{Z},$$
$$\sigma_{h,m}(g) := \sup_{i \in \mathbb{Z}} \sup_{ih < x < (i+1)h} |g^{(m)}(x)| < \infty.$$

Clearly, $V^{m,\infty}(\mathbb{R}) \subset V_h^{m,\infty}(\mathbb{R})$ and $||f^{(m)}||_\infty = \sigma_{h,m}(f)$ for $f \in V^{m,\infty}(\mathbb{R})$.

Lemma 12 *For $m \ge 2$, it holds*

$$V_h^{m,\infty}(\mathbb{R}) = V^{m,\infty}(\mathbb{R}) + S_{h,m},$$

i.e., every $g \in V_h^{m,\infty}(\mathbb{R})$ has a representation

$$g = f + g_h, \quad f \in V^{m,\infty}(\mathbb{R}), \quad g_h \in S_{h,m}. \quad (23)$$

In particular, (23) holds with

$$f(x) = \frac{1}{(m-1)!}\int_0^x (x-t)^{m-1}G_m(t)dt, \quad x \in \mathbb{R}, \quad (24)$$

where $G_m \in L^\infty(\mathbb{R})$ is defined by $G_m(x) = g^{(m)}(x)$ for $x \in (ih, (i+1)h)$, $i \in \mathbb{Z}$ (and other f in (23) differ from (24) by an additive polynomial of degree $m-1$).

As the consequence of Theorem 2 and Lemma 12 we obtain the following result.

Theorem 13 *Let $m \ge 2$. Assume that $g \in V_h^{m,\infty}(\mathbb{R})$ satisfies the inequality*

$$|g(x)| \le c(1 + |x|^r), \quad x \in \mathbb{R}, \quad (25)$$

where $r \ge 0$ and $c \ge 0$. Then

$$|g(x) - (Q_{h,m}g)(x)| \le \sigma_{h,m}(g)|E_{h,m+1}(x)|, \quad x \in \mathbb{R}, \quad (26)$$

$$||g - Q_{h,m}g||_\infty \le \Phi_{m+1}\pi^{-m}h^m\sigma_{h,m}(g). \quad (27)$$

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