On Numerical Solution of Boundary Integral Equations of the Plane Elasticity Theory by Singular Integral Approximation Methods

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Abstract: - A singular integral approximation method is offered for numerical solution of some basic problems of plane theory of elasticity. A question of investigation of the corresponding numerical scheme is considered.

Key-Words: - Theory of Elasticity, Singular Integral, Approximation, Numerical Scheme, Stress, Displacement

1 Introduction

A boundary integral equations method represents one of the effective methods of research and solution of many applied problems and, among them, problems of elasticity theory. In particular, it concerns also to numerical solution of similar problems which in the given case consists in application of one or another approximate methods to solution of corresponding boundary integral equations. In the existing literature Fredholm integral equations are usually meant under such equations.

However, taking into account the character of dependence of kernels of such equations on the boundaries of the considered domains, calculating schemes founded on certain Cauchy type singular integral approximation, seem more effective in the case of arbitrarily given domains. On the base of such approximations, numerical solution of the initial problem leads to linear algebraic system, whose coefficients are sufficiently easily realizable even in the cases, when the boundary is not given by its exact equation (but graphically or as a table, which usually takes place in practice). On the other side, we note that foundation of the schemes of such kind is generally difficult and in different cases, individual research becomes necessary.

2 Problem Formulation

In the present paper, the mentioned question is considered for a concrete numerical solution scheme for some basic problems of plane elasticity theory in the case of finite simply connected domains. Here we will start from the known Sherman-Lauricella boundary integral equation (see [1], [2]), which can be written as follows (for definiteness, the first elasticity problem, according to the accepted in [1] terminology, is considered below):

$$\left.\begin{aligned}
&\omega_{(1)}(t_0)+\frac{1}{\pi}\int_L[\omega_{(1)}(t)(1-\cos 2\vartheta)-\omega_{(2)}(t)\sin 2\vartheta]d\vartheta+\\
&+\operatorname{Re}\frac{1}{2\pi i}\left(\frac{1}{t_0}-\frac{1}{\overline{t}_0}+\frac{t_0}{\overline{t}_0^{2}}\right)\operatorname{Re}\int_L\frac{\omega_{(1)}(t)+i\omega_{(2)}(t)}{t^2}dt=f_1(t_0)\\
&\omega_2(t_0)-\frac{1}{\pi}\int_L\left[\omega_1(t)\sin 2\vartheta-\omega_{(2)}(t)(1+\cos 2\vartheta)\right]d\vartheta+\\
&+\operatorname{Im}\frac{1}{2\pi i}\left(\frac{1}{t_0}-\frac{1}{\overline{t}_0}+\frac{t_0}{\overline{t}_0^{2}}\right)\operatorname{Re}\int_L\frac{\omega_{(1)}(t)+i\omega_{(2)}(t)}{t^2}dt=f_2(t_0).
\end{aligned}\right\} \tag{1}$$

Here L is a boundary of the considered domain, representing a closed contour on a complex variable plane z (everywhere further we will mean that the contour belongs to a Liapunov class and that the origin is inside), $\omega(t)=\omega_{(1)}(t)+i\omega_{(2)}(t)$ is an unknown function and $f(t_0)=f_1(t_0)+if_2(t_0)$ is a given function (defined on L in a known way by a stress vector), $\vartheta=\vartheta(t_0,t)=$ $=\arg(t-t_0)$.

On the base of transformations (see also [3])

$$\begin{aligned}
&\operatorname{Re}\frac{1}{\pi i}\int_L[\omega_{(1)}(t)(t-\cos 2\vartheta)-\omega_{(2)}(t)\sin 2\vartheta]d\vartheta=\omega_{(1)}(t_0)+\\
&\quad+\operatorname{Re}\frac{1}{\pi i}\int_L\left\{\left[\omega_{(1)}(t)-\omega_{(1)}(t_0)\right](1-\cos 2\vartheta)-\right.\\
&\quad\left.-\left[\omega_{(2)}(t)-\omega_{(2)}(t_0)\right]\sin 2\vartheta\right\}\frac{dt}{t-t_0},
\end{aligned} \tag{2}$$

$$\begin{aligned}
&-\operatorname{Re}\frac{1}{\pi i}\int_L\left\{\omega_{(1)}(t)\sin 2\vartheta-\omega_{(2)}(t)(1+\cos 2\vartheta)\right\}d\vartheta=\\
&=\omega_{(2)}(t_0)-\operatorname{Re}\frac{1}{\pi i}\int_L\left\{\left[\omega_{(1)}(t)-\omega_{(1)}(t_0)\right]\sin 2\vartheta-\right.\\
&\quad\left.-\left[\omega_{(2)}(t)-\omega_{(2)}(t_0)\right](1+\cos 2\vartheta)\right\}\frac{dt}{t-t_0}
\end{aligned} \tag{3}$$

we can write the initial system (1) in Cauchy type singular integrals which will be meant further in (1). It is important to us, that the integration in these integrals is done with respect to complex variable t and at this it is not required to consider derivatives of the function $\vartheta(t_0,t)$ (whose calculation complicates the realization of the numerical schemes in the case of arbitrarily given domains). In result, construction of practically convenient numerical schemes for solution of the given equation depends on possibility of construction of effective approximating processes for indicated integrals.

3 Approximation of Integral Equations System (1)

In this section we give an approximate scheme for numerical solution of system (1), based on certain approximation of singular integrals in (2), (3) (and regular integral contained in (1)).

As in [4], we introduce a system of knots $\{\tau_j\}_{j=0}^{n}$ on the contour L partitioning it into arcs $\tau_\sigma\tau_{\sigma+1}$, $0\le\sigma\le n-1$. In the proofs cited here an equal partitioning of the contour is meant (however, according to the remark in [5] basic results remain valid under somewhat more general partitioning). Naturally, under $\tau_\sigma\tau_{\sigma+1}$ we mean the least arc of L with endpoints τ_σ, $\tau_{\sigma+1}$ lying in the positive direction on L. We apply quadrature formulas from [4] to singular integrals in (2), (3) from [4]:

$$\operatorname{Re}\frac{1}{\pi i}\int_L\Big\{\big[\omega_{(1)}(t)-\omega_{(1)}(t_0)\big](1-\cos 2\vartheta)-$$

$$-\big[\omega_{(2)}(t)-\omega_{(2)}(t_0)\big]\sin 2\vartheta\Big\}\frac{dt}{t-t_0}\approx\omega_{(1)}(t_0)+$$

$$+\operatorname{Re}\big(p_{\nu-1}+2p_\nu+p_{\nu+1}\big)\left\{\frac{\omega_{(1)}(\tau_{\nu+1})-\omega_{(1)}(\tau_\nu)}{\tau_{\nu+1}-\tau_\nu}\right[1-$$

$$-\cos 2\vartheta\big(\tau_{\nu+1},t_0\big)\big]-\left.\frac{\omega_{(2)}(\tau_{\nu+1})-\omega_{(2)}(\tau_\nu)}{\tau_{\nu+1}-\tau_\nu}\sin 2\vartheta\big(\tau_{\nu+1},t_0\big)\right\}+$$

$$+\operatorname{Re}\sum_{\sigma=1}^{n-2}\big(p_{\nu+\sigma}+p_{\nu+\sigma+1}\big)\left\{\frac{\omega_{(1)}(\tau_{\nu+\sigma+1})-L_{n\nu}(\omega_{(1)};t_0)}{\tau_{\nu+\sigma+1}-t_0}\right[1-$$

$$-\cos 2\vartheta(\tau_{\nu+\sigma+1},t_0)]-\left.\frac{\omega_{(2)}(\tau_{\nu+\sigma+1})-L_{n\nu}(\omega_{(2)};t_0)}{\tau_{\nu+\sigma+1}-t_0}\sin 2\vartheta(\tau_{\nu+\sigma+1},t_0)\right\},$$

$$t_0\in\tau_\nu\tau_{\nu+1}\ (\nu=1,2,...,n),\ p_j=\frac{\tau_{j+1}-\tau_j}{2\pi i}\ \ (\tau_{n+1}=\tau_1)\quad(4)$$

(similarly, for corresponding integrals in connection of the second equation of the system (1)), where $L_{n\nu}(\omega_{(q)};t_0)$ $(q=1;2)$ represents a linear interpolation Lagrange polynomial for the functions $\omega_{(q)}$ by points τ_ν, $\tau_{\nu+1}$. It is clear that constructed in this way expressions make sense also at the points t_0, coincided with the knots $\{\tau_j\}$ and by that they are defined on the whole contour L.

However it should be noted that at passing of the parameter t_0 over the endpoints τ_ν, $\tau_{\nu+1}$, the corresponding sums (operator-functions) may have finite breaks. Meanwhile from the viewpoint of foundation of numerical schemes, constructed in similar way, continuity of approximating operator-functions on the whole contour is essential for considered integral equations. It can be shown that approximation of singular integrals can be constructed on the basis of simple modification of indicated above approximating sums. Concretely for the sums involved in (4), it may be done applying a linear interpolating operator to the latter with knots on each arc $\tau_\nu\tau_{\nu+1}$ ($\nu=1,2,...,n$; $\tau_{\nu+1}=\tau_1$). We obtain a piece-wise continuous operator-function, defined on the entire L, continuous on it and, moreover, satisfying on it to the Holder condition (with index 1).

At this we note that an accuracy estimate of the quadrature formulas constructed in the indicated above way for the considered singular integrals can be established similarly to that in [4] (which further was used in [5]). In this connection it should be noted that in [5] a Dirichlet problem, which can be reduced to a boundary integral equation of the form

$$\varphi(t_0)+\operatorname{Re}\frac{1}{\pi i}\int_L\frac{\varphi(t)}{t-t_0}dt=f(t_0),$$

is considered (where unlike (1), the singular integral approximation rate depends only on differential properties of the unknown function $\varphi(t)$ and on the chosen quadrature formula, and in some cases it may appear rather high).

In the considered case it must be taken into account that because of presence of the functions $\sin 2\vartheta$, $\cos 2\vartheta$ in the singular integrals, the smoothness of their densities is essentially determined by the smoothness of the contour L. By that, if the vector-function $\omega=\big(\omega_{(1)}\omega_{(2)}\big)$ belongs e.g. to the Holder class $H(\alpha)$ $(0<\alpha\le1)$ on L, then under the accepted above assumption that L belongs to the Lyapunov class, we can assert a convergence of the indicated quadrature formulas with the rate $O(n^{-\alpha}\ln n)$. For sufficiently smooth functions ω a convergence rate $O\big(n^{1+\delta}\ln n\big)$ can be reached, where δ $(0<\delta\le1)$ is determined via the Lyapunov index. More exact estimates for sufficiently smooth functions ω and contours L may be obtained with the help of more accurate quadrature formulas indicated in [4].

As for regular integral in (1), ordinary quadrature formulas of appropriate accuracy may be applied to them. In particular, in the given case we put

$$\int_L\frac{\omega_{(1)}(t)+i\omega_{(2)}(t)}{t^2}dt\approx$$

$$\approx\sum_{\sigma=0}^{n-1}(p_{\nu+\sigma}+p_{\nu+\sigma+1})\frac{\omega_{(1)}(\tau_{\nu+\sigma+1})+i\omega_{(2)}(\tau_{\nu+\sigma+1})}{\tau_{\nu+\sigma+1}^2}$$

where p_σ are mentioned in (4) coefficients. According to known estimation methods we can see that the error of this formula in the class $H(\alpha)$ is $O\left(n^{-\alpha}\right)$. In cases of sufficiently smooth functions ω the convergence rate can be $O\left(n^{-1+\delta}\right)$ (under the accepted above assumptions). For completeness here we remember that for sufficiently smooth integrand functions this formula guarantees the rate of convergence $O\left(n^{-2}\right)$ (even in cases of arbitrarily piece-wise smooth contours L).

4 On Investigation of a Numerical Scheme

In spite of the fact that the singular integral approximation method leads us to easily realizable schemes, as it was noted above, in general case it does not guarantee the solvability of the approximating equations constructed on such base. Below some considerations in this direction are carried out.

Assuming as earlier $0<\beta<\delta$ we denote by H_β a space of vector-functions $(\omega_{(1)},\omega_{(2)})$ satisfying on L to the Holder condition with the index β with usual definition of a norm in this space. By S we denote the operator generated in the given system by integrals with singular kernels $(t-t_0)^{-1}$ and by Q — the operator corresponding to the regular part. Similarly by S_n and Q_n we denote operators approximating S and Q respectively. Using these notations, we represent the initial and approximating systems in the following forms:

$$(K\omega)(t_0)\equiv 2\omega(t_0)+\operatorname{Re}(S\omega)(t_0)+(Q\omega)(t_0)=f(t_0) \quad (5)$$

$$(K_n\omega_n)(t_0)\equiv 2\omega_n(t_0)+\operatorname{Re}(S_n\omega_n)(t_0)+(Q_n\omega_n)(t_0)=f(t_0) \quad (6)$$

where $\omega_n=(\omega_{n(1)},\omega_{n(2)})$ is an unknown approximate solution. A proof of its existence (usually for sufficiently big n) is an essential part of foundation of the scheme. At this the known fact ([1], see also [2]) that equation (5) is uniquely solvable is used. Below, using approaches formulated in [5], we present shortly some fragments of the proof, that starting from certain n the corresponding homogeneous equation (everywhere below we keep the same notation ω_n for its solution)

$$\left(K_n\omega_n\right)(t_0)=0 \quad (7)$$

has only trivial solution.

Proceeding to presentation, firstly we note that mentioned above error estimates of the considered integrals, evidently, are also valid for the solution ω_n of the equation (6). On the base of these estimates we can write

$$\max_{t_0\in L}\left|\left(K\omega_n\right)(t_0)-\left(K_n\omega_n\right)(t_0)\right|=O\left(\frac{\ln n}{n^\beta}\right)\left\|\omega_n\right\|_{H_\beta} \quad (8)$$

Further, for ω_n we consider

$$h_{n\beta}(\omega_n)=\max_q \max_{\substack{1\le j,\nu\le n\\ j\ne\nu}}\frac{\left|\omega_{n(q)}(\tau_j)-\omega_{n(q)}(\tau_\nu)\right|}{\left|\tau_j-\tau_\nu\right|^\beta} \quad (q=1;2).$$

In the corresponding proofs it is essential that for the solution ω_n of the equation (7) the norm $\left\|\omega_n\right\|_{H_\beta}$ can be estimated by $h_{n\beta}(\omega_n)$. Establishment of such relation is based on direct estimations of the sums involved in the expression of the operator-function K_n. At first, starting from equation (7), we will express the values of the solution $\omega_n(t_0)$ at the knots and, by that, the quotients

$$\frac{\omega_{n(q)}(\tau_j)-\omega_{n(q)}(\tau_\nu)}{\left|\tau_j-\tau_\nu\right|^\beta} \quad (q=1;2)$$

by these sums. At this, taking into account the fact that the part $S_n\omega_n(t_0)$ of the sum, related to the approximation of singular integrals, contains the function ω_n only via its difference quotients, it can be estimated directly by $h_{n\beta}(\omega_n)$.

In order to obtain such estimates, we use essentially, an approach from [5], noting at this, that in this case the corresponding sums are partitioned (for $t_0=\tau_\nu,\tau_j$ respectively) into sums, one of which contains differences

$$\frac{\omega_{n(q)}(\tau_{\nu+\sigma+1})-\omega_{n(q)}(\tau_j)}{\tau_{\nu+\sigma+1}-\tau_j}-\frac{\omega_{n(q)}(\tau_{\nu+\sigma+1})-\omega_{n(q)}(\tau_\nu)}{\tau_{\nu+\sigma+1}-\tau_\nu}$$
$$(q=1;2)$$

multiplied by expressions of the forms $1\pm\cos 2\vartheta(\tau_{\nu+\sigma+1},t_0)$, $\sin 2\vartheta(\tau_{\nu+\sigma+1},t_0)$ and the rest, composed from difference quotients

$$\frac{\omega_{n(q)}(\tau_{\nu+\sigma+1})-\omega_{n(q)}(t_0)}{\tau_{\nu+\sigma+1}-t_0} \quad (t_0=\tau_\nu,\tau_j;\, q=1;2),$$

multiplied by differences of forms

$$\cos 2\vartheta(\tau_{\nu+\sigma+1},\tau_j)-\cos 2\vartheta(\tau_{\nu+\sigma+1},\tau_\nu),$$
$$\sin 2\vartheta(\tau_{\nu+\sigma+1},\tau_j)-\sin 2\vartheta(\tau_{\nu+\sigma+1},\tau_\nu).$$

Using $\max_{1\le\mu\le n}\left|p_\mu\right|=O(n^{-1})$ and some simple transformations (see, in particular, [5]) of the corresponding sums, according to the same equation (7) we get

$$\max_{\nu\ne j}\frac{\left|\omega_n(\tau_j)-\omega_n(\tau_\nu)\right|}{\left|\tau_j-\tau_\nu\right|^\beta}=O(\ln n)h_{n\beta}(\omega_n)+O(1)M(\omega_n)$$
$$(\nu, j=1,2,...,n),$$

where $M(\omega_n)=\max_{q=1;2}\max_{t\in L}\left|\omega_{n(q)}(t)\right|$. Using this estimate it may be shown (see [5]) that in the previous estimate the knots can be changed by arbitrary points $t_1,t_2\in L$ $(t_1\ne t_2)$. In result, taking into account the presence of the operator H_n in equation (7), under accepted by us assumptions we come to a certain estimate, which may be written in the following form

$$\left\|\omega_n\right\|_{H_\beta}=O(\ln n)h_{n\beta}(\omega_n)+O(1)M(\omega_n). \quad (9)$$

The value $M(\omega_n)$ may be also estimated by $h_{n\beta}(\omega_n)$. For this, first of all in (8) we will use estimate (9). We can write

$$\begin{aligned}&\left(K\omega_n\right)(t_0)-\left(K_n\omega_n\right)(t_0)=\\&=O\left(\frac{\ln^2 n}{n^\beta}\right)h_{n\beta}(\omega_n)+O\left(\frac{\ln n}{n^\beta}\right)M(\omega_n)\end{aligned} \quad (10)$$

Using the known fact (see [1]), that under the mentioned above conditions on L, the operator K is continuously reversible in the space C (it is reversible in the space H_β too), and on the base of equation (7) and relation (10) it is possible to obtain the estimate

$$M(\omega_n) = O\left(\frac{\ln^2 n}{n^\beta}\right) h_{n\beta}(\omega_n) . \qquad (11)$$

From estimates (9), (11) it follows that the mentioned above statement about solution ω_n of equation (7) will be true if we prove that all $h_{n\beta}(\omega_n) = 0$ for sufficiently big n. The proof of this statement (rather cumbersome) consists in some transformations and estimations of expressions, involved in equation (7), using indicated above statements. Stopping shortly on presentation of basic approaches in this direction, everywhere below under $\{j_n\} \uparrow \infty$, $n^{-1} j_n \to 0$ $(n \to \infty)$ we will mean a sequence of natural numbers, more exact choose of which will be given later.

Firstly we note that via (11) and assumption on the contour L, we can write

$$\max_{|j-\nu|>j_n} \frac{\left|\omega_n(\tau_j) - \omega_n(\tau_\nu)\right|}{\left|\tau_j - \tau_\nu\right|^\beta} = O\left(n^\beta \big/ j_n^\beta \right) M(\omega_n) =$$

$$= O\left(\frac{\ln^2 n}{j_n^\beta}\right) h_{n\beta}(\omega_n) . \qquad (12)$$

In this connection now we will consider in detail cases of such ν, j, that $|j - \nu| \le j_n$. Without restriction of generality we put $j = \nu + \lambda$ for $\lambda = 1, 2, ..., j_n$.

From equation (7) we have

$$\omega_n(\tau_{\nu+\lambda}) - \omega_n(\tau_\nu) =$$

$$= -\frac{1}{2}\Big\{ \mathrm{Re}\left[(S_n\omega_n)(\tau_{\nu+\lambda}) - (S_n\omega_n)(\tau_\nu) \right] +$$

$$+ (Q_n\omega_n)(\tau_{\nu+\lambda}) - (Q_n\omega_n)(\tau_\nu) \Big\} = 0 . \qquad (13)$$

The main part of the following considerations for obtaining needed estimates of sums involved in (13), practically is reduced to consideration of the expressions of the form

$$\frac{\left| \mathrm{Re}\left[(S_n\omega_n)(\tau_{\nu+\lambda}) - (S_n\omega_n)(\tau_\nu) \right] \right|}{\left|\tau_{\nu+\lambda} - \tau_\nu\right|^\beta} \qquad (14)$$

(at $\lambda = 1, 2, ..., j_n$). Taking into account $\tau_{\nu+n-\mu} = \tau_{\nu-\mu}$ $(0 \le \mu \le n)$, it appears more convenient to use the knots $\{\tau_j\}_{j=1}^n$ in the form $\{\tau_{\nu\pm\sigma}\}_{\sigma=1}^{\sigma_n}$, where $\sigma_n = \left[n\big/2 \right]$ for even n and $\sigma_n = \left[n\big/2 \right] - 1$ for odd n.

We partition each of the sums $(S_n\omega_n)(\tau_\nu)$, $(S_n\omega_n)(\tau_{\nu+\lambda})$ from (14) into two sums: $\Sigma_{\nu 1}$, $\Sigma_{\nu 2}$ and $\Sigma_{\nu+\lambda 1}$, $\Sigma_{\nu+\lambda 2}$, respectively, where $\Sigma_{\nu 1}$, and $\Sigma_{\nu+\lambda 1}$ contain terms, corresponding to indices $\nu \pm \sigma$ $(1 \le \sigma \le 2 j_n)$ and $\Sigma_{\nu 2}$, $\Sigma_{\nu+\lambda 2}$ contain the rest. Concerning the estimation of the sum $\Sigma_{\nu+\lambda 1}$ - $\Sigma_{\nu 1}$ we note that for $1 \le \sigma \le 2 j_n$ an asymptotic representation, similar to that indicated in [5], is true (recalling that the contour L belongs to the Lyapunov class).

$$\frac{p_{\nu\pm\sigma-1} - p_{\nu\pm\sigma}}{\tau_{\nu\pm\sigma} - \tau_{\nu+\mu}} = \frac{1}{\pi i(\pm\sigma - \mu)}\left[1 + O\left(j_n^\delta n^{-\delta} \right) \right] . \qquad (15)$$

$$(\mu = 0, 1, ..., \lambda)$$

Using this presentation in the sums and noting that the main parts of the right hand sides in formulas (15) are imaginary values (and $\omega_{n(1)}$, $\omega_{n(2)}$ are real functions), we have (similarly to [5]) that

$$\mathrm{Re}\left(\Sigma_{\nu+\lambda 1} - \Sigma_{\nu 1} \right) = O\left(j_n^{\delta+\beta} n^{-(\delta+\beta)} \right) h_{n\beta}(\omega_n) .$$

As for the estimation of $\Sigma_{\nu+\lambda 2} - \Sigma_{\nu 2}$, it essentially settles on the same principles which were used in [5] while estimating analogous sums. However, in the given case this brings to significantly more cumbersome transformations, conditioned by presence in the considered sums of expressions, containing values of the functions $\sin 2\vartheta(t,t_0)$, $\cos 2\vartheta(t,t_0)$. Execution of the corresponding considerations along with the previous results and equality (13), leads us to the following asymptotic relations

$$\frac{1}{2\left|\tau_{\nu+\lambda} - \tau_\nu\right|^\beta} \Big\{ \left[\omega_{n(1)}(\tau_{\nu+\lambda}) - \omega_{n(1)}(\tau_\nu) \right](1 + \varepsilon_{n1}^{(1)}) +$$

$$+ \left[\omega_{n(2)}(\tau_{\nu+\lambda}) - \omega_{n(2)}(\tau_\nu) \right] \varepsilon_{n2}^{(1)} \Big\} +$$

$$+ \left[O\left(n^{-\delta} j_n^{\delta+\beta} \right) + O\left(j_n^{-\beta} \ln^2 n \right) \right] h_{n\beta}(\omega_n) = 0 \quad (n \to \infty) ,$$

$$\frac{1}{2\left|\tau_{\nu+\lambda} - \tau_\nu\right|^\beta} \Big\{ \left[\omega_{n(2)}(\tau_{\nu+\lambda}) - \omega_{n(2)}(\tau_\nu) \right](1 + \varepsilon_{n2}^{(2)}) +$$

$$+ \left[\omega_{n(1)}(\tau_{\nu+\lambda}) - \omega_{n(1)}(\tau_\nu) \right] \varepsilon_{n1}^{(2)} \Big\} +$$

$$+ \left[O\left(n^{-\delta} j_n^{\delta+\beta} \right) + O\left(j_n^{-\beta} \ln^2 n \right) \right] h_{n\beta}(\omega_n) = 0 \quad (n \to \infty) .$$

Imposing on the sequence $\{j_n\}$ additional requests (taking into account $\delta > \beta$)

$$\left\{ j_n^{-\beta} \ln^2 n \right\} \to 0 , \ \left\{ n^{-\delta} j_n^{\delta+\beta} \right\} \to 0 \ \ (n \to \infty)$$

and recalling (12), we can see that the relation $h_{n\beta}(\omega_n) = 0$ it valid for sufficiently big n.

From the proved above (as in [5]) it follows that the operator K_n is continuously reversible and the norms $\left\| K_n^{-1} \right\|_{H_\beta}$ increase at $(n \to \infty)$ not faster that $\ln n$. From this the estimate and convergence of the scheme follow.

As is known, the main goal of solution of problems of the elasticity theory is definition of values of stress and displacement at points of the domain D. In the given case, after having found the solution $\omega_n(t)$ $(t \in L)$ to the equation (6), approximate meanings of the mentioned values can be determined by the function $\omega_n(t)$ applying certain formulas known in the literature as Kolosov-Muskhelishvili complex potentials. This finally is connected with calculation of the Cauchy type integrals (see [1]) with kernel $(t - z)^{-1}$. Densities of such integrals (potentials) depend in known way on solution of corresponding boundary integral equations (in the given case, under the approximate solution, the function $\omega_n(t)$ is meant). According to this and [1] the corresponding potentials may be represented as

$$\frac{1}{2\pi i}\int_L \frac{\omega_n(t)}{t-z}dt\,,\ \frac{1}{2\pi i}\int_L \frac{\bar{\omega}_n(t)}{t-z}dt - \frac{1}{2\pi i}\int_L \frac{\bar{t}\,\omega'_n(t)}{t-z}dt\,.$$

Calculation of these integrals and their derivatives is required.

Omitting here representation of the mentioned potentials by $\omega_n(t)$ in detail (which in the given case is not principal), for generality we note that in general case the question is reduced to approximate calculation of integrals of type (for the sake of simplicity we denote the density in all integrals by $\varphi(t)$)

$$\frac{1}{2\pi i}\int_L \frac{\varphi(t)}{t-z}dt\,,\ \frac{1}{2\pi i}\int_L \frac{\varphi(t)}{(t-z)^2}dt \quad (z\in D)\,.$$

Since z represents an interior point of the domain D, ordinary quadrature formulas can be applied to approximate calculation of the indicated integrals. Such formulas give satisfactory accuracy for the values z, comparatively not close to the boundary L. But the accuracy of these formulas decreases at tending z to L. In this connection in the paper [6] some special quadrature formulas are considered for approximate calculation of Cauchy type integrals. They have rather more complicated structure, than ordinary quadrature formulas, though, free of indicated disadvantage. For definiteness here we will give in unfolded form one such formula from those for approximate calculation of Cauchy integrals of the form $\frac{1}{2\pi i}\int_L \frac{\varphi(t)}{t-z}dt$:

$$\begin{aligned}
&\frac{1}{2\pi i}\int_L \frac{\varphi(t)}{t-z}dt \approx L_\mu(\varphi;\tau)+\frac{1}{2}\Big\{p_{\mu-1}+2p_\mu+p_{\mu+1}+\\
&+\frac{1}{\pi i}(z-\tau)\left[\frac{z-\tau_{\mu-1}}{\tau_\mu-\tau_{\mu-1}}\ln\frac{\tau_\mu-z}{\tau_{\mu-1}-z}+\ln\frac{\tau_{\mu+1}-z}{\tau_\mu-z}+\right.\\
&\left.+\frac{z-\tau_{\mu+2}}{\tau_{\mu+1}-\tau_{\mu+2}}\ln\frac{\tau_{\mu+2}-z}{\tau_{\mu+1}-z}\right]\Big\}\frac{\varphi(\tau_{\mu+1})-\varphi(\tau_\mu)}{\tau_{\mu+1}-\tau_\mu}+\\
&+\frac{1}{2}\sum_{\sigma=1}^{n-2}\Big\{p_{\mu+\sigma}+p_{\mu+\sigma+1}+\frac{1}{\pi i}(z-\tau)\times\\
&\times\left[\frac{z-\tau_{\mu+\sigma}}{\tau_{\mu+\sigma+1}-\tau_{\mu+\sigma}}\ln\frac{\tau_{\mu+\sigma+1}-z}{\tau_{\mu+\sigma}-z}+\frac{z-\tau_{\mu+\sigma+2}}{\tau_{\mu+\sigma+1}-\tau_{\mu+\sigma+2}}\times\right.\\
&\left.\times\ln\frac{\tau_{\mu+\sigma+2}-z}{\tau_{\mu+\sigma+1}-z}\right]\Big\}\frac{\varphi(\tau_{\mu+\sigma+1})-L_\mu(\varphi;\tau)}{\tau_{\mu+\sigma+1}-\tau}\,,\ \tau\in L\,.\\
&(\tau\in\tau_\mu\tau_{\mu+1}\,,\ \mu=0,1,...,n-1) \qquad (16)
\end{aligned}$$

(with properly chosen branch of the logarithmic function) where p_j are the above indicated coefficients, $L_\mu(\varphi;\tau)$ is a Lagrange piece-wise linear interpolation polynomial constructed on the arcs $\tau_\mu\tau_{\mu+1}$, using the knots τ_μ, $\tau_{\mu+1}$ $(\mu=0,1,...,n-1)$. At this under the variable parameter τ one of nearest to z on the contour L is meant while tending z to L.

A quadrature formula similar to (16) for Cauchy type integral with a kernel $(t-z)^{-2}$ is also constructed in [6] on the same principles.

We recall that for the sake of definiteness the first basic problem of elasticity theory was considered here, though the corresponding scheme can be applied to the second basic problem too. Formally such scheme can be used also for a mixed problem of elasticity theory, though in the given case the foundation of the scheme turns out to be difficult (due to properties of the right hand side of the corresponding integral equation). Note that everywhere above finite (simply connected) domains were meant, though corresponding considerations for such infinite domains do not differ essentially from the previous ones. As for multiply connected domains, difficulties appeared in this case mainly have technical character.

5 Conclusion

From the above said it is clear that possibility of foundation of such schemes depends essentially on individual properties of operators and approaches to their investigation. It is known, that unlike Fredholm integral equations, in approximate schemes based on approximation of singular integrals, the closeness of the initial and approximating operators is not always a reason to assert even solvability of approximating equations while arbitrarily increasing the number of partitions of the corresponding contours. It was already mentioned partially in the beginning.

Note that a bit different scheme for numerical solution of the first and second basic problems (and some boundary problems) was developed earlier by one of the authors (with co-authors), and a package of applied programs ([7]) was made on its base. In the mentioned work the corresponding boundary integral equation was considered in the form (equivalent to that considered here):

$$\begin{aligned}
&\omega(t_0)+\frac{1}{2\pi i}\int_L \frac{\varphi(t)}{t-t_0}dt+\frac{1}{2\pi i}\int_L \frac{\bar{\omega}(t)}{t-t_0}d\bar{t}+\\
&+\frac{1}{2\pi i}\int_L \frac{h(t_0,t)}{t-t_0}\omega(t)dt+\frac{1}{\pi i}\left(\frac{1}{t_0}-\frac{1}{\bar{t}_0}-\frac{\bar{t}_0}{t_0^{\,2}}\right)\mathrm{Re}\int_L \frac{\varphi(t)}{t^2}dt=\\
&=f(t_0) \qquad (17)
\end{aligned}$$

(the first basic problem), where

$$h(t_0,t)=\frac{d\bar{t}}{dt}-\frac{\bar{t}-\bar{t}_0}{t-t_0}\quad (t\neq t_0)\,,\ \ h(t_0,t_0)\equiv 0\,.$$

In the scheme from [7] for approximate solution to equation (17) certain quadrature formulas (similar to indicated in [4]) were used for integrals with kernels $(t-t_0)^{-1}$ (and ordinary quadrature formulas for corresponding regular integrals). At this, approximate values of $h(t_0,t)$ were found with the help of numerical differentiation formulas, selected in proper way. Constructed in the indicated way approximate scheme was used in practical calculations. However, in spite of rather good approximation of equation (17) by this scheme, foundation of the corresponding numerical process did not appear justifiable.

We also note that in the given case the calculation of complex potentials (and thus, calculation of stress and displacement components) was done on the base of approximation of their densities (and their derivatives) by spline type expressions. Constructed on such basis formulas ensure rather good approximation at the points located more or less far from the boundary of the domain. However, unlike the formulas from [6], they stop being effective near the boundary.

We note also, that having in view practical purpose of the elaboration mentioned in [7], more accurate quadrature formulas were used for singular integrals (and similarly for regular integrals). Such formulas may be used without trouble in approaches offered in this paper. The indicated above approximation formulas were used here only from the viewpoint of simplicity of the corresponding considerations.

In this connection it may be shown that applying certain transformations, the accuracy of such (offered here) schemes may be increased. In order to make it clear we remember that approximation of singular (and regular) integrals was based on approximation of the values involved in the products of the unknown functions $\omega_{(1)}(t)$, $\omega_{(2)}(t)$ and $\sin 2\vartheta(t_0,t)$, $\cos 2\vartheta(t_0,t)$ (and also in the product of $\omega(t)$ and t^{-2} in regular integral). It is clear that, in general, differential properties of these products essentially depend on the same properties of the contour L. By that the accuracy rate of such approximation is substantively determined by these properties (note that in the scheme, mentioned in [7], the similar circumstance is connected with approximation of the integral

$$\int_L \left(\frac{d\overline{t}}{dt} - \frac{\overline{t}-t_0}{t-t_0} \right) \frac{\varphi(t)}{t-t_0} dt ,$$

involved in the corresponding equation of both the first and the second basic problems).

Below we will stop shortly on a possibility of modification of the considered here scheme, where the mentioned circumstance is someway accounted. This consists in simple modification of the approximate formulas for singular and regular integrals used here. In particular, concerning the singular integrals in [7] this means an approximation only of initial functions $\omega_{(1)}(t)$, $\omega_{(2)}(t)$ by method, indicated in [4] (considering by that the functions $\sin 2\vartheta(t_0,t)$, $\cos 2\vartheta(t_0,t)$ as weights). Besides we will approximate the regular integral also by approximation of the same unknown function (with corresponding weights t^{-2}). In result we come to approximate equations system, which differs from the previous one in corresponding coefficients (subject to further calculations). Namely, in the coefficients of the part, corresponding to singular integrals, expressions of type

$$p_{\sigma k}^{(1)} = \frac{1}{\pi i} \int_{\tau_\sigma \tau_{\sigma+1}} l_{\sigma k}(t) \sin 2\vartheta(t,t_0) dt ,$$

$$p_{\sigma k}^{(2)} = \frac{1}{\pi i} \int_{\tau_\sigma \tau_{\sigma+1}} l_{\sigma k}(t) \cos 2\vartheta(t,t_0) dt$$

will appear, where, as earlier, $l_{\sigma k}(t)$ $(k=1;2)$ denote the coefficients of the Lagrange piece-wise interpolation polynomial (constructed on the arcs $\tau_\sigma \tau_{\sigma+1}$). For approximate calculation of such integrals each arc $\tau_\sigma \tau_{\sigma+1}$ is partitioned by points t_1, t_2,..., t_N $(N > n)$ into equal arcs $t_\mu t_{\mu+1} \in \tau_\sigma \tau_{\sigma+1}$ $(\mu = 1,2,...,N-1)$ for considered values of σ. Approximating the functions $\sin 2\vartheta(t_0,t)$, $\cos 2\vartheta(t_0,t)$ in these integrals by the Lagrange piece-wise interpolation polynomial, constructed on the arcs $t_\mu t_{\mu+1}$, the approximate calculation of $p_{\sigma k}^{(1)}$, $p_{\sigma k}^{(2)}$ is reduced to calculation of integrals (on arcs $t_\mu t_{\mu+1}$) from polynomials, representing products of coefficients, constructed by two considered knots systems. We can do the similar considerations for regular integrals.

It is clear that if N is sufficiently big (much bigger than n), then the error of final approximation of system (1) is determined mainly by error of approximation of the unknown function $\omega(t)$. According to that and said above, in some (possible) cases the accuracy of such transformed schemes may be higher.

Finally, we note that generally, application of the singular integral approximation method appears rather effective to some other problems of complex variables functions theory too, namely, to problems of conformal mapping of domains. In this connection, we can note e.g. [8], [9].

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