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On the steady fall of a rigid body in Oseen flow - weighted approach

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Abstract: The aim of this paper is to prove the existence of the strong solution to the problem of the Oseen flow around a rotating body in weighted spaces.

Key Words: *rigid body, steady fall, strong solution, Lizorkin theory, Marcinkiewicz theory, weighted spaces, Muckenhoupt class*

1 Introduction

One of the important problems in fluid mechanics is to study of the Navier-Stokes flow past a rotating obstacle. Among a series of results on qualitative properties of the problem or realted linear problems let us mention T. Hishida [H1],[H2],[H3], G. P. Galdi [G1],[G2], R. Farwig, T. Hishida, D. Müller [FHM], R. Farwig [Fa1], [Fa2], Š. Nečasová [Ne1], [Ne2], S. Kračmar, Š. Nečasová, P. Penel [KNPe], R. Farwig, M. Krbec, ˇ S. Nečasová [FKN1],[FKN2], S. Kračmar ,Š. Nečasová, [KN], M. Geissert, M. Hieber, H. Heck [GHH].

We investigate the modified Oseen problem which is the simplified form of the problem of the fall of the rigid body in viscous fluid. We consider a coordinate system which is attached to the body. We assume that the body also rotate and the angular velocity ω is in the direction of gravitational field g, for simplicity we choose $\omega = \lambda g$.

2 Mathematical preliminaries

The Lebesgue spaces are denoted by $L^p(\mathbb{R}^n)$, $1 \le p \le \infty$, and equipped the norms $\|\cdot\|_{0,p}$. By $W^{k,p}(\mathbb{R}^N)$, $k \ge 0$ an integer, $1 \le p \le \infty$, we denote the usual Sobolev spaces with the norms $\|\cdot\|_{k,p}$. Further, we define the homogeneous Sobolev spaces $D^{m,q}(\mathbb{R}^n)$ equipped with the norm $\|\nabla \cdot\|_{m-1,q}$. Denote by $\mathcal{S}(\mathbb{R}^n)$ the space of functions of rapid decrease. For $u \in \mathcal{S}(\mathbb{R}^n)$ we denote by $\widehat{u}$ its Fourier transform. Given a function $\Phi : \mathbb{R}^n \to \mathbb{R}$, let us consider the integral transform

$$Tu \equiv h(x) = \frac{1}{(2\pi)^{n/2}} \int_{R^n} e^{ix\xi} \phi(\xi)\widehat{u}(\xi)dx,\ u \in \mathcal{S}(\mathbb{R}^n). \tag{2.1}$$

We denote the vector space

$$\mathbf{W}^{\cdot 1,q}(\Omega) = \{p \in L^1_{loc}(\bar{\Omega}); \nabla p \in L^q(\Omega)^n\},$$

is called a homogenneous Sobolev space. Here $\|\nabla p\|_q$ is only a seminorm, and the quotient space $W^{\cdot 1,q}(\Omega)/N_1$ modulo N_1, the space of constants, becomes a Banach space. The vector space

$$\tilde{\mathbf{W}}^q(\Omega) = \{u \in L^q_{loc}(\bar{\Omega}); \|\nabla^2 u\|_q < \infty,\ \|\partial_2 u\|_q < \infty\},$$

where $\|\nabla^2 u\ \| = (\sum_{j,k=1}^n \|\partial_j\partial_k u\|_q^q)^{1/q}$ will be endowed with the seminorm $\|\nabla^2 u\|_q + \|\partial_2 u\|_q$. It is easy to see that $\|\nabla^2 u\|_q + \|\partial_2 u\|_q = 0$ if and only if $u \in N_2 = \{b_1x_1 + a + b_3x_3 + ... + b_nx_n; a, b_1, b_3, ..., b_n \in \mathbf{R}\}$. Then $\tilde{\mathbf{W}}^q(\Omega)/N_2$ becomes a Banach space with norm $\|\nabla^2 u\|_q + \|\partial_2 u\|_q$, where u is understood as a class modulo N_2. For vector fields we get correspondingly the spaces $\tilde{\mathbf{W}}^q(\Omega)^n$ and $\tilde{\mathbf{W}}^q(\Omega)^n/N_2^n$, $W^{\cdot 1,q}/N_1$, N_1 is the space of constants.

We shall consider the weighted Lebesgue space

$$L_w^q(\mathbb{R}^n) = L_w^q = \Big\{ u \in L_{loc}^1(\mathbb{R}^n) : \|u\|_{q,w} = \left(\int_{\mathbb{R}^n} |u(x)|^q w(x)\,dx \right)^{1/q} < \infty \Big\},$$

where $w \in L_{loc}^1$ is a weight function. A *weight* or a *weight function* will be always an a.e. non-negative and locally integrable function. In order to apply estimates for singular integral operators, multiplier operators, the weight function w will be supposed to satisfy the Muckenhoupt A_p-condition.

Definition *Let $\mathcal{R}$ be a colection of bounded sets in $\mathbb{R}^n$. A weight function $0 \le w \in L_{loc}^1$ belongs to the Muckenhoupt class $A_q(\mathbb{R}^n, \mathcal{R})$, $1 \le q < \infty$, if there exists a constant $C > 0$ such that*

$$\sup_Q \left(\frac{1}{|R|} \int_R w(x)\,dx \right) \left(\frac{1}{|R|} \int_R w^{-1/(q-1)}\,dx \right)^{q-1} \le C < +\infty, \ \textit{for any } R \in \mathcal{R}$$

if $1 < q < \infty$, and

$$\sup_{x \in R, R \in \mathcal{R}} \frac{1}{|R|} \int_{\mathcal{R}} w(x)\,dx \le Cw(x_0), \quad \textit{for a.a. } x_0 \in \mathbb{R}^N,$$

if $q = 1$, respectively.

Lemma 2.1. *Let $\Phi : \mathbf{R}^n \to \mathbf{R}$ be continuous together with the derivative $\frac{\partial^n \phi}{\partial \xi_1 ... \partial \xi_n}$ and all preceding derivatives for $|\xi_i| > 0$, $i = 1, \dots, n$. Then, if for some $\beta \in [0, 1)$ and $M > 0$*

$$|\xi_1|^{k_1+\beta} \dots |\xi_n|^{k_n+\beta} \Big| \frac{\partial^k \phi}{\partial \xi_1^{k_1} \dots \partial \xi_n^{k_n}} \Big| \le M,$$

where k_i is zero or one and $K = \sum_{i=1}^n k_i = 0, 1, \dots, n$, the integral transform (2.2) defines a bounded linear operator from $L^q(\mathbf{R}^n)$ into $L^r(\mathbf{R}^n)$, $1 < q < \infty$, $1/r = 1/q - \beta$ and we have $\|Tu\|_r \le c\|u\|_q$, with a constant c depending only on M, r and q.

For more details see [Ga1].

Lemma 2.2 Let $1 < q < \infty$ and $w \in A_q$. Then the following statement holds true: Let $m \in C^n(\mathbb{R}^n \setminus \{0\})$ satisfy the pointwise Hörmander-Mikhlin multiplier condition

$$|\xi|^{|\alpha|}\,|D^\alpha m(\xi)| \le c_\alpha \quad \text{for all} \quad \xi \in \mathbb{R}^n \setminus \{0\}$$

and all multi-indices $\alpha \in \mathbb{N}_0^n$ with $|\alpha| \le n$. Then the multiplier operator $u \mapsto \mathcal{F}^{-1}(m\widehat{u})$, $u \in \mathcal{S}(\mathbb{R}^n)$, can be extended to a bounded linear operator from L_w^q to L_w^q.

(ii) Let $m \in C^n$ in each "quadrant" of R^n and such that $\|m\|_\infty \le B$,

$$\sup_{x_{k+1},...x_k} \int_{\mathcal{J}} \Big| \frac{\partial^k m(x)}{\partial x_1 ... \partial x_k} \Big| dx_1 ... dx_k \le B$$

for $0 < k \le n$, $\mathcal{J}$ any dyadic interavl in R^k, and any permutation of $(x_1, ..., x_n)$.If $1 < p < \infty$ and $w \in A_p(R^n, \mathcal{R}_\setminus)$ then m is a bounded multipler from $L_w^p(R^n)$ to $L_w^p(R^n)$.

Proof: see [GCRF], [Ku].

Lemma 2.3 Let $1 < p < \infty$ and let

$$\sigma_\alpha = (1 + |x|)^\alpha$$

$$\eta_{\alpha,\beta} = (1 + |x|)^\alpha (1 + s(x))^\beta$$

Then

$$\sigma_\alpha \in A_p \text{ if } -1 < \alpha < (p-1)$$

$$\eta_{\alpha,\beta} \in A_p \text{ if } -1 < \beta < p-1, -1 < \alpha+\beta < (p-1).$$

Proof: see [Ku],[KNPo],[FKN1].

3 Oseen problem in the whole space without weights

We investigate the problem

$$\begin{cases} R\tilde{v}\cdot\nabla u+\lambda g\times u-\mu\Delta u+\nabla p=Rf \\ \nabla\cdot u=h \end{cases} \tag{3.1}$$

The estimates for (3.1) will be obtained if we make replacements

$$\begin{aligned} f &\to f/\mathcal{R}, \\ p &\to p/\mathcal{R}, \\ h &\to h/\mathcal{R}, \\ x_i &\to \mathcal{R}x_i. \end{aligned}$$

Now, we investigate the problem

$$\begin{cases} \tilde{v}\cdot\nabla u+\lambda g\times u-\mu\Delta u+\nabla p=f \\ \nabla\cdot u=h \end{cases} \tag{3.2}$$

Theorem 3.1. Given $f\in W^{m,q}(\mathbf{R}^n)$, $h\in W^{m+1,q}(\mathbf{R}^n)$, $m\geq 0$, $1<q<\infty$, there exists a pair of functions v,π with $v\in D^{m+2,q}$, $\pi\in D^{m+1,q}$, for any $m>0$ and satisfying

$$\begin{cases} \frac{\partial v}{\partial x_2}-\mu\Delta v+\omega\times v+\nabla\pi=f, \\ \nabla\cdot v=h. \end{cases} \tag{3.3}$$

Moreover, for $l\in[0,m], i=1,3$ the quantitilies $|v_i|_{1,q}$ $\|v_i\|_q$, $\|\frac{\partial v}{\partial x_2}\|_{l,q}$, $|v|_{l+2,q}$, $|\pi|_{l+1,q}$ are finite and satisfy the estimate

$$\begin{aligned} &R\left\|\frac{\partial v}{\partial x_2}\right\|_{l,q}+|v|_{l+2,q}+|p|_{l+1,q}+ \\ &+\|v_i\|_{l,q}+|v_i|_{l+1,q} \\ &\leq c(R\|f\|_{l,q}+R\|h\|_{l+1,q}+\|h\|_{l,q}),\ i=1,3. \end{aligned} \tag{3.4}$$

If $1<q<4$ then $R\|\frac{\partial v_2}{\partial x_l}\|_{l,\frac{4q}{4-q}}$ is finite and

$$\begin{aligned} &R\left\|\frac{\partial v}{\partial x_2}\right\|_{l,q}+\|v_i\|_{l,q}+|v_i|_{1,q} \\ &+R^{1/4}\left\|\frac{\partial v_2}{\partial x_l}\right\|_{l,\frac{4q}{4-q}}+|v|_{l+2,q}+|\pi|_{l+1,q} \\ &\leq c(R\|f\|_{l,q}+R\|h\|_{l+1,q}+\|h\|_{l,q}),\ i=1,3. \end{aligned} \tag{3.5}$$

If $1<q<2$ then $|v_2|_{\frac{2q}{2-q}}$ is finite and

$$\begin{aligned} &(|v_i|_{l,q}+R^{1/2}|v_2|_{l,\frac{2q}{2-q}})+\left(|v_i|_{l+1,q}+\left|\frac{\partial v_2}{\partial x_l}\right|_{l,q}\right) \\ &+\left|\frac{\partial v}{\partial x_2}\right|_{l,q}+|v|_{l+2,q}+|\pi|_{l+1,q} \\ &\leq c(R|f|_{l,q}+R|h|_{l+1,q}+|h|_{l,q}),\quad i=1,3. \end{aligned} \tag{3.6}$$

Proof: We will sketch the proof.

We already have solved problem for the homogeneous Oseen problem. We can assume that $v=u+w$, $\nabla\cdot u=0$, $\nabla\cdot w=g$. Then we have to solve

$$\begin{cases} \frac{\partial w}{\partial x_2}+\omega\times w-\Delta w+\nabla\tau=0, \\ \nabla\cdot w=h. \end{cases} \tag{3.7}$$

Applying the Fourier transformation and Lizorkin theorem we get

$$\|w_i\|_r\leq c\|h\|_r,\quad i=1,3, \tag{3.8}$$

$$|w|_{1,r}\leq c\|h\|_r, \tag{3.9}$$

$$|w|_{2,r}\leq c|h|_{1,r}, \tag{3.10}$$

$$|\tau|_{1,r}\leq c\|h\|_{1,r}, \tag{3.11}$$

$$\left\|\frac{\partial w}{\partial x_2}\right\|_r\leq\|h\|_r, \tag{3.12}$$

$$\left\|\frac{\partial w}{\partial x_l}\right\|_r\leq\|h\|_r, \tag{3.13}$$

$$\|w_2\|_{\frac{3r}{3-r}}\leq\|h\|_r,\qquad 1<r<3. \tag{3.14}$$

Now, putting together with homogeneous case, we get $1<q<\infty$

$$\left\|\frac{\partial v}{\partial x_2}\right\|_q+|v|_{2,q}+|p|_{1,q}\leq C(\|f\|_q+\|h\|_{1,q}), \tag{3.15}$$

$$|u_2|_{1,s_1}\leq c(\|f\|_q+\|h\|_{s_1}), \tag{3.16}$$

$$|v_i|_{1,q}\leq c(\|f\|_q+\|h\|_{1,q}),\quad i=1,3, \tag{3.17}$$

where $s_1=\frac{4q}{4-q}$. Applying the Sobolev imbedding we get

$$\|g\|_{s_1}\leq\|h\|_{1,q}\qquad 1<q<4. \tag{3.18}$$

Then

$$\|v_2\|_{1,s_1}\leq c(\|f\|_q+\|h\|_{1,q}). \tag{3.19}$$

Finally, since

$$|v_2|_{\frac{3r}{3-r}}\leq|h|_{\frac{3r}{3-r}} \tag{3.20}$$

and let $\frac{3r}{3-r}=\frac{2q}{2-q}$, we get $r_1=\frac{6q}{6-5q}$ and

$$|v_2|_{\frac{2q}{2-q}}\leq|h|_q, \tag{3.21}$$

then

$$|v_2|_{\frac{2q}{2-q}}\leq|h|_{\frac{6q}{6-5q}}\leq c\|h\|_{1,q}. \tag{3.22}$$

4 Weighted approach in R^n

We consider the radial weight functions of the form

$$M(x) = (1+|x|)^{\alpha}, x \in \mathbf{R^n},$$

with $0 < \alpha < 1$, $n \geq 2$. Multiplying the Oseen equation (3.5) with M we obtain the following equations for Mu

$$\begin{cases} -\Delta(Mu) + \partial_2(Mu) + \nabla(Mp) + \omega \times (Mu) = \\ Mf + F(M), \\ \text{div } (Mu) = Mg + (\nabla M) \cdot g, \end{cases} \tag{4.1}$$

where

$$F(M) = -2(\nabla M)(\nabla u) - (\Delta M)u + (\partial_2 M)u + (\nabla M)p.$$

We cannot apply Theorem 3.1 directly since Mu could leave the spaces which we consider in our theorem.To avoid this problem we introduce a cut-off procedure. Let $\varphi \in C_0^{\infty}(\mathbf{R^n})$ be a function with $0 \leq \varphi \leq 1$ for $|x| \leq 1$, $\varphi(x) = 1$ for $|x| \leq 1$, $\varphi(x) = 0$ for $|x| \geq 2$ and set $\varphi_j(x) = \varphi(j^{-1}x)$, $M_j = \varphi_j M$ for all $j \in N$. Replacing M by M_j we get the main result

Theorem 4.1
Let $n = 3$, $0 < \alpha < 1$, $\alpha' > \alpha + \frac{1}{2}$, $q > 1$ such that

$$1 < \alpha + \frac{3}{q} < \frac{10}{4}$$

and put $M(x) = (1+|x|)^{\alpha}$, $M'(x) = (1+|x|)^{\alpha'}$, $x \in \mathbf{R}^n$. Assume that $f \in L^q(\mathbf{R}^n)^n, g \in W^{1,q}(\mathbf{R}^n)$ such that $\|(M'f, M'g, M'\nabla g)\|_q < \infty$, and let $(u,p) \in \tilde{\mathbf{W}}^q(\mathbf{R}^n)^n \times \mathbf{W}^{\cdot 1,q}(\mathbf{R}^n)$ be such a solution of the equations

$$-\Delta u + \partial_2 u + \nabla p + \omega \times u = f, \quad divu = g.$$

Then, after redefining modulo $N_2^n \times N_1$, (u,p) satisfies the estimate

$$\begin{cases} \|(\nabla^2(Mu), \partial_2(Mu), \nabla(Mp), Mu_j, \partial(Mu_j)\|_q \\ +\|M\nabla^2 u, M\partial_2 u, M\nabla p\|_q \leq \\ C\|(M'f, M'g, M'\nabla g\|_q, \ j = 1,3 \\ \\ \|u_2\|_{q_1} + \|\frac{\partial u_2}{\partial x_l}, p\|_{q_2} + \\ \|\nabla^2 u, \partial_2 u, \nabla p, \frac{\partial u_j}{\partial x_l}, u_j\|_{q_3} \leq \\ \\ c\|(M'f, M'g, M'\nabla g\|_q, \ j = 1,3, l = 1,3 \end{cases} \tag{4.2}$$

where $\frac{1}{q_1} = \frac{1}{q} - \frac{1}{n} + \frac{\alpha}{n}$, $\frac{1}{q_3} = \frac{1}{q_1} + \frac{2}{n+1}$, $\frac{1}{q_2} = \frac{1}{q_3} - \frac{1}{n}$, $c = c(n,q,\alpha,\alpha') > 0$ is a constant and assuming $\|(Mf, Mg, M\nabla g)\|_q + \|(f,g,\nabla g)\|_{q_4} < \infty$. we get

$$\begin{cases} \|(\nabla^2(Mu), \partial_2(Mu), \nabla(Mp))\|_q + \\ +\|(M\nabla^2 u, M\partial_2 u, M\nabla p)\|_q \leq \\ \leq C(\|(Mf, Mg, M\nabla g)\|_q + \|(f,g,\nabla g)\|_{q_4}), \end{cases} \tag{4.3}$$

where $\frac{1}{q_4} = \frac{1}{q} - \frac{1}{n} + \frac{\bar{\beta}}{n} + \frac{2}{n+1}$ with a $\bar{\beta}$ satisfying $\alpha < \bar{\beta} < \frac{10}{4} - \frac{3}{q}$ and $\frac{1}{2} + \bar{\beta} < \alpha'$ it holds.

Proof:

Since $0 < \alpha < 1$, $\alpha' > \alpha + \frac{1}{2}$, $\frac{6}{5-2\alpha} < q < \frac{3}{1-\alpha}$ it implies $\frac{1}{q_1} = \frac{1}{q} - \frac{1}{n} + \frac{\alpha}{n} > 0$, $\frac{1}{q_3} = \frac{1}{q_1} + \frac{1}{2} = \frac{1}{q_2} + \frac{1}{n} = \frac{1}{q} + \frac{1}{6} + \frac{\alpha}{3} < 1$ and $\frac{1}{q_3} < \frac{1}{q} + \frac{\alpha'}{3}$. It follows that, $1 < q_3 < q_2 < q_1 < \infty$. Setting $\alpha" = \frac{1}{2} + \alpha$ we get $\|M'^{-1}\|_{\frac{3}{\alpha"}} < \infty$ and since $\frac{1}{q_3} = \frac{1}{q} + \frac{\alpha"}{3}$.

$$\|f\|_{q_3} = \|M'^{-1} M'f\|_{q_3} \leq \|M'^{-1}\|_{\frac{3}{\alpha"}} \|M'f\|_q < \infty. \tag{4.4}$$

Similarly, $\|g\|_{q_3} < \infty$ and $\|\nabla g\|_{q_3} < \infty$. Therefore Theorem 3.2 yields a solution $(u,p) \in \widetilde{W}^{q_3}(R^3)^3 \times \dot{W}^{1,q_3}(R^3)$ of (3.1) satisfying

$$\begin{cases} \|(\nabla^2 u, \partial_2 u, \nabla p, u_j, \frac{\partial u_j}{\partial x_l})\|_{q_3} \leq c_1\|(f,g,\nabla g)\|_{q_3} \\ \leq c_2\|(M'f, M'g, M'\nabla g)\|_q. \end{cases} \tag{4.5}$$

Since $1 < q_3 < 2$ and $\frac{1}{3} < \frac{1}{q} + \frac{\alpha}{3} < \frac{1}{q} + \frac{\alpha"}{3} = \frac{1}{q_3}$, $1 < q_3 < 3$, we may use Theorem 2.2 to get (u,p) modulo $N_2^n \times N_1$ such that

$$\begin{aligned} &\|u_2\|_{q_1} \leq C\|(f,g,\nabla g)\|_{q_3}, \\ &\|\frac{\partial u_2}{\partial x_l}\|_{q_2} \leq C\|\nabla^2 u\|_{q_3}, \\ &\|p\|_{q_2} \leq C\|\nabla p\|_{q_3}. \end{aligned} \tag{4.6}$$

Then

$$\|u_2\|_{q_1} + \|(\frac{\partial u_j}{\partial x_l}, p)\|_{q_2} \leq c\|(M'f, M'g, M'\nabla g)\|_q. \tag{4.7}$$

Choosing $\bar{\beta}$ such that $\alpha < \bar{\beta} < \frac{10}{4} - \frac{3}{q}, \frac{1}{2} + \bar{\beta} < \alpha'$, $r_1 = (\frac{1}{q} - \frac{1}{n} + \frac{\bar{\beta}}{3})^{-1}$, $r_3 = (\frac{1}{r_1} + \frac{1}{2})^{-1}$, $r_2 = (\frac{1}{r_3} - \frac{1}{3})^{-1}$, $\bar{\alpha}" = \frac{1}{2} + \bar{\beta}$ correspondingly to $q_1, q_2, q_3, \alpha"$. Then the arguments above yield the same inequalities (4.5),(4.6) with q_1, q_2, q_3 replaced by $\bar{q}_1, \bar{q}_2, \bar{q}_3$, in particular $(u,p) \in \widetilde{W}^{r_3}(R^n)^n \times$

$\dot{W}^{1,\bar{r}_4}(R^n)$. Multilplying (3.1) by $M_j = \psi_j M, j \in N$, with ψ_j as above, we obtain the equations

$$\begin{cases} -\Delta(M_j u) + \partial_2(M_j u) + \nabla(M_j p) = M_j f + F(M_j), \\ div(M_j u) = M_j g + (\nabla M_j) \cdot u. \end{cases} \tag{4.8}$$

Since $M_j u, M_j p$ have compact support we may apply Theorem 3.3. First we will estimate the expressions on the right of (4.8) independently of $j \in N$. The functions ψ_j have the following elementary properties:

$$\begin{array}{l} \lim_{j\to\infty} \psi_j = 1 \text{ for all } x \in R^n, \\ \text{supp } \nabla\psi_j \subseteq \{x \in R^n; j \le |x| \le 2j\}, \\ |\nabla\psi_j(x)| \le C(1+|x|)^{-1}, \\ |\nabla^2\psi_j(x)| \le C(1+|x|)^{-2}, \end{array}$$

where c is independent of $j,\ x$. Further

$$\nabla M_j = (\nabla M)\Psi_j + M\nabla\Psi_j,$$

which gets

$$\begin{array}{l} |\nabla M_j(x)| \le C(1+(x))^{\alpha-1}, \\ |\nabla^2 M_j(x)| \le C(1+|x|)^{\alpha-2} \end{array}$$

for all $j \in N, x \in R^3$.

Hence for (4.5),(4.7) $r_1,\ r_2,\ r_3$ we get the following estimate

$$\|(\partial_2 M_j)u\|_q \le \|\partial_2 M_j\|_{(\frac{1}{n}-\frac{\bar{\beta}}{n})^{-1}} \|u\|_{r_1} \le c\|u\|_{r_1}. \tag{4.9}$$

Since $\frac{1}{q} = (\frac{1}{3} - \frac{\bar{\beta}}{3}) + \frac{1}{r_1}$ and $\sup_j \|\partial_2 M_j\|_{(\frac{1}{3}-\frac{\bar{\beta}}{3})^{-1}} < \infty$ it follows

$$\|(\partial_2 M_j)u\|_q \le C\|(M'f, M'g, M'\nabla g)\|_q, \tag{4.10}$$

with C not depending on j. Further we get

$$\begin{array}{l} \|(\nabla M_j)(\nabla u)\|_q \le \|\nabla M_j\|_{\left(\frac{1}{n}-\frac{\beta"}{n}\right)^{-1}} \|\nabla u\|_{r_2} \\ \le C_1\|\nabla u\|_{r_2} \le C_2\|(M'f, M'g, M'\nabla g)\|_q. \end{array} \tag{4.11}$$

Similarly, we obtain

$$\|(\nabla M_j)p\|_q \le C(\|M'f, M'g, M'\nabla g)\|_q,$$

$$\|(\Delta M_j)u\|_q \le C\|M'f, M'g, M'\nabla g)\|_q,$$

$$\|(\nabla M_j)u\|_q \le C\|(M'f, M'g, M'\nabla g)\|_q,$$

and

$$\|\nabla \text{div}(M_j u)\|_q \le C\|(M'f, M'g, M'\nabla g)\|_q.$$

Applying Theorem 3.2. and using (4.8) from the last estimates we obtain

$$\begin{array}{l} \|(\nabla^2 M_j u), \partial_2(M_j u), \nabla(M_j p))\|_q \le \\ \le C_1(\|(Mf, Mg, M\nabla g)\|_q + \\ \|(M'f, M'g, M'\nabla g)\|_q), \end{array} \tag{4.12}$$

with C_1 not depending on j. Therefore $\|(\nabla^2(M_j u), \partial_2(M_j u), \nabla(M_j)\|_q$ is bounded uniformly is $j \in N$ and using weak convergence properties for $j \to \infty$ yields

$$\begin{array}{l} \|(\nabla^2(Mu), \partial_2(Mu), \\ \nabla(M_p)\|_q \le C\|(M'f, M'g, M'\nabla g)\|_q. \end{array} \tag{4.13}$$

Moreover the above estimates leads to

$$\begin{array}{l} \|(M_j\nabla^2 u, M_j\partial_2 u, M_j\nabla p)\|_q \\ \le C_1(\|(\nabla^2(M_j u), \\ \partial_2(M_j u), \nabla(M_j p)\|_q + \\ +\|((\nabla M_j)(\nabla u), \\ (\nabla^2 M_j)u, (\partial_2 M_j)u, (\nabla M_j)p)\|_q) \\ \le c_2\|(M'f, M'g, M'\nabla g)\|_q, \end{array} \tag{4.14}$$

for all $j \in N$; therefore we get

$$\|(M\nabla^2 u, M\partial_2 u, M\nabla p)\|_q \le C\|(M'f, M'g, M'\nabla g)\|_q. \tag{4.15}$$

Using (4.5), (4.7), (4.13) we get (4.2).

We consider homogeneous Oseen problem

$$\begin{cases} R\tilde{v} \cdot \nabla u + \lambda g \times u - \mu\Delta u + \nabla p = Rf \\ \nabla \cdot u = 0 \end{cases} \tag{4.16}$$

Theorem 4.1. *Given $f \in L^q_w(\mathbf{R}^n)$, $1 < q < \infty$, with $w = \sigma_\alpha, \eta_{\alpha,\beta}$ there exists a pair of functions (u,p) with $u \in D^{2,q}$, $\nabla p \in L^q$, $u_1, u_3 \in L^q$, $\frac{\partial u_1}{\partial x_2}$, $\frac{\partial u_2}{\partial x_2}$, $\frac{\partial u_3}{\partial x_2} \in L^q$, satisfying (4.16) and moreover*

$$\begin{array}{l} R\left\|\frac{\partial u}{\partial x_2}\right\|_{q,w} + R\left\|\frac{\partial u_1}{\partial x_l}\right\|_{q,w} + R\left\|\frac{\partial u_3}{\partial x_l}\right\|_{q,w} + \\ +\|u_1\|_{q,w} + \|u_3\|_{q,w} + |\nabla p|_{q,w} + |\Delta u|_{q,w} \\ \le c_1 R\|f\|_{q,w}. \end{array} \tag{4.17}$$

Proof: *It follows Theorem 3.1, Lemma 2 and from results [FKN1].*

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