Spine Fin Efficiency – A Three Sided Pyramidal Fin of Equilateral Triangular Cross-Sectional Area

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Abstract: A model is presented for determining the fin efficiency of a spine fin with the geometry of a three sided pyramid of equilateral triangular cross-sectional area (from here on referred to as TSPECA). It is assumed that heat transfer is one-dimensional. The differential equation describing the system is solved using finite differences. The fin efficiency is then calculated using numerical integration, the trapezoidal rule. The results are compared against an equivalent fin of constant cross-sectional area. It is found that the errors are considerable when approximating the fin geometry as that of constant cross-section - as high as 19%. It is recommended that the spine fin be modeled as variable in cross-section throughout its length.

Key Words: fin efficiency, finite differences, spine fin, convection, conduction, heat transfer.

1 Introduction
The interest in determining the fin efficiency for TSPECA is based on a previous work by Carranza [1]. Carranza models the spine fin as a three sided pyramid, but determines fin efficiencies for geometries that are not equilateral triangular in cross-section. Each side of the triangle has a different length. Moreover, the fins analyzed are of fixed dimensions. In total, only two fins with different sets of physical parameters are studied.

For this reason, a general method for determining the fin efficiency is presented. A robust model is outlined that calculates the fin efficiency of a TSPECA for a variety of spine lengths, widths, materials and temperatures. Any physical or thermal parameter can be varied. The problem is simply restricted to a pyramid with equilateral triangular cross-section.

Before beginning the analysis, it is necessary to review why the fin efficiency is required. In heat transfer studies, \( h_o \) is measured experimentally for finned surfaces through the Fourier heat transfer equation, where the overall heat transfer coefficient is defined relative to the pipe inner radius:

\[
U_i = \frac{1}{\frac{1}{h_i} + \frac{r_i \ln r_o / r_i}{k_m} + \frac{A_i}{h_o(A_b + A_f \eta)}}
\] (1)

As one can see, \( h_o \) is determined for a finned pipe through the use of Equation 1. All of the physical parameters in Equation 1 are measured. The \( h_i \) is easily calculated from established correlations.

The problem with Equation 1 is that \( \eta \) is a function of \( h_o \). That is, the \( h_o \) must be estimated to determine \( \eta \). Once \( \eta \) is quantified, then \( h_o \) is calculated from Equation 1 and checked against the initial estimate. For this reason, a systematic method for determining \( \eta \) is required. \( \eta \) is a function of fin geometry, fin material, temperature profile
and much more. All of these issues are addressed in this work.

2 Geometry

The geometry of the spine in question is that of a three sided pyramid (see Figure 1). The footprint of the spine, at the base, is that of an equilateral triangle. It is then assumed that the sides slowly decrease in size as they approach the tip of the spine. The rate of decrease is assumed to be a linear function of the spine height. Thus, the cross-sectional area of the spine is always that of an equilateral triangle for any cross-section throughout the length of the spine.

It must be noted that the cross-sectional area of the spine is changing throughout its length. The cross-sectional area is zero at the spine tip. Thus, most models for determining the spine efficiency cannot be used for this spine geometry since they assume constant cross-sectional area.

Therefore, a set of special geometrical equations are established for this spine geometry - TSPECA. The first step is to assign a spatial coordinate system to the model (see Figure 2). At the base, the cross-sectional area of the spine is shaped like an equilateral triangle. Each side of the triangle has a length “a”. The height of the spine is assigned the term “b”.

As was stated, the sides of the triangle decrease linearly as they approach the tip of the spine. The length of the sides of the triangle are zero at the spine tip. The geometry must be addressed mathematically. To do so, a parameter is required that determines the triangle’s side length as a function of “x”, \( l(x) \). This is done by making \( l(x) \) a dependent variable of the independent variables \( a \) and \( b \). Hence, the following equation is proposed for \( l(x) \):

\[
l(x) = a - \frac{a}{b}x = a - \beta x
\]  

(2)

Knowing the length of the sides of the equilateral triangle as a function of “x”, the perimeter of the spine is known as a function of “x”:

\[
P = 3l(x) = 3\frac{dA_s}{dx} = 3(a - \beta x)
\]  

(3)

Lastly, the cross-sectional area of any equilateral triangle is determined from simple geometry:
\[ A_c = \frac{i^2 \sqrt{3}}{4} \]  \hfill (4)

Substituting Equation 2 into Equation 4, the final equation for the cross-sectional area of the spine fin is derived.

\[ A_c(x) = \frac{\sqrt{3}}{4} (a^2 - 2a\beta x + \beta^2 x^2) \]  \hfill (5)

In heat transfer studies, an important quantity is the rate of change of cross-sectional area as a function of fin length. This is a rate term used later when the heat balance is conducted around the fin. For the sake of uniformity, the equation is presented here.

\[ \frac{dA_c}{dx} = \frac{\sqrt{3}}{2} (-a\beta + \beta^2 x) \]  \hfill (6)

### 3 Theory

#### 3.1 Transport Equations

Prior to calculating the fin efficiency, it is necessary to find the temperature profile through the fin. This is done with the aid of an energy balance [2]:

\[ A_c \frac{d^2T}{dx^2} + \frac{dA_c}{dx} \frac{dT}{dx} - \frac{h_v}{k_m} \frac{dA_c}{dx} (T(x) - T_\infty) = 0 \]  \hfill (7)

The boundary conditions for the spine fin are \( T(0)=T_b \) and \( T(b)=T_\infty \).

Equation 7 is best solved if done via dimensionless variables. Dimensionless variables make the solution more general and applicable. Furthermore, note that Equation 7 is a universal equation that applies to many different types of fins. Equations 2 - 6 are substituted into Equation 7 making it applicable to a TSPECA.

\[ (X-1) \frac{d^2\Theta}{dX^2} + 2 \frac{d\Theta}{dX} + a\Theta = 0 \]  \hfill (8)

where \( \Theta(0) = 1 \) and \( \Theta(1) = 0 \).

The dimensionless variables in Equation 8 are defined in Equations 9 and 10.

\[ \Theta = \frac{T - T_\infty}{T_b - T_\infty} \]  \hfill (9)

\[ X = \frac{x}{b} \]  \hfill (10)

![Figure 2. Spatial Coordinate system used in model development.](image-url)
Equation 8 is $\alpha$. Nevertheless, values for $\alpha$ are arbitrarily specified. Values for $\alpha$ are selected between 0 and 10. Thus, a completely different temperature profile is calculated for each value of $\alpha$.

4 Results

4.1 Temperature Profile

The temperature profile throughout the fin is obtained from Equation 8 and presented in Figure 3. It is clear that as $\alpha$ increases in magnitude, the heat transfer through the spine increases. That is, as heat transfer is enhanced, most of the heat is exchanged before reaching the tip of the spine. The fin acts as a channel that sends heat to the tip from the base. As $\alpha$ increases, the heat is lost before reaching the tip. Therefore, the spine’s temperature (especially near the tip) begins to approach the temperature of the convective bulk fluid.

This phenomena is most easily seen by examining the definition of $\alpha$:

$$\alpha = 4\sqrt{3} \frac{h_o b}{\beta k_m} = 4\sqrt{3} \frac{h_o b^2}{ak_m}$$ (11)

As $\alpha$ gets larger, the following heat transfer mechanisms are occurring within the spine fin:

- $h_o$ is increasing. As $h_o$ increases, the rate of heat lost by convection increases. Heat does not reach the fin tip; and for this reason, the temperature at the tip is closer to the temperature of the convective fluid.
- $b$ is increasing. As $b$ increases, the heat must travel farther to reach the tip. More thermal resistance is experienced and this increases the likelihood that heat is transferred by convection rather than by conduction.
- $a$ is decreasing. As $a$ decreases, the cross-sectional area for heat flow reduces. Thus, heat does not have a suitable avenue for reaching the fin tip.
- $k_m$ is decreasing. As $k_m$ decreases, thermal resistance to heat transfer increases. And once again, heat is inhibited from reaching the tip.

It is important to note that $\alpha$ is simply the ratio of the thermal conductance due to convection divided by the thermal conductance due to conduction for a straight fin with constant equilateral triangular cross-sectional area. To see this, it is necessary to restate the definition of thermal conductance:

$$\frac{Q}{\Delta T} = \frac{1}{R_{\text{conv}}} = K_{\text{conv}} = h_o A_o = h_o 3ab$$ (12)

$$\frac{Q}{\Delta T} = \frac{1}{R_{\text{cond}}} = K_{\text{cond}} = \frac{k_m A_c}{L} = \frac{k_m a^2 \sqrt{3}}{4b}$$ (13)

Now, by simply dividing Equation 12 by Equation 13, Equation 11 is derived.

Thus, it is clear that as $\alpha$ increases, the heat transfer due to convection increases. Less heat reaches the spine tip and causes the temperature of the spine to more closely approximate the temperature of the convective bulk fluid.

![Figure 3. $\Theta$ versus $X$ for specified values of $\alpha$.](image)
4.2 Fin Efficiency

The fin efficiency is now determined for TSPECA. This is done by using the strict definition of the fin efficiency:

\[
\eta = \frac{Q}{Q_{\text{max}}} = \frac{\int_{A=0}^{A=A_{\text{f}}} h_x(T(x)-T_{\infty})dA}{h_x A_x(T_b-T_{\infty})} \quad (14)
\]

Formally, the fin efficiency is defined as the actual heat transferred by the fin divided by the heat the fin would transfer if it were at one homogenous temperature, \(T_b\).

Equation 14 is not yet in the proper form for developing useful results. Therefore, it is transformed through the use of Equations 3, 9, and 10 into a function of dimensionless variables.

\[
\eta = 2 \int_{X=0}^{X=1} \Theta(1-X)dX \quad (15)
\]

In this form, Equation 15 is specifically setup for determining the fin efficiency of a TSPECA. Equation 15 is solved numerically using the Trapezoidal rule and the data generated via Equation 8. The results are presented in Figure 4 as a function of \(\alpha\).

Note that the efficiency for a fin with constant cross-sectional area is also presented in Figure 4. The fin has the same geometrical dimensions as TSPECA, given in Figure 1. The relationships for calculating the efficiency of a fin with constant cross-sectional area are well established and given in Equations 16 and 17 [3]:

\[
\eta = \frac{\tanh(mb)}{mb} \quad (16)
\]

\[
m^2 = \frac{h_o P}{k_m A_c} \quad (17)
\]

From simple observation, it is difficult to see a common basis on which to compare Equations 15, 16, and 17. However, when Equations 2, 3, and 4 are substituted into Equation 17, a convenient basis is found (note, \(l(x)=a\), \(P(x)=3a\) and \(A_c=a^2 \times 0.5 / 4\)).

\[
mb = \sqrt{\alpha} \quad (18)
\]

Thus, the efficiencies for TSPECA and a fin of constant cross-sectional area are both plotted as a function of the same independent variable, \(\alpha\).

![Figure 4. \(\eta\) versus \(\alpha\): a comparison of fin efficiencies for TSPECA and an equivalent fin of constant cross-sectional area.](image)

5 Conclusion

A model is presented for determining the fin efficiency of a TSPECA. The results are presented in Figure 4.

It is important to note that in many heat transfer applications the fin efficiency is approximated using Equations 16 and 17. The equations are applied often to helical fins, plate fins, and spine fins in HVAC applications - despite they are valid only for straight fins of constant cross-sectional area. See Carranza for a complete overview and description [4].

This work, however, shows that there is a large discrepancy between the efficiency for a fin of constant cross-sectional area and a TSPECA. In Figure 4, the error between the
two methods is 12.1% and 19.2% for values of $\alpha$ at 1 and 2, respectively.

Therefore, it is recommended that a detailed attempt be made at modeling the physical geometry of any fin in question with as much accuracy as possible; and that simplistic assumptions, such as constant cross-sectional area, be kept to a minimum. Otherwise, significant errors in the fin efficiency are made.

For this reason, a model for a TSPECA is presented. The fin efficiencies are determined from a mathematical system that is detailed and precise.

6 Nomenclature

$A =$ area
$K =$ thermal conductance
$L =$ length of conductance
$P =$ perimeter
$Q =$ heat transfer rate
$R =$ thermal resistance
$T =$ temperature
$U =$ overall heat transfer coefficient
$a =$ fin parameter specified in Figure 1
$b =$ fin parameter specified in Figure 2
$h =$ individual heat transfer coefficient
$k =$ thermal conductivity
$l =$ fin geometrical parameter specified in Equation 2
$m =$ fin efficiency parameter specified in Equation 17
$r =$ pipe radius
$x =$ spatial coordinate
$y =$ spatial coordinate
$\Theta =$ dimensionless temperature
$\chi =$ dimensionless length
$\alpha =$ ratio of conductance due to convection per conductance due to conduction
$\beta =$ ratio of $a$ to $b$
$\eta =$ fin efficiency

c = cross-sectional
$f =$ fin
$i =$ inside
$m =$ metal
$max =$ maximum
$o =$ outside
$s =$ surface
$\approx =$ bulk fluid conditions

References:

6.1 Subscripts

$b =$ base