

# An effective use of a CAS in an advanced calculus course

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*Abstract:* - In this study, it is observed how a CAS programme can help student to increase their mathematical understanding. CAS have a lot of great properties that can be used for educational purposes. For example, you can produce 2D, 3D and even animated graphs. Symbolic calculations and some advanced mathematical level calculations can also be done. CAS is a name of a group of software such as Mathematica, Maple, Derive and Mathlab. In this study, Maple software has been chosen to design a discovery learning environment. A group of students has been directed to discover the relations among calculus concepts by using 3D graphs produced by Maple. A semi-structured interview has been employed in this study to collect data. At the end of this paper, mathematical acquisitions of students have been discussed.

*Key-words:* - Mathematics education, Calculus, Computer algebra systems, CAS, Maple, Integral, level curve

## 1 Introduction

Computer algebra systems are originally created for the use of applied mathematicians and engineers. In the last two decades, it has been also researched how CAS can be used while teaching mathematics [1,2,3].

We are all aware of the importance of constructivist approaches in mathematics learning. Imagine a mathematics course! “The texts are well structured, nicely formatted and polished, and offer little leeway both for the teacher and the learner. Definitions and theorems are well structured and easily understandable. The teacher has enough both mathematical and pedagogical knowledge to transfer his or her information to the students”. Even if there is no problem for all most everyone in this environment, most of the time the historical background of mathematical concepts is forgotten.

Generally, the end users are not aware that the built-up of mathematics has always been a creative process which most of the time involves lost of experimentation. We don't realize how much hard work was done by great mathematicians in the past. Students tend to think mathematical discovery only comes by stroke of genius. Students are convinced that they will never have such luck, and this adds the general uneasiness about mathematics and its teaching.

Many educators therefore want to follow a traditional paradigm using sequence “*Definition* → *theorem* → *proof* → *corollary* → *application*” by an approach which is more historic using the discovery chain “*Problem* → *experiment* → *conjecture and idea of proof*” [4].

A CAS allows lots of experimenting by the students, thus helping to find reasonable conjectures [5,6]). As a result of a great body of research, it can be understood that using CAS is more meaningful when it was

integrated with constructivist principles in the same environment [7,8].

### 2 Problem Situation

Calculus concepts should have been taught in a carefully designed learning environment, because these concepts constitute a very important base for almost all applied sciences.

In a technology based constructivist learning environment, students can find the opportunity to discover, making conjectures and construct their own mathematical knowledge. Many researchers used a CAS like Maple, Mathematica or Derive making students to discover by visualizing the calculus concepts [5,9,10,11,12].

Considering all of these points, in this paper it is found out that a CAS can have an effect on a discovery application in an advanced calculus course.

### 3 Methodology

In this study, a project was given to a group of students studying at an average Turkish university. The project title was “*constructing a real life problem that can be solved by using double integration.*” These students were taking an elective Maple course during the time of the implementation of the project.

After constructing and solving the problem, students were encountered a contradiction about the solution of their own problem. Instead of giving the right solution directly to them, it was tried to make students to reconsider their own solutions by using some visualisation created by Maple. This procedure was carried out by using a semi-structured interview.

#### 3.1 Students’ Problem

In the project, students studied some problems from textbooks and constructed the following problem in their own context;

“A teetotum, formed by a cone with a height of 12 cm and a semi-sphere with a radius of 6 cm, has to be created. Evaluate the volume of teetotum.”

#### 3.1.2 Students’ Solution

To form a teetotum, the base of cone and semi-sphere must coincide. Therefore, radius of base of cone and radius of sphere must be equal.

First, let’s evaluate the volume by using ordinary formulas;

$$\text{Volume of semi-sphere} + \text{Volume of cone} = \frac{4}{6}\pi r^3 + \frac{\pi r^2 h}{3} = \frac{4\pi \cdot 216}{6} + \frac{\pi \cdot 36 \cdot 12}{3} = 288\pi \text{ cm}^3 \dots\dots\dots(1)$$

We feel secure ourselves by this result, because we already know the formulas of volumes of sphere and cone.

Now, let’s try to evaluate by using double integration;

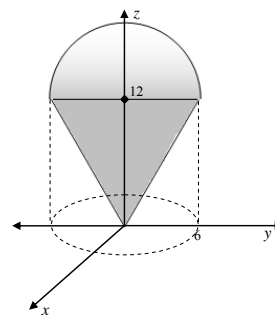


figure-1

Let’s place the teetotum on IR<sup>3</sup> as in Figure-1. We can easily understand that the equation of sphere with the center of (0, 0, 12) and radius of 6 cm, is

$$x^2 + y^2 + (z - 12)^2 = 36 \dots(2)$$

and above part of it can be expressed by

$$z = \sqrt{36 - (x^2 + y^2)} + 12 \dots(3)$$

We also know that the equation of the cone which is above part of xy plane is

$$z = \sqrt{x^2 + y^2} \dots\dots\dots(4)$$

Now, we can evaluate the volume between the surfaces numbered (2) and (3) as below by the help of polar coordinate transformation;

$$V = \int_0^{2\pi} \int_0^6 (\sqrt{36 - r^2} + 12 - r) r dr d\theta = 432\pi \text{ cm}^3 \dots\dots\dots(5)$$

Our students were surprised because the results in (1) and (5) are not equal. At this step, these students were directed by some visualization. By this way, students would have the chance of both making connection between calculus concepts and consolidating their Maple knowledge.

### 4 Presentation of Findings

In this section, a semi-structured interview between teacher and students A, B and C has been reported.

**Teacher:** -First, let’s check your calculations by Maple;

```
> int(int((sqrt(36-r^2)+12-r)*r, r=0..6), theta=0..2*Pi);
```

$$\int_0^{2\pi} \int_0^6 (\sqrt{36 - r^2} + 12 - r) r dr d\theta = 432 \pi$$

```
> 4*Pi*6^3/6 + Pi*6^2*12/3;
288 pi
```

-We can say that your calculations are true. There is something wrong conceptually. I suggest you to check the graphs represented by the equations (3) and (4).

**Student A:** -Yeah! I have already tried it. But, I could not manage to see the graphs.

**Teacher:** -You must pay attention to the type(s) of equations (2) and (3). These equations are explicit. You can use “plot” command to draw on maple. But, to see the semi-sphere exactly you should use the equation (2). Of course, this equation is implicit. Therefore you can

use the command “**implicitplot**” and you must determine lower bound of  $z$ -range as 12 in order not to display lower part of sphere.

-You are trying to display two different types of graphs on a same coordinate. Therefore, you must remember the use of command “**display**”.

**Student B:** -It was a bit confusing. But after some discussions with my friends, we managed to see the graphs as below;

```
> with(plots):
> A:=implicitplot3d(x^2+y^2+(z-12)^2=36,
x=-7..7, y=7..7, z=12..20, numpoints=5000):
> B:=plot3d(sqrt(x^2+y^2), x=-7..7, y=-7..7, view=0..20):
> display(A,B);
```

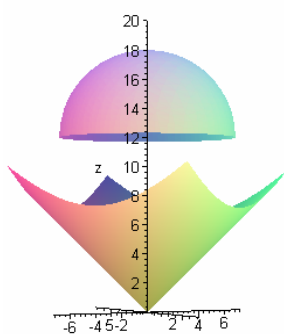


figure-2

**Teacher:** -What is your comment about this figure?

**Student C:** -This figure can not form a teetotum. Our equations are wrong.

**Student B:** -According to figure, semi-sphere equation (equation (3)) is right. But, there is something wrong with the equation of cone (equation (4)).

While constructing the equation, students had written the equation of cone only remembering the standard cone equation. However, this equation was not the expected equation. Thanks to figure-2, students have realized their mistake.

**Teacher:** -Yes! How can you redefine the equation of cone to form a teetotum with the semi-sphere?

Student had some approaches to find the right equation. Some of them were correct, some of them were not. One approach was selected and students were directed to find the equation.

**Teacher:** -Let’s remember the concept “*level curve of a surface*”. What is the curve level of the surface  $z = \sqrt{x^2 + y^2}$  for the value of  $z = 12$ ?

**Student A:**  $-12 = \sqrt{x^2 + y^2} \Rightarrow 12^2 = x^2 + y^2$ .

Therefore, the level curve is the circle  $12^2 = x^2 + y^2$

**Teacher:** -How can you assess this situation by using level curve?

**Student A:** -A circle which is 12 cm high from  $xy$  plane. Its radius is 12 cm and center is at the origin.

**Teacher:** -All right! What must be our expected level curve which is 12 cm high from  $xy$  plane?

**Student C:** -Let’s redraw figure-2 like below;

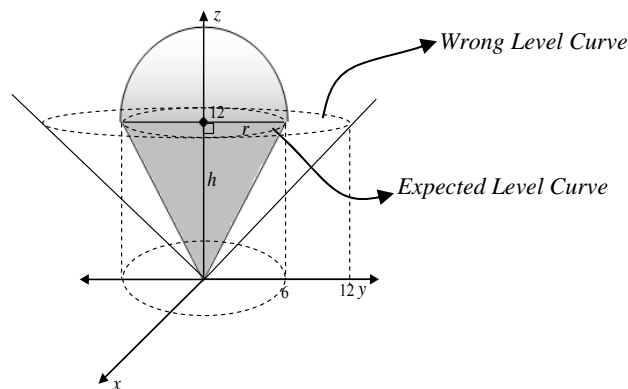


Figure-3

The expected level curve’s equation must be  $6^2 = x^2 + y^2$  for the value of  $z = 12$ .

**Teacher:** -OK! How can you redefine the cone’s equation to obtain the level curve  $6^2 = x^2 + y^2$  for the value  $z = 12$  ?

**Student B:** -After some cooperation we have concluded that;

Let’s substitute 12 instead of  $z$  in the equation (4);

We obtain  $12 = \sqrt{x^2 + y^2}$  .....(6)

But we are trying to reach the equation;

$6 = \sqrt{x^2 + y^2}$  .....(7)

To obtain the equation (7) from equation (6) we can multiply left hand side of the equation (6) by  $\frac{1}{2}$ . Then let’s resubstitute  $z$  instead of 12;

$\frac{1}{2}z = \sqrt{x^2 + y^2} \Rightarrow z = 2\sqrt{x^2 + y^2}$  ..... (8)

Therefore, equation (8) must be our expected cone equation.

**Teacher:** -If you want to check your result, you can use the Maple code group which draws the figure-2 after altering the cone equation.

**Student A:** -Yeah! I can make that modification as highlighted below;

```
> with(plots):
> A:=implicitplot3d(x^2+y^2+(z-12)^2=36,
x=-7..7, y=7..7, z=12..20, numpoints=5000):
> B:=plot3d(2*sqrt(x^2+y^2), x=-7..7,
y=-7..7, view=0..20):
> display(A,B);
```

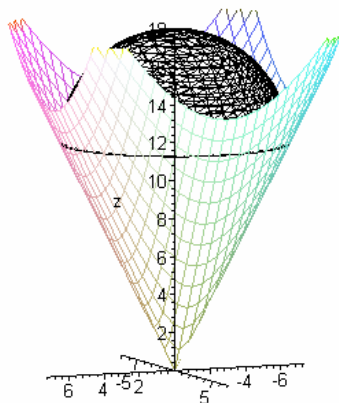


figure-4

OK! We reached the shape of teetotum at last.

**Student B:** -Let me make some calculations again by using our new equation;

```
> int(int(sqrt(36-r^2)+12-2*r)*r, r=0..6, theta=0..2*Pi);
```

$$\int_0^{2\pi} \int_0^6 (\sqrt{36-r^2} + 12 - 2r) r dr d\theta = 288\pi$$

This result is exactly the same as the result in (1).

## 5 Results and Discussion

In this research, an application, which can be considered as an example of how a CAS can be used to increase students' conceptual understanding, has been presented.

Calculus concepts like limit, derivative and integral are usually hard to understand due to these concepts' abstract nature. Many researchers have used a CAS to make some concepts more tangible by producing dynamic or static graphs. Most of these studies have showed the positive effect of CAS to conceptual understanding [10,13,14]. Only some findings of a few number of research could not show positive effects of CAS [15]. In this kind of studies, researchers have indicated that there is an inappropriate use of computers.

This study's findings are also similar to findings of previous research mentioned above. First, some important student views;

- Using our new knowledge from the Maple course to discover something in Calculus was a great experience and pleasure for us.
- Using double integral to solve a question in our own context was more meaningful.

- Using graphs and finding same results with the result that we found by our high school formulas were more concrete and understandable.

- Maybe, one of the most important things for us is that "in this kind of courses we feel ourselves as a real university student."

Besides all of these positive views, students have found a chance to examine some advanced level mathematical skills. Students;

- Not only did students use the level curves to graph a surface, but also they realized and used the real meaning of level curves in real context,

- Made a conjecture from the 3D graph produced by Maple,

Furthermore, the students had the chance to improve their Maple skills while studying on their main project. Students had to use some advanced Maple commands like "implicitplot" and "display". While using "implicitplot", students had to also consider what an implicit function is.

This research was a specific qualitative case study. The findings of this study are also supported by several quantitative studies like MacCrae's research. McCraes have concluded that even if CAS may have some weak points, it has a great impact on students' conceptual understanding [16].

In conclusion, it can be easily suggested that this kind of applications should be extended to all steps of calculus courses.

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