On Large Deformations of Elastic Half Rings

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Abstract: The physical and mathematical models of free-clamped elastic half ring, used as gripper and spring element, with an acting force at the free end are constructed, based on the balance conditions of forces and Euler's constitutive law for the bending moment. By turning the model into a pendulum equation, the large deformation of the half ring, especially the multiplicity of the solutions of the corresponding model is discussed. With the aid of manifold method, the approximate critical values for bifurcation and loss of stability are found, the configurations of the deformed elastic half ring for different parameter values are obtained.

Key-words: large deformation, elastic half ring, bifurcation, loss of stability

1 Introduction

Study on deformation of elastic rod has been an active part in elastic theory, because of its wide use in practical equipment and applied technologies, for instance gripper structure in micro machinery and spring elements in the moving system of micro robots. The existing results on elastic deformations and motion caused by them are mainly about small deformations, owning to the nonlinearity of large deformation and unusual phenomenon accompanied with it, such as bifurcation and loss of stability. This paper will be concentrated on the large deformations of elastic half ring used frequently as gripper and spring element. By constructing the physical and mathematical models, the multiplicity and bifurcation as well as loss of stability brought about by the large deformation are studied with manifold method and numerical treatment. This paper gives a special attention to the critical value for bifurcation and loss of stability because of its theoretical and practical significance, the configurations of deformed elastic half ring are also discussed numerically.

2 Physical and Mathematical Model

For simplicity we consider a gripper consisting of two elastic clamped-free half rings whose physical model is shown in Fig. 1a, where R_1 and R_2 are elastic rods being half ring in undeformed form, clamped at 0. The principle of the gripper is: the gripper is opened by the forces $\vec{F}_l$ and $\vec{F}_r$ acting at the free ends, when the forces are moved, a body of weight G is hold by the elastic force. Owing to the symmetry of Fig. 1a, only the part shown in Fig. 1b is to be discussed. In [1, 2] some methods of handling small deformations of elastic rings and straight rods were given.

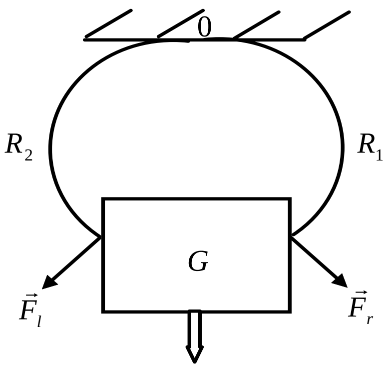

Fig.1a Elastic gripper system

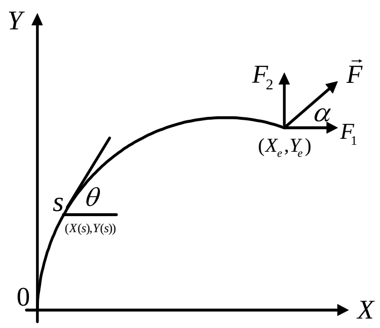

Fig.1b Simplified form of Fig.1a

Without loss of generality, the elastic half ring in Fig.1b is assumed to be non-extensible and undergoes only pure bending, the configuration of deformed elastic half ring can then be parameterized by its arc length $s \in [0, L]$ in the form of $(X(s), Y(s))$ with $s = 0$ at the clamped end, $s = L$ at the free end. Based on the curve geometry, balance conditions of the forces and Euler's constitutive law for the bending moment [3], $(X(s), Y(s))$ is described by the following system:

$$\begin{aligned}
&\frac{dX}{ds} = \cos\theta(s), \frac{dY}{ds} = \sin\theta(s), \\
&\frac{d\theta}{ds} = \frac{1}{EI}\{F_2[X_e - X(s)] + F_1[Y(s) - Y_e]\} + \kappa, \\
&0 \le s \le L, \\
&X(0) = Y(0) = 0, \theta(0) = \frac{\pi}{2},
\end{aligned} \tag{1}$$

where I is the moment of inertia and E is the modulus of elasticity of the material, $\kappa = \pi/L$ is the curvature of the undeformed elastic half ring, we assume I, E are constants in Eq.(1). $|\vec{F}| = F$, other symbols appeared in Eq.(1) are as shown in Fig. 1b.

3 Method and Result

After making the following non-dimensional transformation

$$X = Lx, \ \ Y = Ly, \frac{L^2 F}{EI} = f, s = L\sigma = L/\sqrt{f}\, t,$$
$$\theta(s) = \vartheta(\sigma) = \psi(t) + \alpha - \pi$$

and differentiating the third equation in Eq.(1) with respect to t give the following pendulum equation

$$\begin{aligned}
&\psi'' = -\sin\psi, \ t \in [0, \sqrt{f}\,] \\
&\psi(0) = \frac{3\pi}{2} - \alpha, \quad \psi'(\sqrt{f}\,) = -\pi/\sqrt{f}\,.
\end{aligned} \tag{2}$$

At the same time system (1) is shown up as

$$\frac{dX}{d\sigma} = \cos\vartheta(\sigma), \frac{dY}{d\sigma} = \sin\vartheta(\sigma),$$
$$\frac{d\vartheta}{d\sigma} = f(y(\sigma)\cos\alpha - x(\sigma)\sin\alpha) + \omega_0\sqrt{f}, \quad (3)$$
$$0 \le \sigma \le 1,$$
$$x(0) = 0, y(0) = 0, \vartheta(0) = \frac{\pi}{2},$$

where $\omega_0 = \psi'(0)$ and

$\omega_0\sqrt{f} = L\kappa + p[x_e \sin\alpha - y_e \cos\alpha]$.

Obviously, for solution of (3), the value of ω_0 in (3) should be found first. Using the manifold method introduced in [4], instead of direct discussing Eq.(3), the corresponding initial value problem of Eq.(2)

$$\psi'' = -\sin\psi, \psi(0) = 3\pi/2 - \alpha, \psi'(0) = \omega_0 \quad (4)$$

is treated. It is clear that if a solution $\psi(t,\alpha,\omega_0)$ of Eq.(4) is to be a solution of Eq.(2), the following equation

$$\psi'(\sqrt{f}, \alpha, \omega_0) = -\pi/\sqrt{f} \quad (5)$$

should be satisfied. We handle the problem in the following way: solving Eq.(4) to get its solutions for all possible ω_0, from these solutions choosing the ones for which Eq.(5) is valid, then the values of ω_0 related to those solutions are the values needed for solving Eq.(3). After the values of ω_0 being obtained and substituted into Eq.(3), Eq.(3) becomes an initial value problem for two order ordinary differential equation.

Using some mathematical software, for instance Mathematica, with the following program，we can get the schematic expression of relation between f and ω_0 for some selected α.

```
Remove["Global`*"]; Off[General::spell1];
Off[FindRoot::frsec]; Off[FindRoot::frmp];
ende[p_, om_, alpha_] := Module[{x, y},
Evaluate[{x[N[Sqrt[p]]], y[N[Sqrt[p]]]}] /.
    NDSolve[{Derivative[1][x][t] == y[t],
    Derivative[1][y][t] == -Sin[x[t]], x[0] ==
    1.5*Pi-alpha, y[0] == om}, {x, y},
  {t, 0, N[Sqrt[p]]}]];
Alpha=the given value;
ContourPlot[ende[p, om, alpha][[1,2]]
   +Pi/Sqrt[p] , {p, 0.1, 50}, {om, -2, 2},
   ContourShading -> None, Contours ->{0,0},
  PlotLabel->ToString[alpha]]
```

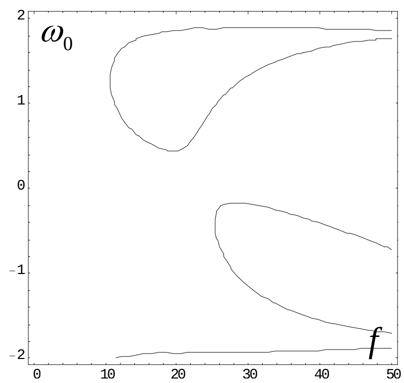

Fig.2a Relation f *vs* ω_0 for $\alpha = -0.98$

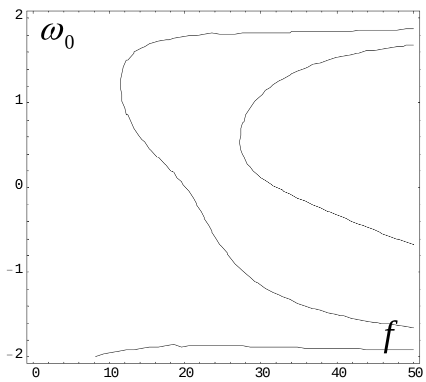

Fig.2b Relation f *vs* ω_0 for $\alpha = -0.79$

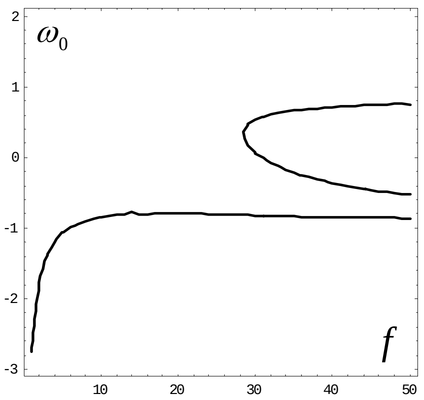

Fig.2c Relation f *vs* ω_0 for $\alpha = 0.25\pi$

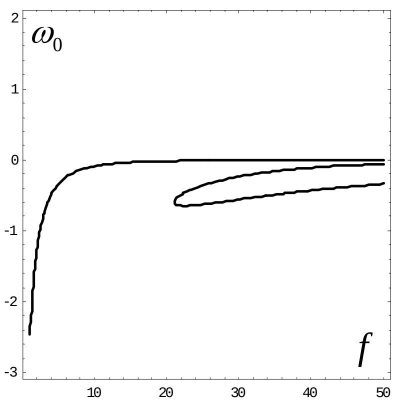

Fig.2d Relation f *vs* ω_0 for $\alpha = 0.5\pi$

The above figures show clearly that for each fixed α, the relation between f and ω_0 is not always one-to-one. For enough large given f, one f will correspond to more than one ω_0, and the greater f is, the more ω_0 corresponds. That means for these values of f, Eq.(3) has more than one solutions. In these cases the uniqueness conditions for Eq.(3) are broken, the stability of the system is therefore lost.

From the above figures we can also see that for smaller f, the relation between f and ω_0 is exactly one-to-one. But for each α, there are some values of f for which the number of deformed configurations will be changed, these values are called bifurcation value or critical value. The approximate critical values for $\alpha = 0.25\pi, 0.5\pi$ are $f = 28.3$ and 20.8 separately, at these values the number of deformed configurations changes from 1 to 3. From Fig. 2a and Fig.2b we see that between $\alpha = -0.98$ and -0.79 there will be a ω_0 for which the f *vs* ω_0 figure has a pitch-fork bifurcation point.

For numerical discussion of the deformed elastic half ring, we use Eq. 4 ,5 and Matehmatica to get the values of ω_0 first.

```
Remove["Global`*"]
Off[General::spell1]
Off[FindRoot::frsec]
Off[FindRoot::frmp]
al = 0.5*Pi; gamma=0.5*Pi;
ende[om_, p_] := Module[{x, y},
Evaluate[{x[N[Sqrt[p]]], y[N[Sqrt[p]]]}] /.
   NDSolve[{Derivative[1][x][t] == y[t],
   Derivative[1][y][t] == -Sin[x[t]], x[0]
   ==Pi+gamma-al, y[0] == om},
    {x, y}, {t, 0, N[Sqrt[p]]}]];
tab1 = Table[{om, ende[om, p][[1,2]]
+2*gamma/Sqrt[p]}, {om, -3, 2, 0.001}];
g1 = Interpolation[tab1];
g1[om];
Table[{p,FindRoot[g1[om]==0,{om,-1,-0.5]
[[1,2]]}]
Table[{p,FindRoot[g1[om]==0,{om,-0.5,-0.3}][
[1,2]]}]
Table[{p,FindRoot[g1[om]==0,{om,-0.2,
0}][[1,2]]}].
```

Then substitute the values of ω_0 into Eq.(3) to obtain the corresponding numerical solutions of it.

```
Remove["Global`*"]; Off[General::spell1]
Off[FindRoot::frsec]; Off[FindRoot::frmp]
gamma=0.5*Pi;
al = 0.5*Pi; om = om_0;
NDSolve[{Derivative[1][u][t] == v[t],
   Derivative[1][v][t] == -Sin[u[t]],
   u[0] == Pi+gamma-al, v[0] == om}, {u[t],
   v[t]}, {t, 0, Sqrt[p]}]
```

```
ParametricPlot[{u[t], v[t]} /. %, {t, 0, Sqrt[p]},
PlotLabel -> ToString[p],
 Axes -> True, Frame -> True]
NDSolve[{Derivative[1][x][t] == Cos[th[t]],
  Derivative[1][y][t] == Sin[th[t]],
  Derivative[1][th][t] == p*(Cos[al]*y[t]
  -Sin[al]*x[t]) +om*Sqrt[p],
  x[0] == 0, y[0] == 0, th[0] == gamma}, {x[t],
  y[t], th[t]}, {t, 0, 1}];
  ParametricPlot[{x[t], y[t]} /. %, {t, 0, 1},
  PlotLabel -> ToString[p], Frame -> True,
 Axes -> None]
```

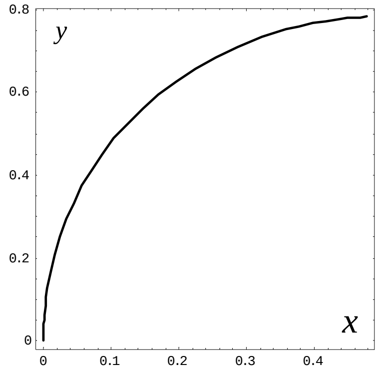

Fig.3a Configuration of deformed elastic half ring for $f = 5$

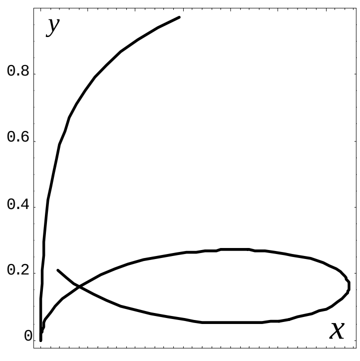

Fig.3b Configuration of deformed elastic half ring for $f = 20.9855$

Take $\alpha = 0.5\pi$ for example. Fig.3a shows the unique and stable deformed elastic half ring for $f = 5$, Fig.3b shows the two configurations for $f = 20.9855$ and Fig.3c shows the three deformed configurations for $f = 22$. The system (3) for $f = 20.9855$ and $f = 22$ is unstable with respect to f.

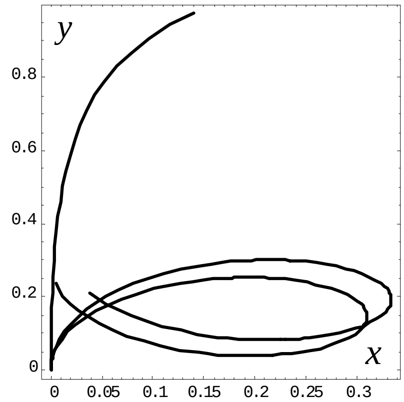

Fig.3c Configuration of deformed elastic half ring for $f = 22$

4 Acknowledgement

This project was supported by the Natural Science Foundation of Beijing Municipal Government (No.1042007) and the Scientific Research Foundation for Returned Overseas Chinese Scholars, State Education Ministry (No. Lxkyjj200416)

References

[1] J. Steigenberger, K. Zimmermann and U. Schulte On large deformations of elastic rings via phase-plane discussion, Archive of Applied Mechanics, No.70, 2000, pp. 489-507.

[2] C.Y. Wang, Asymptotic formula for the flexible bar, Mech. Mach. Theory, 34 ,1999, pp. 645-655

[3] S.P. Timoshenko, J.M. Gere, *Mechanics of Materials*, Nostrand Reinhold Company, 1972.

[4] X. J. Yin, X.E Wang, On circular elastic rods as nonlinear springs and gripper elements, Internat. J. Engrg. Sci. Vol.40, No.3, 2002, pp. 231-238