Proceedings of the 5th WSEAS International Conference on Applications of Electrical Engineering, Prague, Czech Republic, March 12-14, 2006 (pp219-224)

Almost Periodic Solutions of Non-Autonomous Beverton-Holt Difference Equations

David Cheban[1], Cristiana Mammana[2]

[1] State University of Moldova
A. Mateevich Street 60, MD–2009 Chişinău, Moldova

[2] University of Macerata
str. Crescimbeni 20, I–62100 Macerata, Italy

Abstract: The article is devoted to the study of almost periodic solutions of difference Beverton-Holt equation. We prove that such equation admits an invariant continuous section (an invariant manifold). Then, we obtain the conditions for the existence of an almost periodic solution. We study this problem in the framework of non-autonomous dynamical systems (cocycles). The main tool in the study of almost periodic solutions in our work are the continuous invariant sections (selectors) of cocyle.).

Key words: Non-autonomous dynamical systems with discrete time, skew–product flow, almost periodic solutions, Beverton-Holt difference equation.

AMS subject classification: primary 34C35, 34D20, 34D40, 34D45, 58F10, 58F12, 58F39; secondary 35B35, 35B40.

1 Introduction

In the qualitative theory of differential and difference equations non-local problems play the important role. It refers to questions of boundedness, periodicity, almost periodicity, asymptotic behaviour, dissipativity etc. The present work belongs to this direction and is dedicated to the study of almost periodic solutions of non-autonomous Beverton-Holt difference equations. Almost periodic solutions of difference equations arise in numerouse theories, from Dynamical Systems [4, 10], Dynamical Economics [8], Chaos [1], Physics [14] and their references.

Below we will give a new approach concerning the study of almost periodic difference equations. We study the problem of almost periodicity in the framework of non-autonomous dynamical systems (cocyles) with discrete time. The main tool in the study of almost periodic solutions in our work are the continuous invariant sections (selectors) of cocyle.

This paper is organized as follows.

In Section 2 we give some notions and facts from the theory of non-autonomous dynamical systems (cocycles). In particularly, we present the important for our work notion of continuous section of non-autonomous dynamical systems.

Section 3 is dedicated to notion of almost periodic motion of dynamical systems. This section contains a very important construction (see example 3.8) of non-autonomous dynamical system generated by non-autonomous difference equation.

In section 4 we present the main result of our paper (Theorem 4.7) whiche give the sufficient conditions of existence at least one almost periodic solution of non-autonomous Beverton-Holt difference equation.

2 Continuous Invariant Sections of Non-Autonomous Dynamical Systems

Let S be a group of real (R) or integer (Z) numbers, T ($S_+ \subseteq T$) be a semigroup of the additive group S.

Definition 2.1 *Let (X, h, Y) be a bundle fiber [3, 9]. The mapping $\gamma : Y \to X$ is called a section (selector) of the bundle fiber (X, h, Y), if $h(\gamma(y)) = y$ for all $y \in Y$.*

Remark 2.2 *Let $X := W \times Y$. Then $\gamma : Y \to X$ is a section of the bundle fiber (X, h, Y) ($h := pr_2 : X \to Y$), if and only if $\gamma = (\psi, Id_Y)$ where $\psi : W \to W$.*

Definition 2.3 *Let (X, T_1, π) and (Y, T_2, σ) ($S_+ \subseteq T_1 \subseteq T_2 \subseteq S$) be two dynamical systems. The*

Proceedings of the 5th WSEAS International Conference on Applications of Electrical Engineering, Prague, Czech Republic, March 12-14, 2006 (pp219-224)

mapping $h : X \to Y$ is called a homomorphism (respectively isomorphism) of the dynamical system (X, T_1, π) on (Y, T_2, σ), if the mapping h is continuous (respectively homeomorphic) and $h(\pi(x,t)) = \sigma(h(x), t)$ ($t \in T_1,\ x \in X$).

Definition 2.4 *A triplet $\langle (X, T_1, \pi), (Y, T_2, \sigma),\ h\rangle$, where h is a homomorphism of (X, T_1, π) on (Y, T_2, σ) and (X, h, Y) is a bundle fiber [3, 9], is called a non-autonomous dynamical system.*

Let W, Y be two metric spaces and (Y, T_2, σ) be a semi-group dynamical system on Y.

Definition 2.5 *Recall [13] that a triplet $\langle W, \varphi, (Y, T_2, \sigma)\rangle$ (or briefly φ) is called a cocycle over (Y, T_2, σ) with the fiber W, if φ is a mapping from $T_1 \times W \times Y$ to W satisfying the following conditions:*

1. *$\varphi(0, x, y) = x$ for all $(x, y) \in W \times Y$;*
2. *$\varphi(t+\tau, x, y) = \varphi(t, \varphi(\tau, x, y), \sigma(\tau, y))$ for all $t, \tau \in T_1$ and $(x, y) \in W \times Y$;*
3. *the mapping φ is continuous.*

Let $X := W \times Y$, and define the mapping $\pi : X \times T_1 \to X$ by the equality: $\pi((u, y), t) := (\varphi(t, u, y), \sigma(t, y))$ (i.e. $\pi = (\varphi, \sigma)$). Then it is easy to check that (X, T_1, π) is a dynamical system on X, which is called a skew-product dynamical system [2], [13]; but $h = pr_2 : X \to Y$ is a homomorphism of (X, T_1, π) onto (Y, T_2, σ) and hence $\langle (X, T_1, \pi), (Y, T_2, \sigma), h\rangle$ is a non-autonomous dynamical system.

Thus, if we have a cocycle $\langle W, \varphi, (Y, T_2, \sigma)\rangle$ over the dynamical system (Y, T_2, σ) with the fiber W, then there can be constructed a non-autonomous dynamical system $\langle (X, T_1, \pi),\ (Y, T_2, \sigma), h\rangle$ ($X := W \times Y$), which we will call a non-autonomous dynamical system generated (associated) by the cocycle $\langle W, \varphi, (Y, T_2, \sigma)\rangle$ over $(Y,\ T_2,\ \sigma)$.

Example 2.6 Consider the equation

$$x_{n+1} = f(\sigma(n, y), x_n)\ (y \in Y), \tag{1}$$

where (Y, T_2, σ) ($T_2 \subseteq Z$) is a dynamical system on Y and $f : Y \times W \to W$ is a continuous mapping.

Denote by $\varphi(n, u, y)$ the solution of equation (1) with initial condition $\varphi(0, u, y) = u$. From the general properties of difference equations it follows that:

(i) $\varphi(0, u, y) = u$ for all $u \in W$ and $y \in Y$;

(ii) $\varphi(n+m, x, y) = \varphi(n, \varphi(m, x, y), \sigma(m, y))$ for all $n, m \in T_1 \subseteq Z$ and $(x, y) \in W \times Y$;

(iii) the mapping φ is continuous.

Thus every equation (1) generate a cocycle $\langle W, \varphi, (Y, T_2, \sigma)\rangle$ over (Y, T_2, σ) with fiber W.

Definition 2.7 *A mapping $\gamma : Y \to X$ is called an invariant section of the non-autonomous dynamical system $\langle (X, T_1, \pi), (Y, T_2, \sigma), h\rangle$, if it is a section of the bundle fiber (X, h, Y) and $\gamma(Y)$ is an invariant subset of the dynamical system (X, T_2, π) (or, equivalently,*

$$\bigcup\{\pi^t\gamma(q)\ :\ q \in (\sigma^t)^{-1}(\sigma^t y)\} = \gamma(\sigma^t y)$$

for all $t \in T_2$ nd $y \in Y$, where $\pi^t := \pi(t, \cdot)$).

Theorem 2.8 *[5, Ch.2,p.83] Let $\langle (X, T_1, \pi),\ (Y,\ T_2,\ \sigma),\ h\rangle$ be a non-autonomous dynamical system and the following conditions be fulfilled:*

(i) the space Y is compact;

(ii) $T_2 = Z$ or R;

(iii) the non-autonomous dynamical system $\langle (X, T_1, \pi), (Y, T_2, \sigma), h\rangle$ is contracting in the extended sense, i.e. there exist positive numbers N and ν such that

$$\rho(\pi(t, x_1), \pi(t, x_2)) \le Ne^{-\nu t}\rho(x_1, x_2) \tag{2}$$

for all $x_1, x_2 \in X$ $(h(x_1) = h(x_2))$ and $t \in T_1$;

(iv) $\Gamma(Y, X) = \{\gamma \mid \gamma : Y \to X$ is a continuous mapping and $h(\gamma(y)) = y$ for all $y \in Y\} \neq \emptyset$.

Then

(i) there exists a unique invariant section $\gamma \in \Gamma(Y, X)$ of the non-autonomous dynamical system $\langle (X, T_1, \pi), (Y, T_2, \sigma), h\rangle$;

(ii) the following inequality holds

$$\rho(\pi(t, x), \pi(t, \gamma(h(x)))) \le Ne^{-\nu t}\rho(x, \gamma(h(x))) \tag{3}$$

for all $x \in X$ and $t \in T$.

Proceedings of the 5th WSEAS International Conference on Applications of Electrical Engineering, Prague, Czech Republic, March 12-14, 2006 (pp219-224)

3 Almost periodic motions of dynamical systems

Let (X, Z_+, π) be a dynamical system.

Definition 3.1 *A number $m \in Z_+$ is called an ε-almost period of the point $x \in X$, if $\rho(\pi(m + n, x), \pi(n, x)) < \varepsilon$ for all $n \in Z_+$.*

Definition 3.2 *The point x is called almost periodic, if for any $\varepsilon > 0$ there exists a positive number $l \in Z_+$ such that on every segment (in Z_+) of length l there may be found an ε-almost period of the point x.*

Denote $M_x = \{\{t_n\} \subset Z_+ | \ \{\pi(t_n, x)\}$ is convergent$\}$.

Theorem 3.3 *([11], [12]) Let (X, Z_+, π) and (Y, Z_+, σ) be two dynamical systems. Assume that $h : X \to Y$ is a homomorphism of (X, Z_+, π) onto (Y, Z_+, σ). If a point $x \in X$ is almost periodic, then the point $y := h(x)$ is also almost periodic and $M_x \subseteq M_y$.*

Definition 3.4 *A solution $\varphi(n, u, y)$ of equation (1) is said to be almost periodic, if the point $x := (u, y) \in X := E \times Y$ is an almost periodic point of the skew-product dynamical system (X, Z_+, π), where $\pi := (\varphi, \sigma)$, i.e. $\pi(n, (u, y)) := (\varphi(n, u, y), \sigma(n, y))$ for all $n \in Z_+$ and $(u, y) \in E \times Y$.*

Let E be a Banach space with the norm $|\cdot|$.

Lemma 3.5 *Suppose that $u \in C(Y, E)$ satisfies the condition*

$$u(\sigma(n, y)) = \varphi(n, u(y), y) \tag{4}$$

for all $n \in Z_+$ and $y \in Y$. Then the map $h : Y \to X$, defined by

$$h(y) := (u(y), y) \tag{5}$$

for all $y \in Y$, is a homomorphism of (Y, Z_+, σ) onto (X, Z_+, π), where $X := E \times Y$ and $\pi := (\varphi, \sigma)$.

Proof. This assertion follows from equalities (4) and (5).

Remark 3.6 *A function $u \in C(Y, E)$ with property (4) is called a continuous invariant section (or an integral manifold) of non-autonomous difference equation (1).*

Theorem 3.7 *If a function $u \in C(Y, E)$ satisfies condition (4) and a point $y \in Y$ is almost periodic, then the solution $\varphi(n, u(y), y)$ of equation (1) also is almost periodic.*

Proof. This statement follows from Theorem 3.3 and Lemma 3.5.

Example 3.8 Consider the equation

$$u_{n+1} = f(n, u_n) \tag{6}$$

where $f \in C(Z_+ \times E, E)$; here $C(Z_+ \times E, E)$ is the space of all continuous functions $Z_+ \times E \to E$) equipped with metric defined by equality

$$d(f_1, f_2) := \sum_{1}^{+\infty} \frac{1}{2^n} \frac{d_n(f_1, d_2)}{1 + d_n(f_1, d_2)},$$

where $d_n(f_1, d_2) := \max\{\rho(f_1(k, u), f_2(k, u)) \mid k \in [0, n],\ |u| \le n\}$, there is defined a distance on $C(Z_+ \times E, E)$ which generates the topology of pointwise convergence with respect to $n \in Z_+$ uniformly with respect to u on every bounded subset from E.

Along with equation (6), we will consider the H-class of equation (6)

$$v_{n+1} = g(n, v_n) \quad (g \in H(f)), \tag{7}$$

where $H(f) = \overline{\{f_m \mid m \in Z_+\}}$ and the over bar denotes the closure in $C(Z_+ \times E, E)$, and $f_m(n, u) = f(n + m, u)$ for all $n \in Z_+$ and $u \in E$. Denote by $(C(Z_+ \times E, E), Z_+, \sigma)$ the dynamical system of translations. Here $\sigma(m, g) := g_m$ for all $m \in Z_+$ and $g \in C(Z_+ \times E, E)$.

Let Y be the hull $H(f)$ of a given function $f \in C(Z_+ \times E, E)$ and denote the restriction of $(C(Z_+ \times E, E), R, \sigma)$ on Y by (Y, Z_+, σ). Let $F : E \times Y \to E$ be a continuous map defined by $F(u, g) = g(0, u)$ for $g \in Y$ and $u \in E$. Then equation (7) can be rewritten in this form:

$$u_{n+1} = F(\sigma(n, y), u_n)$$

where $y := g$ and $\sigma(n, y) := g_n$.

Definition 3.9 *The function $f \in C(Z_+ \times E, E)$ is said to be almost periodic if $f \in C(Z_+ \times E, E)$ is a almost periodic point of the dynamical system of translations $(C(Z_+ \times E, E), Z_+, \sigma)$.*

If the function $f \in C(Z_+ \times E, E)$ is almost periodic, then the set $Y := H(f)$ is the compact minimal set of the dynamical system $(C(Z_+ \times E, E), Z_+, \sigma)$ consisting of almost periodic functions.

Proceedings of the 5th WSEAS International Conference on Applications of Electrical Engineering, Prague, Czech Republic, March 12-14, 2006 (pp219-224)

4 Almost periodic solutions of Beverton-Holt equation

The periodic Beverton-Holt equation

$$x_{n+1} = \frac{\mu K_n x_n}{K_n + (\mu - 1)x_n} \quad (8)$$

($K_{n+k} = K_n$) has been studied by Jim Cushing and Shandelle Henson [6] and Saber Elaydi and Robert J. Sacker [7].

Below we will suppose that the following conditions hold:

(C1) the squence $\{K_n\}_{n\in Z}$ is almost periodic;

(C2) $\alpha < \beta$ are two positive constants such that $\alpha \le K_n \le \beta$ for all $n \in Z$;

(C3) $\mu > (\frac{\beta}{\alpha})^2$.

Denote by $\varphi(n, u, f)$ the solution of equation

$$x_{n+1} = f(n, x_n) \quad (9)$$

with initial condition $\varphi(0, u, f) = u$.

Lemma 4.1 *Let $f_i : Z_+ \times R_+ \to R_+$ $(i = 1, 2)$. Suppose that the following conditions hold:*

(i) $u_1, u_2 \in R_+$ and $u_1 \le u_2$ (respectively, $u_1 < u_2$);

(ii) $f_1(n, x) \le f_2(n, x)$ (respectively, $f_1(n, x) < f_2(n, x)$) for all $n \in Z_+$ and $x \in R_+$;

(iii) the function f_2 is monotone non-decreasing (respectively, strictly monotone increasing) with respect to variable $x \in R_+$.

Then $\varphi(n, u_1, f_1) \le \varphi(n, u_2, f_2)$ (respectively, $\varphi(n, u_1, f_1) < \varphi(n, u_2, f_2)$) for all $n \in Z_+$.

Proof. Let $u_1 \le u_2$ (respectively, $u_1 < u_2$), then we have

$$\varphi(1, u_1, f_1) = f_1(0, u_1) \le f_2(0, u_1) \le f_2(0, u_2) = \varphi(1, u_2, f_2).$$

(respectively, $\varphi(1, u_1, f_1) = f_1(0, u_1) < f_2(0, u_1) < f_2(0, u_2) = \varphi(1, u_2, f_2)$). Suppose that $\varphi(k, u_1, f_1) \le \varphi(k, u_2, f_2)$ (respectively, $\varphi(k, u_1, f_1) < \varphi(k, u_2, f_2)$) for all $k \le n$, then we obtain

$$\varphi(n + 1, u_1, f_1) = f_1(n, \varphi(n, u_1, f_1))$$

$$\le f_2(n, \varphi(n, u_2, f_2)) = \varphi(n + 1, u_2, f_2)$$

(respectively, $\varphi(n + 1, u_1, f_1) = f_1(n, \varphi(n, u_1, f_1)) < f_2(n, \varphi(n, u_1, f_1)) < f_2\ (n,\ \varphi(n,\ u_2,\ f_2)) = \varphi(n + 1, u_2, f_2)$).

Lemma 4.2 *Let $f(n, x) := \frac{\mu K_n x}{K_n+(\mu-1)x}$ for all $n \in Z_+$ and $x \in R_+$. Then the following statements hold:*

(i) $f(n, x) \ge 0$ for all $n \in Z_+$ and $x \in R_+$;

(ii) $f'_x(n, x) = \frac{\mu K_n^2}{(K_n+(\mu-1)x)^2} > 0$ for all $n \in Z_+$ and $x \in R_+$;

(iii) $f''_{x^2}(n, x) = -\frac{2\mu(\mu-1)K_n^2}{(K_n+(\mu-1)x)^3} < 0$ for all $n \in Z_+$ and $x \in R_+$.

Proof. This statement is straightforward.

Lemma 4.3 *Let $f(n, x) := \frac{\mu K_n x}{K_n+(\mu-1)x}$ for all $n \in Z_+$ and $x \in R_+$. Then the following statements hold:*

(i) $f(n, x) - x = \frac{(\mu-1)x(K_n-x)}{K_n+(\mu-1)x}$ for all $n \in Z_+$ and $x \in R_+$;

(ii) $f(n, K_n) = K_n$ for all $n \in Z$;

(iii) $f(n, x) - \frac{\mu}{\mu-1}K_n = -\frac{\mu K_n^2}{(K_n+(\mu-1)x)(\mu-1)} < 0$ for all $n \in Z_+$ and $x \in R_+$.

(iv) (a) $\varphi(n, u, f) \le u$ for all $u \ge \beta$;

(b) $\varphi(n, u, f) \ge u$ for all $u \le \alpha$.

Proof. The first three statements are obvious. Let $u \ge \beta$ (respectively, $u \le \alpha$), then

$$\varphi(1, u, f) - u = \frac{(\mu - 1)u(K_1 - u)}{K_1 + (\mu - 1)u} \le 0$$

(respectively, $\varphi(1, u, f) - u \ge 0$). Suppose that $\varphi(k, u, f) \le u$ (respectively, $\varphi(k, u, f) \ge u$) for all $k \le n$, then we obtain $\varphi(n + 1, u, f) = \varphi(1, \varphi(n, u, f), \sigma(n, f))$ ($\sigma(n, f)(k, x) := f(k + n, x)$ for all $k \in Z$ and $x \in R_+$). By Lemma 4.1 we have $\varphi(1, \varphi(n, u, f), \sigma(n, f)) \le \varphi(1, u, \sigma(n, f))$ (respectively, $\varphi(1, \varphi(n, u, f), \sigma(n, f)) \ge \varphi(1, u, \sigma(n, f))$) because $\sigma(n, f)'_x(m, x) = f'_x(n + m, x) \ge 0$ for all $n.m \in Z$ and $x \in R_+$. for all $k \in Z$ and $x \in R_+$Since

$$\varphi(1, u, \sigma(n, f)) = \frac{\mu K_{n+1} u}{K_{n+1} + (\mu - 1)u},$$

then

$$\varphi(1, u, \sigma(n, f)) - u = \frac{(\mu - 1)u(K_{n+1} - u)}{K_{n+1} + (\mu - 1)u} \le 0$$

because $K_{n+1} \le \beta$ (respectively, $\varphi(1, u, \sigma(n, f) - u \ge 0$ because $K_{n+1} \ge \alpha$). Thus $\varphi(n + 1, u, f) = \varphi(1, \varphi(n, u, f), \sigma(n, f)) \le u$ (respectively, $\varphi(n + 1, u, f) = \varphi(1, \varphi(n, u, f), \sigma(n, f)) \ge u$).

Corollary 4.4 *Let* $f(n,x) := \frac{\mu K_n x}{K_n+(\mu-1)x}$ *for all* $n \in Z_+$ *and* $x \in R_+$, *then*

(i)
$$\limsup_{n\to+\infty} |\varphi(n,u,f)| \le \frac{\mu}{\mu-1}\beta \qquad (10)$$
for all $u \in R_+$;

(ii) $\alpha - h \le \varphi(n,u,f) \le \beta + h$ *for all* $n \in Z_+$ *and* $u \in [\alpha-h, \beta+h]$, *where* $0 < h < \frac{\beta-\alpha}{2}$.

Proof. By Lemma 4.3 we have
$$\varphi(n+1,u,f) - \frac{\mu}{\mu-1}K_n$$
$$= -\frac{\mu K_n^2}{(K_n+(\mu-1)x)(\mu-1)} < 0$$
for all $n \in Z_+$ and, consequently,
$$\limsup_{n\to+\infty} |\varphi(n,u,f)| \le \limsup_{n\to+\infty} \frac{\mu}{\mu-1}K_n \le \frac{\mu}{\mu-1}\beta.$$

The second statement of Corollary follows directly from Lemma 4.3.

Lemma 4.5 *Let* $f(n,x) := \frac{\mu K_n x}{K_n+(\mu-1)x}$ *for all* $n \in Z_+$ *and* $x \in R_+$ *and* $0 < h < \min\{\frac{\mu}{\mu-1}\frac{\beta-\alpha}{2}, \frac{\mu}{\mu-1}\alpha - \frac{\mu^{1/2}}{\mu-1}\beta\}$, *then*
$$|f_x'(n,x)| \le k(h) := \frac{\mu\beta^2}{(\mu\alpha - h(\mu-1))^2} < 1 \qquad (11)$$
for all $x \in [\alpha-h, \beta+h]$.

Proof. If $0 < h < \min\{\frac{\mu}{\mu-1}\frac{\beta-\alpha}{2}, \frac{\mu}{\mu-1}\alpha\}$ then
$$\frac{1}{(K_n+(\mu-1)x)^2} \le \frac{1}{(\mu\alpha - h(\mu-1))^2}$$
for all $x \in [\alpha-h, \beta+h]$ because $\alpha \le K_n \le \beta$ ($\forall$ $n \in Z$). Thus we have
$$f_x'(n,x) \le \frac{\mu\beta^2}{(\mu\alpha - h(\mu-1))^2} := k(h).$$
Since $k(0) = \frac{\beta^2}{\mu\alpha^2} < 1$, then $0 < k(h) < 1$ for sufficiently small positive h. It easy to check that $k(h) = 1$ iff $h_{1,2} = \frac{\mu}{\mu-1}\alpha \pm \frac{\mu^{1/2}}{\mu-1}\beta$ and $k(h) > 1$ for all $h \in (h_1, h_2)$. Consequently $0 < k(h) < 1$ for all $0 < h < \min\{\frac{\mu}{\mu-1}\frac{\beta-\alpha}{2}, \frac{\mu}{\mu-1}\alpha - \frac{\mu^{1/2}}{\mu-1}\beta\}$.

Corollary 4.6 *Let* $f(n,x) := \frac{\mu K_n x}{K_n+(\mu-1)x}$ *for all* $n \in Z_+$ *and* $x \in R_+$ *and* $0 < h < \min\{\frac{\mu}{\mu-1}\frac{\beta-\alpha}{2}, \frac{\mu}{\mu-1}\alpha - \frac{\mu^{1/2}}{\mu-1}\beta\}$, *then*
$$|\varphi(n,u_1,f) - \varphi(n,u_2,f)| \le k(h)^n |u_1 - u_2| \qquad (12)$$
for all $u_1, u_2 \in [\alpha-h, \beta+h]$ *and* $n \in Z_+$.

Proof. According to Lagrange's formula we have
$$\varphi(n+1,u_1,f) - \varphi(n+1,u_2,f) \qquad (13)$$
$$= f_x'(n, \varphi(n,u_1,f) + \theta_n(\varphi(n,u_2,f)$$
$$-\varphi(n,u_1,f)))(\varphi(n,u_1,f) - \varphi(n,u_2,f)),$$
where $\theta_n \in (0,1)$ for all $n \in Z_+$. If $u_1, u_2 \in [\alpha-h, \beta+h]$, then by Corollary 4.4 we have $\varphi(n,u_i,f) \in [\alpha-h, \beta+h]$ ($\forall$ $n \in Z_+$ and $i = 1,2$) and, consequently, $\varphi(n,u_1,f) + \theta_n(\varphi(n,u_2,f) - \varphi(n,u_1,f)) \in [\alpha-h, \beta+h]$ for all $n \in Z_+$. Thus
$$|f_x'(n, \varphi(n,u_1,f) + \theta_n(\varphi(n,u_2,f) \qquad (14)$$
$$-\varphi(n,u_1,f)))| \le k(h)$$
for all $n \in Z_+$. From the relations (13)-(14) we obtain
$$|\varphi(n+1,u_1,f) - \varphi(n+1,u_2,f)|$$
$$\le k(h)|\varphi(n,u_1,f) - \varphi(n,u_2,f)|$$
($\forall n \in Z_+$) and, consequently,
$$|\varphi(n,u_1,f) - \varphi(n,u_2,f)| \le k(h)^n |u_1 - u_2|$$
for all $u_1, u_2 \in [\alpha-h, \beta+h]$ and $n \in Z_+$.

Denote by $C(Z, R_+)$ the space of all numerical sequences $M = \{M_n\}_{n\in Z}$ equipped with the distance
$$d(M^1, M^2) := \sum_{k=1}^{\infty} \frac{1}{2^k} \frac{d_k(M^1,M^2)}{1 + d_k(M^1,M^2)},$$
where $M^i := \{M_n^i\}_{n\in Z}$ $(i = 1,2)$ and $d_k(M^1,M^2) := \max\{|M_n^1 - M_n^2| \ : n \in [-k,k]\}$. Let $(C(Z,R), Z, \sigma)$ by the dynamical system of translations on $C(Z,R)$ (i.e. $\sigma(n,M)(k) := M_{n+k}$ for all $k \in Z$) and $H(M) := \overline{\{\sigma(n,M) \ : \ n \in Z\}}$, where by bar we denote the closure in $C(Z,R)$. This means that $\tilde{M} \in H(M)$ iff there exists a sequence $\{m_k\} \subset Z$ such that $\tilde{M}_n = \lim_{k\to+\infty} M_{n+m_k}$ for every $n \in Z$.

Theorem 4.7 *Let* $f(n,x) := \frac{\mu K_n x}{K_n+(\mu-1)x}$ *for all* $n \in Z$, $x \in R_+$, $0 < h < \min\{\frac{\mu}{\mu-1}\frac{\beta-\alpha}{2}, \frac{\mu}{\mu-1}\alpha - \frac{\mu^{1/2}}{\mu-1}\beta\}$ *and the conditions (C1)-(C3) hold. Then the equation (8) admits at least one almost periodic solution* $\varphi(n,u_0,f)$.

Proof. Let $f(n,x) := \frac{\mu K_n x}{K_n+(\mu-1)x}$ $(\forall\ n \in Z$ and $x \in R_+)$ and $Y := H(f)$, where $H(f) := \overline{\{\sigma(n,f)\ :\ n \in Z\}}$ and by bar we denote the closure in $C(Z \times R_+, R_+)$. It easy to see that $g \in H(f)$ iff there exists a sequence $\tilde{K} \in H(K)$ $(\tilde{K} := \{\tilde{K}_n\}_{n\in Z})$ such that $g(n,x) := \frac{\mu \tilde{K}_n x}{\tilde{K}_n+(\mu-1)x}$ for all $n \in Z$ and $x \in R_+$.

Consider the equation

$$x_{n+1} = f(n, x_n) \tag{15}$$

and denote by $(R_+, \varphi, (H(f), Z, \sigma))$ the cocycle generated by equation (15) (see Examples 2.6 and 3.8). Let $\langle (X, Z_+, \pi), (Y, Z, \sigma), h\rangle$ be the non-autonomous dynamical system generated by cocycle φ (i.e. $Y := H(f)$, (Y, Z, σ) is the shift dynamical system on Y, $X := R_+ \times Y$, $\pi := (\varphi, \sigma)$ and $h := pr_2 : X \to Y$). Note that the set $\mathcal{X} := [\alpha - h, \beta + h] \times Y \subseteq X$ by Corollary 4.4 is invariant with respect to dynamical system (X, Z_+, π). Thus we can consider the non-autonomous subsystem $\langle (\mathcal{X}, Z_+, \pi), (Y, Z, \sigma), h\rangle$ of system $\langle (X, Z_+, \pi), (Y, Z, \sigma), h\rangle$. According to Corollary 4.6 the non-autonomous dynamical system $\langle (\mathcal{X}, Z_+, \pi), (Y, Z, \sigma), h\rangle$ is contracting because $\rho(\pi(n, x_1), \pi(n, x_2)) \le k(h)^n \rho(x_1, x_2)$ for all $x_1, x_2 \in \mathcal{X}$ $(h(x_1) = h(x_2))$. By Theorems 2.8 and 3.7 there exists a continuous function $u : H(f) \to [\alpha - h, \beta + h]$ such that $u(\sigma(n, g)) = \varphi(n, u(g), g)$ for all $g \in H(f)$ and $n \in Z_+$ and the solution $\varphi(n, u(f), f)$ of equation (15) is almost periodic.

References

[1] Allais M., Probabilites, frequences et hasard. *J. Soc. Stat.,* Paris, t. 124, No. 2, 1983, pp.70-221.

[2] Alekseev V. M., *Symbolic Dynamics.* Naukova Dumka, Kiev, 1986. The 11th Mathematical School.

[3] Bourbaki N., *Variétés Différentielles et Analitiques (Fascicule de résultats).* Herman, Paris, 1971.

[4] Broer H. W., Huitema G. B. and Sevryuk M. B., *Quasi-Periodic Motions in Families of Dynamical Systems.* Lect. Notes in Math. No. 1645, Springer-Verlag, Berlin, 1996.

[5] Cheban D. N., *Global Attractors of Nonautonomous Dissipstive Dynamical Systems.* World Scientific, 2004.

[6] Cushing J and Henson S., A Periodically Forced Beverton-Holt Equation. *J. Difference Eq. and Appl.,* 8:1119-1120, 2002.

[7] Elaydi S. and Sacker R., Global Stability of Periodic Orbits of Non-Autonomous Difference Equations and Population Biology. *Journal of Differential Equations,* to appear.

[8] Hommes C. H., Periodic, Almost Periodic and Chaotic Behaviour in Hick's Non-Linear Trade Cycle Model. *Econom. Lett.,* 41(1993), pp.391-397.

[9] Husemoller D., *Fibre Bundles.* Springer–Verlag New York Heidelberg Berlin 1994.

[10] Pennequin D., Existence of Almost Periodic Solutions of Discrete Time Equations. *Discrete and Continuous Dynamical Systems,* Vol. 7, No. 1, 2001, pp.51-60.

[11] Shcherbakov B. A., *Topological Dynamics and Poisson's Stability of Solutions of Differential Equations.* Kishinev, Shtiintsa, 1972. (in Russian)

[12] Shcherbakov B. A., *Poisson's Stability of Motions of Dynamical Systems and Solutions of Differential Equations.* Kishinev, Shtiintsa, 1985. (in Russian)

[13] Sell G. R., *Topological Dynamics and Ordinary Differential Equations.* Van Nostrand-Reinhold, London, 1971.

[14] Zaslavski A. J., Exitence and Structure of Optimal Solutions of Variational Problems. *Recent Developments in Optimization and Nonlinear Analysis,* Contemporary mathematics, Vol. 204, Am. Math. Soc., Providence, Rhode Island, (1997), pp.247-278.