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Flow Complexity, Multiscale Flows, and Turbulence

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Abstract: - Fundamental theoretical ideas are considered toward the formulation of a measure of flow complexity that is useful both fundamentally and in applications. General considerations of geometric fractional dimensions, which can be functions of scale, offer a means to quantify the physical complexity of multiscale flows including turbulence. An analytical example of the dependence of the geometric dimension on scale is investigated. It is motivated by its physical significance and relevance to turbulent flows both at large scales and small scales for flow conditions associated with large Reynolds numbers. A basic element for a flow complexity measure must be its ability to incorporate the entire range of multiscale flow behavior. Key theoretical ingredients, based on the geometric dimension and its dependence on scale, are investigated to develop a flow complexity measure that exhibits this multiscale capability.

Keywords: - Geometric Dimensions, Fractals, Distribution of Scales, Turbulence, Flow Optimization, Adaptive Flows, Evolutionary Flows.

1. Introduction

One of the most basic and challenging features of multiscale flows, including turbulence at large Reynolds numbers, is the presence of dynamics throughout a wide range of scales (Catrakis, 2000; Sreenivasan, 1991). Such phenomena are known to be highly complex, on the basis of many observations. However, measures of their complexity are still under development (Bar-Yam, 1997; Gell-Mann, 1995; Gell-Mann & Lloyd, 1996; Richardson, 1926; Reynolds, 1883). The development of a quantitative physical measure, or measures, of flow complexity requires advances in the understanding of the multiscale behavior itself. Recent advances on a general theoretical framework that relates geometric dimensions to the distribution of scales (Catrakis, 2000, 2004), provide a means toward the realization of such a flow complexity measure.

The fundamental and applied needs for a flow complexity measure, in problems involving multiscale flows and turbulence, arise naturally from the challenges of quantifying multiscale flow aspects such as the information content. A possible measure of the geometrical complexity of multiscale flows is the fractal dimension (Catrakis *et al.*, 2002; Catrakis & Dimotakis, 1998, 1996; Sreenivasan & Meneveau, 1986; Mandelbrot, 1975). To quantify the complexity of the multiscale behavior across the entire range of scales, however, requires considerations of the dimension as a function of scale (Catrakis, 2004, 2000).

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2. Theoretical Considerations

Theoretically, the following are desirable aspects, useful properties, and key questions related to the development of a flow complexity measure that should be kept in mind:

First Set of Ideas: The flow complexity measure should incorporate not only the small scales but the large scales as well, i.e. the behavior across the entire range of scales. Thus, it should quantify the total level of complexity from the behavior as a function of scale.

Second Set of Ideas: The measure should be useful to quantitatively compare flows at different conditions and in different geometries, or even if they are at the same conditions and/or in the same geometries. It should be useful both for the instantaneous structure of the flow and its average characteristics. What are the flow complexity dynamical equations that result from the basic equations of motion? What are the limiting behaviors of the flow complexity as various flow parameters, such as the Reynolds number, are increased?

Third Set of Ideas: The measure should be useful for investigating aspects associated with maximization/minimization, optimization, self-optimization, and related concepts. What are the optima, i.e. maxima/minima, principles or variational principles for the flow complexity? Why is the flow complexity not larger or smaller than it is in each physical flow? To what extent can the flow complexity be increased or decreased in adaptive flows and evolutionary flows? To what extent do multiscale flows self-optimize?

The first, second, and third set of ideas above emphasize three key features desirable of a useful measure of flow complexity: the ability to incorporate the entire range of scales; the utility to compare quantitatively flows of different complexity; and the suitability to facilitate optimization studies. The fractal dimension, at least as originally considered in turbulence (Sreenivasan, 1991; Mandelbrot, 1975), only applies to the small scales, for example.

The general framework on geometrical complexity, developed by Catrakis (2004, 2000), addresses the multiscale behavior across the entire range of scales, i.e. both large scales and small scales. The large-scale behavior can be expected to be strongly scale dependent in turbulence. Consider the full range of scales:

$$\lambda_{\min} \ \lesssim \ \lambda \ \lesssim \ \lambda_{\max}\,, \tag{1}$$

for a multiscale object in a d-dimensional Euclidean space with a topological dimension d_{t}. For a multiscale object, the geometric dimension $D_d(\lambda)$ at a scale λ is:

$$d_{\mathrm{t}} \ \leq \ D_d(\lambda) \ \leq \ d \tag{2}$$

without necessarily assuming self-similarity. The geometric dimension $D_d(\lambda)$ is a function of scale λ, can be fractional, and is a scale-local measure of the complexity of the object.

A key general analytical relation pioneered and derived by Catrakis (2004, 2000) is the relation between the geometric dimension $D_d(\lambda)$ and the distribution $f_d(\lambda)$ of scales:

$$D_d(\lambda) \ = \ d \ - \ \frac{\lambda\, f_d(\lambda)}{\displaystyle\int_0^{\lambda} f_d(\lambda')\,\mathrm{d}\lambda'} \tag{3}$$

with $\lambda_{\min} = 0$ and $\lambda_{\max} = \infty$, and its inverse:

$$f_d(\lambda) = \frac{d - D_d(\lambda)}{\lambda}\,\exp\left\{-\int_{\lambda}^{\infty}\left[d - D_d(\lambda')\right]\frac{\mathrm{d}\lambda'}{\lambda'}\right\} \tag{4}$$

where $f_d(\lambda)$ is the probability density of the distance between a randomly-chosen location, within a reference region bounding the object, and the nearest part of the multiscale object.

Two paradigms in the study of multiscale behavior, which are useful for insight into the development of a measure of flow complexity, are the random walk and turbulence. Figure 1 shows a computational simulation of the trajectory of a random walk with 10,000 steps in two dimensions (top) and an experimental result of a three-dimensional turbulent jet with Reynolds number 20,000 (bottom). Both phenomena exhibit complexity that is a function of scale (Takayasu, 1982; Catrakis, 2000).

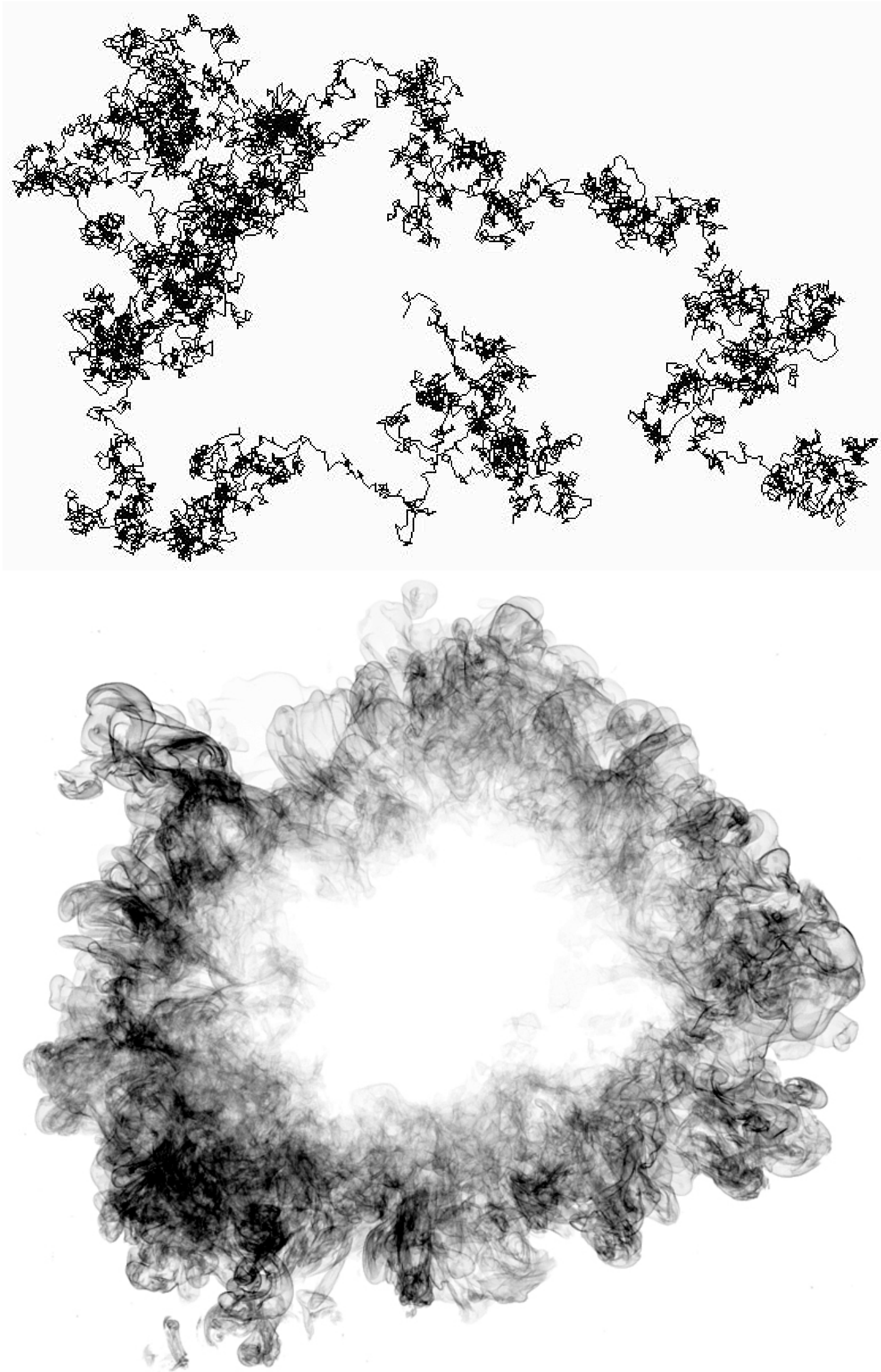

Figure 1: Two paradigms of multiscale behavior: the random walk and turbulence. Top: Computational simulation of a two-dimensional random walk with 10,000 steps of equal length. Bottom: Experimental image of a three-dimensional turbulent jet at Reynolds number 20,000.

3. General Flow Complexity

General considerations, as suggested by Catrakis (2000), indicate the following 3-level hierarchy of geometrical complexity:

Level 1: *Euclidean Complexity* – Complexity is only at a single scale

Level 2: *Fractal Complexity* – Complexity is the same at all scales

Level 3: *General Complexity* – Complexity is a general function of scale

It is the third level, i.e. general complexity, that needs to be addressed in order to develop a flow complexity measure useful for multiscale objects in general, including turbulence. Consider, as an example in one dimension, a multiscale fluctuating signal which has a probability density function $p_1(\lambda)$ of an interval λ between two successive crossings of a certain threshold value. In other words, $p_1(\lambda)$ is the level-crossing probability density. As shown by Catrakis (2000), the geometric dimension $D_1(\lambda)$ is directly related to $p_1(\lambda)$ by the following general analytical relation:

$$D_1(\lambda) \;=\; 1 - \frac{\lambda \int_\lambda^\infty p_1(\lambda')\,\mathrm{d}l}{\int_0^\lambda \int_{\lambda'}^\infty p_1(\lambda'')\,\mathrm{d}\lambda''\,\mathrm{d}\lambda'}\,. \quad (5)$$

In this case, the probability density $f_1(\lambda)$ mentioned in the previous section in equation 3, is related to $p_1(\lambda)$ as $f_1(\lambda) = (\lambda_\mathrm{m})^{-1} \int_\lambda^\infty p_1(\lambda')\,\mathrm{d}\lambda'$ where the mean scale is $\lambda_\mathrm{m} = \int_0^\infty \lambda\, p_1(\lambda)\,\mathrm{d}\lambda$. As an illustration, consider the case of an exponential probability density $p_1(\lambda) = \exp(-\lambda/\lambda_\mathrm{m}) \,/\, \lambda_\mathrm{m}$ corresponding to Poisson behavior. For this case, we first note that $f_1(\lambda) = p_1(\lambda)$. Then the analytical result, as derived by Catrakis (2000), for the geometric dimension as a function of scale is:

$$D_1(\lambda) \;=\; 1 \;-\; \frac{\lambda/\lambda_\mathrm{m}}{e^{\lambda/\lambda_\mathrm{m}} - 1}\,. \quad (6)$$

This functional form is generalizable also to two dimensions and three dimensions. Thus, in d dimensions:

$$D_d(\lambda) \;=\; d \;-\; \frac{\lambda/\lambda_\mathrm{m}}{e^{\lambda/\lambda_\mathrm{m}} - 1}\,, \quad (7)$$

where the topological dimension d_t for these Poisson processes is understood to be $d_\mathrm{t} = d-1$. The large-scale and small-scale limiting values of the geometric dimension are:

$$D_d(\lambda) \;\longrightarrow\; \begin{cases} d, & \text{as } \lambda \to \infty \\ d_\mathrm{t}, & \text{as } \lambda \to 0. \end{cases} \quad (8)$$

This can be thought of as a measure of the geometrical complexity as a function of scale, which is shown in figure 2 (left) for Poisson behavior with $d = 1$. The corresponding distribution of scales is shown in figure 2 (right) as the probability density function $\widetilde{f}[\log_{10}(\lambda/\lambda_\mathrm{m})]$ for which, by conservation of probability, we must have:

$$\frac{\widetilde{f}\left[\log_{10}\left(\frac{\lambda}{\lambda_\mathrm{m}}\right)\right]}{f\left(\frac{\lambda}{\lambda_\mathrm{m}}\right)} \;=\; \frac{\mathrm{d}\left(\frac{\lambda}{\lambda_\mathrm{m}}\right)}{\mathrm{d}\left[\log_{10}\left(\frac{\lambda}{\lambda_\mathrm{m}}\right)\right]}\,, \quad (9)$$

or $\widetilde{f}\,[\log_{10}(\lambda/\lambda_\mathrm{m})] = \ln(10)\;(\lambda/\lambda_\mathrm{m})\; f\,(\lambda/\lambda_\mathrm{m})$, where $\ln(\cdot)$ denotes the natural logarithm.

In the general case, based on the geometric dimension function $D_d(\lambda)$, let us consider an interval of scales:

$$\lambda_1 \;\le\; \lambda \;\le\; \lambda_2\,. \quad (10)$$

Then we can consider a cumulative flow complexity function $\mathcal{C}_{\lambda_1,\lambda_2}$ for the behavior in this range of scales as an integral of the difference between the geometric dimension and the topological dimension with respect to the logarithmic scale:

$$\mathcal{C}_{\lambda_1,\lambda_2} \;=\; \int_{\ln\lambda_1}^{\ln\lambda_2} \left[D_d(\lambda) \;-\; d_\mathrm{t}\right]\,\mathrm{d}\ln\lambda\,, \quad (11)$$

which is a purely dimensionless quantity since we have equivalently the dimensionless form:

$$\mathcal{C}_{\lambda_1,\lambda_2} \;=\; \int_{\lambda_1}^{\lambda_2} \left[D_d(\lambda) \;-\; d_\mathrm{t}\right]\,\frac{\mathrm{d}\lambda}{\lambda}\,. \quad (12)$$

The total complexity is thus $\mathcal{C}_{\lambda_\mathrm{min},\lambda_\mathrm{max}}$ for the full range of scales $\lambda_\mathrm{min} \le \lambda \le \lambda_\mathrm{max}$.

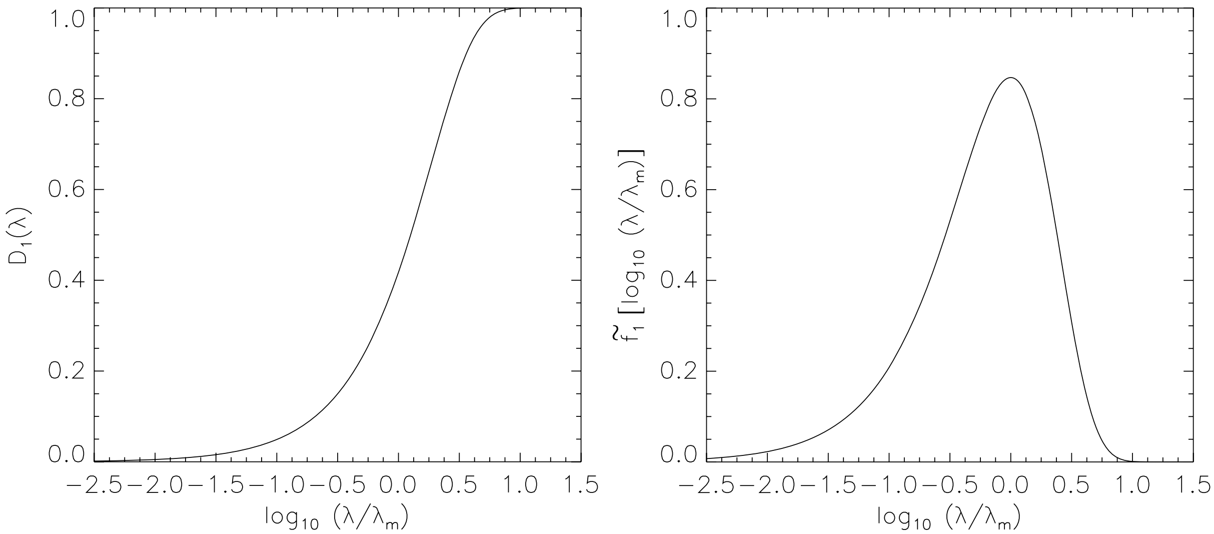

Figure 2: Fundamental measures useful to quantify the average complexity of multiscale objects as a function of scale. As an illustrative example, Poisson statistics are shown in one dimension. Left: Ensemble-averaged geometric dimension D_1 plotted as a function of logarithmic scale. Right: Probability density function $\widetilde{f}$ of logarithmic scale.

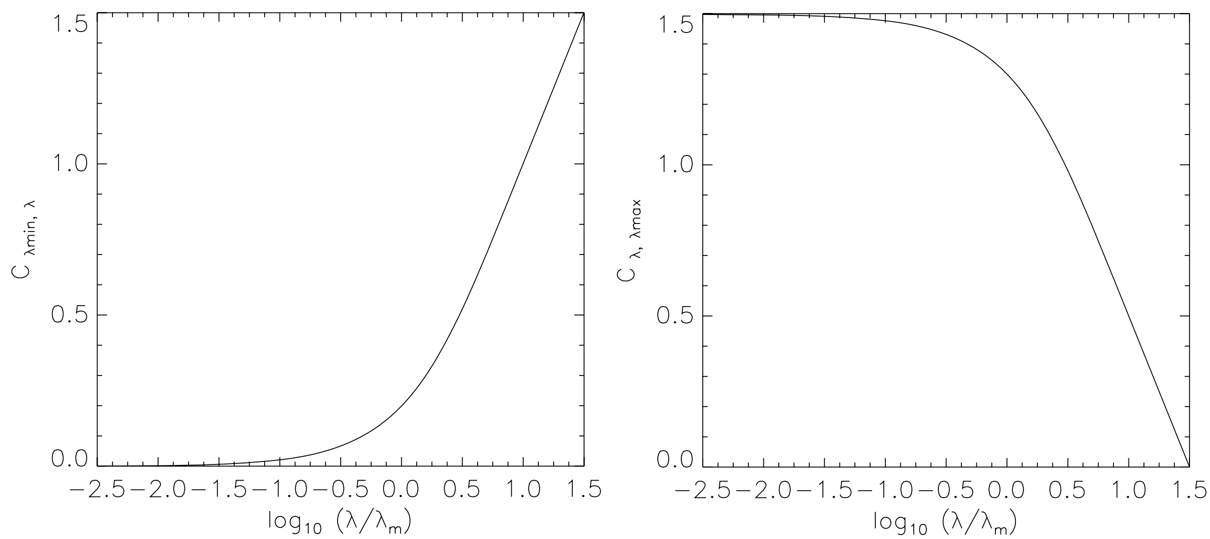

Figure 3: Cumulative complexities $\mathcal{C}_{\lambda_{\min},\lambda}$ and $\mathcal{C}_{\lambda,\lambda_{\max}}$ shown as functions of scale, on the left and right respectively, corresponding to the one-dimensional Poisson statistics in figure 2, with minimum and maximum scales taken as $\log_{10}(\lambda_{\min}/\lambda_{\mathrm{m}}) = -2.5$ and $\log_{10}(\lambda_{\max}/\lambda_{\mathrm{m}}) = 1.5$. The total complexity value is $\mathcal{C}_{\mathrm{total}} = \mathcal{C}_{\lambda_{\min},\lambda_{\max}} \simeq 1.5$, which is the maximum value in both plots in the range $\lambda_{\min} \leq \lambda \leq \lambda_{\max}$, corresponding to this behavior.

Available experimental and computational data, for the random walk as well as for turbulence, support the third level of complexity, i.e. general complexity, because of the observed dependence on scale. Specifically, analysis of the random walk in one dimension (Takayasu, 1982) shows a successive-coverage dimension that is a function of scale, consistent with experiments and computations in three dimensions (Tsurumi & Takayasu, 1986). In turbulence, high-resolution measurements in fully-developed flows indicate the presence of exponential level-crossing distributions of scales (Kailasnath & Sreenivasan, 1993; Sreenivasan *et al.*, 1983) and thus support directly the results presented above.

4. Conclusions

The present considerations indicate that a measure of flow complexity can be quantified in terms of an integral of the geometric dimension as a function of scale. This provides a means to quantify the complexity across the entire range of scales and is useful for comparing different flow conditions, such as for flow optimization, as well as for quantifying the total information content of multiscale flows.

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