On the Idle Time Model in Computer Networks

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Abstract: An open queueing network model in light traffic has been developed. The theorem on the law of the iterated logarithm for the idle time process of customers in an open queueing network has been presented. Finally, we present an application of the theorem - a idle time model from computer network practice.

Key–Words: mathematical models of technical systems, reliability theory, queueing theory, open queueing network, heavy traffic, a law of the iterated logarithm, idle time process of customers.

1 Introduction

One can apply the law of the iterated logarithm to the waiting time of customers, virtual waiting time of a customer, and the queue length of customers to get more important probabilistic characteristics of the queueing theory in heavy traffic (see, for example, [1]-[7], [9]-[11]). A single - phase case, where intervals of time between the arrival of customers to queue are independent identically distributed random variables and there is a single device working independently of the output in heavy traffic, has been completely investigated (see, for example, [1]-[2]). But there are only several papers on the theory of open queueing networks in heavy traffic (see [9]-[10]) and no proof theorems on laws of the iterated logarithm for the main probabilistic characteristics of an open queueing network in heavy traffic (for example, sojourn time of a customer, a virtual waiting time of a customer and idle time process of customers).

So in this paper, we present a theorem on the law of the iterated logarithm for the idle time process of customers in the queueing network. The service discipline is “first come, first served” (FCFS).

We consider open queueing networks with the FCFS service discipline at each station and general distributions of interarrival and service time.

The queueing network we studied has k single server stations, each of which has an associated infinite capacity waiting room.

Every station has an arrival stream from outside the network, and the arrival streams are assumed to be mutually independent renewal processes. Customers are served in the order of arrival and after service they are randomly routed to either another station in the network, or out of the network entirely. Service times and routing decisions form mutually independent sequences of independent identically distributed random variables. The basic components of the queueing network are arrival processes, service processes, and routing processes.

We begin with a probability space (Ω, B, P) on which these processes are defined. In particular, there are mutually independent sequences of independent identically distributed random variables $\left\{z_n^{(j)}, n \geq 1\right\}$, $\left\{S_n^{(j)}, n \geq 1\right\}$ and $\left\{\Phi_n^{(j)}, n \geq 1\right\}$ for $j = 1, 2, ..., k$; defined on the probability space. Random variables $z_n^{(j)}$ and $S_n^{(j)}$ are strictly positive, and $\Phi_n^{(j)}$ have support in $\{0, 1, 2, ..., k\}$. We define $\mu_j = \left(E\left[S_n^{(j)}\right]\right)^{-1}$, $\sigma_j = D\left(S_n^{(j)}\right)$ and $\lambda_j = \left(E\left[z_n^{(j)}\right]\right)^{-1}$, $a_j = D\left(z_n^{(j)}\right)$, $j = 1, 2, ..., k$; with all of these terms assumed finite. Denote $p_{ij} = P\left(\Phi_n^{(i)} = j\right)$, $i, j = 1, 2, ..., k$. The $k \times k$ matrix $P = (p_{ij})$ is assumed to have a spectral radius strictly smaller than a unit (see [9]). The matrix P is called a routing matrix.

In the context of the queueing network, the random variables $z_n^{(j)}$ function as interarrival times (from outside the network) at the station j, while $S_n^{(j)}$ is the nth service time at the station j, and $\Phi_n^{(j)}$ is a routing indicator for the nth customer served at the station j. If $\Phi_n^{(i)} = j$ (which occurs with probability p_{ij}), then the nth customer served at the station i is routed to the station j. When $\Phi_n^{(i)} = 0$, the associated customer leaves the network.

At first let us define $I_j(t)$ as the idle time process of customers at the jth station of the queueing network in time t, $\hat{\beta}_j = \frac{\lambda_j + \sum_{i=1}^{k} \mu_i \cdot p_{ij}}{\mu_j} - 1$, $\hat{\sigma}_j^2 = \sum_{i=1}^{k} p_{ij}^2 \cdot \mu_i \cdot \left(\sigma_j + \left(\frac{\mu_i}{\mu_j}\right)^2 \cdot \sigma_i\right) + \lambda_j \cdot \left(\sigma_j + \left(\frac{\lambda_j}{\mu_j}\right)^2 \cdot a_j\right)$, $j = 1, 2, \cdots, k$ and $t > 0$.

We suppose that the following conditions are fulfilled:

$$\lambda_j + \sum_{i=1}^{k} \mu_i \cdot p_{ij} < \mu_j, \ j = 1, 2, \ldots, k. \tag{1}$$

Note that this conditions quarantees that, with probability one there exists a idle time process of customers and this idle time process of customers is constantly growing.

In addition, we assume throughout that

$$\max_{1 \le j \le k} \sup_{n \ge 1} E\left\{\left(z_n^{(j)}\right)^{2+\varepsilon}\right\} < \infty \ \text{ for some } \ \varepsilon > 0, \tag{2}$$

$$\max_{1 \le j \le k} \sup_{n \ge 1} E\left\{\left(S_n^{(j)}\right)^{2+\varepsilon}\right\} < \infty \ \text{ for some } \ \varepsilon > 0. \tag{3}$$

Conditions (2) and (3) imply the Lindeberg condition for the respective sequences, and are easier to verify in practice (usually $\varepsilon = 1$ works).

One of the results of the paper is a following theorem on the law of the iterated logarithm for the idle time process of customers in an open queueing network (proof can be found in [7]).

Theorem 1 *If conditions (1) - (3) are fulfilled, then*

$$P\left(\overline{\lim_{t\to\infty}} \frac{I_j(t) - \hat{\beta}_j \cdot t}{\hat{\sigma}_j \cdot a(t)} = 1\right) =$$

$$P\left(\underline{\lim_{t\to\infty}} \frac{I_j(t) - \hat{\beta}_j \cdot t}{\hat{\sigma}_j \cdot a(t)} = -1\right) = 1,$$

$j = 1, 2, \ldots, k$ *and* $a(t) = \sqrt{2t \ln \ln t}.$

2 Idle Time Function of Computer Network

Now we present a technical example from the computer network practice. Assume that queues arrive at the computer v_j at a rate λ_j per hour during business hours, $j = 1, 2, \ldots, k$. These queues are served at the rate μ_j per hour by the computer v_j, $j = 1, 2, \ldots, k$. After service in the computer v_j, with probability p_j (usually $p_j \ge 0.9$), they leave the network and with probability p_{ji}, $i \ne j$, $1 \le i \le k$ (usually $0 < p_{ji} \le 0.1$) arrive at the computer v_i, $i = 1, 2, \ldots, k$. Also, we assume the computer v_j is idle when the idle time of waiting for service computer is less than k_j, $j = 1, 2, \ldots, k$.

Therefore, using Theorem 1 we get for $0 < \varepsilon < 1$

$$P\left(\overline{\lim_{t\to\infty}} \frac{I_j(t) - \hat{\beta}_j \cdot t}{\hat{\sigma}_j \cdot a(t)} < 1 - \varepsilon\right) = 0, \ j = 1, 2, \ldots, k. \tag{4}$$

Let us investigate a computer network which consists of the elements (computers) v_j that are indicators of stations X_j, $j = 1, 2, \ldots, k$.

Denote

$$X_j = \begin{cases} 1, & \text{if the element } v_j \text{ is idle} \\ 0, & \text{if the element } v_j \text{ is not idle,} \end{cases}$$

$j = 1, 2, \ldots, k$.

Note that $\{X_j = 1\} = \{I_j(t) < k_j\}$, $j = 1, 2, \ldots, k$.

Denote the structural function of the system of elements connected by scheme 1 from k (see, for example, [8]) as follows:

$$\phi(X_1, X_2, \ldots, X_k) = \begin{cases} 1, & \sum_{i=1}^k X_i \ge 1 \\ 0, & \sum_{i=1}^k X_i < 1. \end{cases}$$

Denote $y = \sum_{i=2}^k X_i$. Estimate the idle function of the system (computer network) using the formula of full conditional probability

$$h(X_1, X_2, \ldots, X_k) = E\phi(X_1, X_2, \ldots, X_k) =$$

$$P(\phi(X_1, X_2, \ldots, X_k) = 1) = P(\sum_{i=1}^{k} X_i \ge 1) =$$

$$P(X_1 + y \ge 1) = P(X_1 + y \ge 1 | y = 1) \cdot$$

$$P(y = 1) + P(X_1 + y \ge 1 | y = 0) \cdot P(y = 0) =$$

$$P(X_1 \ge 0) \cdot P(y = 1) + P(X_1 \ge 1) \cdot P(y = 0)$$

$$\le P(y = 1) + P(X_1 \ge 1) = P(y = 1) + P(X_1 = 1)$$

$$\le P(y \ge 1) + P(X_1 = 1) = P(\sum_{i=2}^{k} X_i \ge 1) +$$

$$P(X_1 = 1) \le \cdots \le \sum_{i=1}^{k} P(X_i = 1) =$$

$$\sum_{i=1}^{k} P(I_i(t) \leq k_i).$$

Thus,

$$0 \leq h(X_1, X_2, \ldots, X_k) \leq \sum_{i=1}^{k} P(I_i(t) \leq k_i). \quad (5)$$

However, for $t \geq \max_{1 \leq j \leq k} \frac{k_j}{\hat{\beta}_j}$ and $0 < \varepsilon < 1$ (see (4))

$$0 \leq \lim_{t\to\infty} P(I_j(t) < k_j) \leq \lim_{t\to\infty} P(I_j(t) <$$

$$(1-\varepsilon) \cdot a(t) \cdot \hat{\sigma}_j + \hat{\beta}_j \cdot t) =$$

$$\lim_{t\to\infty} P\left(\frac{I_j(t) - \hat{\beta}_j \cdot t}{\hat{\sigma}_j \cdot a(t)} < 1 - \varepsilon\right) \leq$$

$$\overline{\lim_{t\to\infty}} P\left(\frac{I_j(t) - \hat{\beta}_j \cdot t}{\hat{\sigma}_j \cdot a(t)} < 1 - \varepsilon\right) =$$

$$P\left(\overline{\lim_{t\to\infty}} \frac{I_j(t) - \hat{\beta}_j \cdot t}{\hat{\sigma}_j \cdot a(t)} < 1 - \varepsilon\right) = 0, \quad (6)$$

$j = 1, 2, \ldots, k.$

Thus (see (6)),

$$\lim_{t\to\infty} P\left(I_j(t) < k_j\right) = 0, \ j = 1, 2, \ldots, k. \quad (7)$$

So, (see (5) and (6)), $h(X_1, X_2, \ldots, X_k) = 0$.

Consequently, we have proved the following theorem on the idle time of computer network.

Theorem 2 $t \geq \max_{1 \leq j \leq k} \frac{k_j}{\hat{\beta}_j}$ *and conditions (1) - (3) are fulfilled, all computer in the network are idle.*

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