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A Flow through a Cascade of Profiles with Natural Boundary Conditions Involving Bernoulli's Pressure on the Outflow

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Abstract: - The paper deals with the analysis of the model of incompressible, viscous, stationary flow through a plane cascade of profiles. We present a classical as well as weak formulation of the problem and prove the existence of a weak solution. We consider the Navier–Stokes equation with Bernoulli's pressure and the corresponding "do nothing" boundary condition on the outflow.

Key-Words: - Navier–Stokes equations, cascade of profiles , weak solution

1 Introduction

The plane flow through an infinite cascade of profiles represents one of the most important mathematical models of real 3D flows through turbines, compressors, pumps and other similar devices. The theory and numerical methods for such flows are already relatively deeply elaborated. In this context, the potential theory and methods of analysis of functions of complex variable were extensively used by E. Meister [15] and M. Feistauer [2]. The well–known Martensen's method, see e.g. [14] and [16], applies results from the theory of integral equations to the solution of an inviscid, irrotational and incompressible flow through a plane cascade of profiles.

The modelling of viscous incompressible flows through a 2D cascade represents a complicated theoretical problem especially due to the variety of boundary conditions. While the boundary conditions on the inflow and on a profile are of the Dirichlet type, the reduction of the problem to one space period leads to a condition of a space–periodicity on another part of the boundary and finally, a different boundary condition is reasonable on the outflow. From the point of the situation on the outlet, the flow through a cascade has similar features as a flow through a channel. J. Heywood, R. Rannacher and S. Turek [7] explicitly did not involve any boundary condition on the outflow into the weak formulation and by means of a backward integration by parts have shown that this induces the so called "do nothing" boundary condition

$$-\nu \frac{\partial \boldsymbol{u}}{\partial \boldsymbol{n}} + p\boldsymbol{n} = \boldsymbol{0}. \qquad (1)$$

Here $\boldsymbol{u} = (u_1, u_2)$ is the velocity, p is the kinematic pressure and $\boldsymbol{n}$ denotes the outer normal to the boundary. However, this approach causes difficulties in attempts to prove the existence of a weak solution because condition (1) does not exclude a backward flow on the assumed outlet and the backward flow can eventually bring back to the channel a non–controlable amount of kinetic energy. Thus, the energy estimate breaks down. This problem can be avoided by appropriate tricks: S. Kračmar and J. Neustupa in [8] and [9] prescribed an additional boundary condition which restricted the kinetic energy brought back on the outflow and they have therefore described and solved the problem by means of variational inequalities of the Navier–Stokes type. P. Kučera and Z. Skalák solved the problem for "small" data, see [11], [10]. In our previous paper [3], we have subtracted the term $\frac{1}{2}(\boldsymbol{u}\cdot\boldsymbol{n})^-\boldsymbol{u}$ (where the superscript $-$ denotes the negative part) from the left–hand side of (1) and we obtained the boundary condition

$$-\nu \frac{\partial \boldsymbol{u}}{\partial \boldsymbol{n}} + p\boldsymbol{n} - \frac{1}{2}(\boldsymbol{u}\cdot\boldsymbol{n})^-\boldsymbol{u} = \boldsymbol{h}. \qquad (2)$$

This condition also enables us to restrict the kinetic energy brought back by the backward flow on the outlet and consequently, to derive the energy estimate and to prove the existence of a weak solution.

In this paper, we present a different approach. We study the steady problem and use the notation

$$\omega(\boldsymbol{u}) = \frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2} \qquad (3)$$

$$q = p + \frac{1}{2}|\boldsymbol{u}|^2 \qquad (4)$$

where $\omega(\boldsymbol{u})$ is the vorticity of the flow and q is the so called Bernoulli pressure. We write the 2D Navier–Stokes system in the form

$$-\,\omega(\boldsymbol{u})\,u_2 \;=\; -\frac{\partial q}{\partial x_1}+\nu\Delta u_1+f_1\,, \tag{5}$$

$$\omega(\boldsymbol{u})\,u_1 \;=\; -\frac{\partial q}{\partial x_2}+\nu\Delta u_2+f_2\,. \tag{6}$$

If we further denote $\boldsymbol{f}=(f_1,f_2)$ and

$$\boldsymbol{u}^{\perp} \;=\; (-u_2,u_1), \tag{7}$$

we can write the system (5), (6) as one vector equation:

$$\omega(\boldsymbol{u})\,\boldsymbol{u}^{\perp} \;=\; -\nabla q+\nu\Delta\boldsymbol{u}+\boldsymbol{f}. \tag{8}$$

This equation will always be accompanied by the condition of incompressibility

$$\operatorname{div}\boldsymbol{u} \;=\; 0. \tag{9}$$

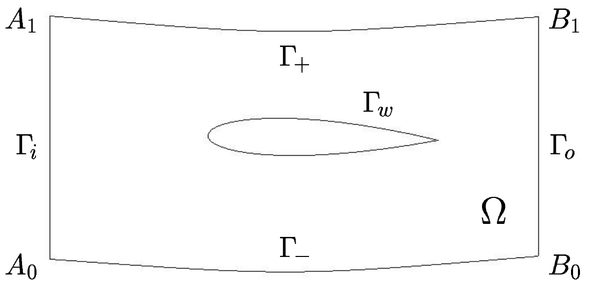

Figure 1: Domain Ω

The geometric configuration of the flow field is apparent from Fig. 1. Our basic domain, which in fact represents one spatial period of the flow field in the profile cascade, is denoted by Ω. The boundary condition used on the inflow Γ_i is a non–homogeneous Dirichlet condition

$$\boldsymbol{u}\,|_{\Gamma_i} \;=\; \boldsymbol{g}, \tag{10}$$

where $\boldsymbol{g}$ represents the known distribution of the velocity. The boundary condition used on the profile Γ_w is the usual no–slip condition

$$\boldsymbol{u}\,|_{\Gamma_w} \;=\; \boldsymbol{0}. \tag{11}$$

Furthermore, we consider the conditions of periodicity

$$\boldsymbol{u}(x_1,x_2+\tau) \;=\; \boldsymbol{u}(x_1,x_2) \tag{12}$$

$$\frac{\partial\boldsymbol{u}}{\partial\boldsymbol{n}}(x_1,x_2+\tau) \;=\; -\frac{\partial\boldsymbol{u}}{\partial\boldsymbol{n}}(x_1,x_2) \tag{13}$$

$$q(x_1,x_2+\tau) \;=\; q(x_1,x_2) \tag{14}$$

for $\boldsymbol{x}\equiv(x_1,x_2)$ on the artificial boundary Γ_-. (The points $(x_1,x_2+\tau)$, for $(x_1,x_2)\in\Gamma_-$, form the curve Γ_+.) The condition used on the outflow Γ_o will naturally arise similarly as the "do nothing" (1) from the weak formulation which will be given in the next section. However, we can note that if the weak solution $\boldsymbol{u}$ is "smooth enough" then it will satisfy

$$-\nu\,\frac{\partial\boldsymbol{u}}{\partial\boldsymbol{n}}+q\boldsymbol{n} \;=\; \boldsymbol{h} \tag{15}$$

on Γ_o. By analogy with [3], it can be shown that a classical (respectively strong) solution of the problem (8)–(14), extended periodically in the direction x_2 with the period τ, is a classical (respectively strong) solution in the infinite and unbounded (in the direction of x_2) cascade of profiles.

2 Weak formulation

We shall use the following function spaces and notation.

– $(.\,,\,.)_0$ is the scalar product of scalar–valued (respectively vector–valued, respectively tensor–valued) functions in $L^2(\Omega)$ (respectively in $L^2(\Omega)^2$, respectively in $L^2(\Omega)^4$).

– $H^1(\Omega)$ is the usual Sobolev space with the scalar product

$$(u,v)_1 \;=\; \int_\Omega (uv+\nabla u\cdot\nabla v)\,\mathrm{d}\boldsymbol{x}.$$

– $H^1(\Omega)^2 \;=\; H^1(\Omega)\times H^1(\Omega)$, the space of vector functions with the scalar product

$$(\boldsymbol{u},\boldsymbol{v})_1 \;=\; \sum_{i=1}^{2}\,(u_i,v_i)_1$$

where $\boldsymbol{u}=(u_1,u_2)$, $\boldsymbol{v}=(v_1,u_2)\in H^1(\Omega)^2$.

– $\mathcal{X}=\{\boldsymbol{v}\in C^\infty(\overline{\Omega})^2;\ \boldsymbol{v}=\boldsymbol{0}$ on $\Gamma_i\cup\Gamma_w$, $\boldsymbol{v}(x_1,x_2+\tau)=\boldsymbol{v}(x_1,x_2)\;\forall\,(x_1,x_2)\in\Gamma_-\}$,

– $\mathcal{V}=\{\boldsymbol{v}\in\mathcal{X};\ \operatorname{div}\boldsymbol{v}=0 \text{ in } \Omega\}$,

– X is the closure of $\mathcal{X}$ in $H^1(\Omega)^2$,

– V is the closure of $\mathcal{V}$ in $H^1(\Omega)^2$.

By the same procedure as in the proof of Theorem 6.6.4 in [12], we can show that

$$X \;=\; \{\boldsymbol{v}\in H^1(\Omega)^2;\ \boldsymbol{v}=\boldsymbol{0} \text{ in } \Gamma_i\cup\Gamma_w,$$
$$\boldsymbol{v}(x_1,x_2+\tau)=\boldsymbol{v}(x_1,x_2) \text{ for } (x_1,x_2)\in\Gamma_-\}.$$

The identities on Γ_i, Γ_w and Γ_- are interpreted in the sense of traces. Similarly as in [4], pages 142–143, we can prove that

$$V \;=\; \{\boldsymbol{v} \in X;\ \operatorname{div} \boldsymbol{v} = 0 \text{ a.e. in } \Omega\}.$$

In the space V, we shall use the norm $|||\,.\,|||$ induced by the scalar product

$$(\boldsymbol{u}, \boldsymbol{v})_V = (\nabla \boldsymbol{u}, \nabla \boldsymbol{v})_0\,. \tag{16}$$

It can be shown that the norm $|||\,.\,|||$ is equivalent with the norm $\|\,.\,\|_1$ in V.

In order to derive formally the weak formulation of the problem (8)–(14), we multiply equation (8) by an arbitrary test function $\boldsymbol{v} = (v_1, v_2) \in V$, integrate over Ω and apply Green's theorem and use the boundary conditions and the periodicity (10)–(14). We finally arrive at the equation

$$a(\boldsymbol{u}, \boldsymbol{v}) \;=\; (\boldsymbol{f}, \boldsymbol{v})_0 + b(\boldsymbol{h}, \boldsymbol{v}), \tag{17}$$

where

$$\begin{aligned}
a_1(\boldsymbol{u}, \boldsymbol{v}) &= (\nabla \boldsymbol{u}, \nabla \boldsymbol{v})_0,\\
a_2(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}) &= \int_\Omega \omega(\boldsymbol{u})\, \boldsymbol{v}^\perp \cdot \boldsymbol{w}\; \mathrm{d}\boldsymbol{x},\\
a(\boldsymbol{u}, \boldsymbol{v}) &= a_1(\boldsymbol{u}, \boldsymbol{v}) + a_2(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{v}),\\
b(\boldsymbol{h}, \boldsymbol{v}) &= -\int_{\Gamma_o} \boldsymbol{h} \cdot \boldsymbol{v}\; \mathrm{d}S.
\end{aligned}$$

All these forms are defined for $\boldsymbol{u},\ \boldsymbol{v},\ \boldsymbol{w} \in H^1(\Omega)^2$, $\boldsymbol{f} \in L^2(\Omega)^2$ and $\boldsymbol{h} \in L^2(\Gamma_o)^2$.

Now the weak problem reads as follows:

Definition 1 *Let the function $\boldsymbol{g} \in H^s(\Gamma_i)^2$ (for some $s \in (\frac{1}{2}, 1]$) satisfy the condition $\mathbf{g}(A_1) = \mathbf{g}(A_0)$. (Recall that A_0 and A_1 are the end points of Γ_i.) Let $\boldsymbol{f} \in L^2(\Omega)^2$ and $\boldsymbol{h} \in L^2(\Gamma_o)^2$. The <u>weak solution</u> of the problem (8)–(14) is a vector function $\boldsymbol{u} \in H^1(\Omega)^2$ which satisfies the identity (17) for all test functions $\boldsymbol{v} \in V$, the condition of incompressibility (9) a.e. in Ω, the boundary conditions (10), (11) in the sense of traces on Γ_i and Γ_w and the condition of periodicity (12) in the sense of traces on Γ_- and Γ_+.*

The pressure term q does not explicitly appear in the definition of the weak solution, however, as it is usual in the theory of the Navier–Stokes equations, it can be defined on the level of distributions or it can be recovered as a function from $H^1(\Omega)$, if the weak solution is sufficiently regular.

We shall further need a suitable extension of the given function $\boldsymbol{g}$ from Γ_i onto the whole domain Ω in the weak formulation as well as in the proof of existence of the weak solution. In the first step, we prolong $\boldsymbol{g}$ onto the whole boundary $\partial\Omega$. The possibility of such a prolongation is guaranteed by the next lemma.

Lemma 2 *There exists an extension of function $\boldsymbol{g}$ from Γ_i onto $\partial\Omega$ (we shall denote the extension again by $\boldsymbol{g}$) such that it belongs to $H^{1/2}(\partial\Omega)^2$ and*

$$\int_{\partial\Omega} \boldsymbol{g} \cdot \mathbf{n}\; \mathrm{d}S \;=\; 0. \tag{18}$$

Moreover, there exists a constant $c_1 > 0$ independent of $\boldsymbol{g}$ such that

$$\|\boldsymbol{g}\|_{1/2;\,\partial\Omega} \;\leq\; c_1\, \|\boldsymbol{g}\|_{s;\,\Gamma_i}\,. \tag{19}$$

The proof can be found in [3]. The norms in (19) are the norms in the Sobolev–Slobodetski spaces $H^s(\Gamma_i)^2$ and $H^{1/2}(\partial\Omega)^2$. The next lemma, which is also taken from [3], shows that $\boldsymbol{g}$ can be extended from $\partial\Omega$ to Ω.

Lemma 3 *A function $\boldsymbol{g} \in H^{1/2}(\partial\Omega)^2$ which satisfies (18) can be extended to a function $\boldsymbol{g}^* \in H^1(\Omega)^2$ such that*

$$\begin{aligned}
\boldsymbol{g}^*|_{\partial\Omega} &= \boldsymbol{g} \quad \textit{(in the sense of traces)}, && (20)\\
\operatorname{div} \boldsymbol{g}^* &= 0 \quad \text{in } \Omega, && (21)\\
\|\boldsymbol{g}^*\|_1 &\leq c_2\, \|\mathbf{g}\|_{1/2;\,\partial\Omega} && (22)
\end{aligned}$$

where the constant $c_2 > 0$ is independent of $\boldsymbol{g}$.

Now we shall construct the weak solution $\boldsymbol{u}$ in the form $\boldsymbol{u} = \boldsymbol{g}^* + \boldsymbol{z}$ where $\boldsymbol{z} \in V$ is a new unknown function. This form of $\boldsymbol{u}$ guarantees that $\boldsymbol{u}$ satisfies the equation (9) and the boundary and periodicity conditions (10)–(12). Substituting $\boldsymbol{u} = \boldsymbol{g}^* + \boldsymbol{z}$ into the equation (17), we get the following problem: *Find a function $\boldsymbol{z} \in V$ such that it satisfies the equation*

$$a(\boldsymbol{g}^* + \boldsymbol{z}, \boldsymbol{v}) \;=\; (\boldsymbol{f}, \boldsymbol{v})_0 + b(\boldsymbol{h}, \boldsymbol{v}) \tag{23}$$

for all $\boldsymbol{v} \in V$.

3 Estimates of the form $a(\boldsymbol{u}, \boldsymbol{v})$

The next two lemmas will give a sufficient condition for coercivity of the form a.

Lemma 4 *There exist positive constants c_3 and c_4 such that*

$$\begin{aligned}
a(\boldsymbol{g}^* + \boldsymbol{z}, \boldsymbol{z}) \;\geq\; |||\boldsymbol{z}|||\, \Big(\nu\, |||\boldsymbol{z}||| - \nu\, c_2\, c_1\, \|\boldsymbol{g}\|_{s;\,\Gamma_i}&\\
-c_3\, \|\boldsymbol{g}\|^2_{s;\,\Gamma_i} - c_4\, \|\boldsymbol{g}\|_{s;\,\Gamma_i}\, |||\boldsymbol{z}|||\Big)&
\end{aligned} \tag{24}$$

for all $\boldsymbol{z} \in V$.

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Proof. Using the definitions of the forms a, a_1 and a_2, we find that

$$a(\boldsymbol{g}^* + \boldsymbol{z}, \boldsymbol{z}) = a_1(\boldsymbol{z}, \boldsymbol{z}) + a_1(\boldsymbol{g}^*, \boldsymbol{z}) + a_2(\boldsymbol{g}^*, \boldsymbol{g}^*, \boldsymbol{z}) + a_2(\boldsymbol{g}^*, \boldsymbol{z}, \boldsymbol{z}) + a_2(\boldsymbol{z}, \boldsymbol{g}^*, \boldsymbol{z}) + a_2(\boldsymbol{z}, \boldsymbol{z}, \boldsymbol{z}).$$

Since $\boldsymbol{z}^\perp \cdot \boldsymbol{z} = 0$ in Ω, the terms $a_2(\boldsymbol{g}^*, \boldsymbol{z}, \boldsymbol{z})$ and $a_2(\boldsymbol{z}, \boldsymbol{z}, \boldsymbol{z})$ vanish. Hence,

$$a(\boldsymbol{g}^* + \boldsymbol{z}, \boldsymbol{z}) \geq a_1(\boldsymbol{z}, \boldsymbol{z}) - |a_1(\boldsymbol{g}^*, \boldsymbol{z})| - |a_2(\boldsymbol{g}^*, \boldsymbol{g}^*, \boldsymbol{z})| - |a_2(\boldsymbol{z}, \boldsymbol{g}^*, \boldsymbol{z})|. \quad (25)$$

We obviously have

$$a_1(\boldsymbol{z}, \boldsymbol{z}) = \nu(\nabla \boldsymbol{z}, \nabla \boldsymbol{z})_0 \geq \nu \, |||\boldsymbol{z}|||^2. \quad (26)$$

Let us further estimate the terms on the right-hand side of (25). If we use the Cauchy inequality, the continuous imbedding of $H^1(\Omega)$ into $L^4(\Omega)$, Green's theorem and the theorem on traces, we successively obtain

$$\begin{aligned} |a_1(\boldsymbol{g}^*, \boldsymbol{z})| &= \nu\,(\nabla \boldsymbol{g}^*, \nabla \boldsymbol{z})_0 \\ &\leq \nu \, \|\boldsymbol{g}^*\|_1 \, |||\boldsymbol{z}|||, \end{aligned} \quad (27)$$

$$\begin{aligned} |a_2(\boldsymbol{g}^*, \boldsymbol{g}^*, \boldsymbol{z})| &= \left| \int_\Omega \omega(\boldsymbol{g}^*) \, \boldsymbol{g}^{*\perp} \cdot \boldsymbol{z} \, \mathrm{d}\boldsymbol{x} \right| \\ &\leq \|\omega(\boldsymbol{g}^*)\|_0 \, \|\boldsymbol{g}^*\|_{L^4} \, \|\boldsymbol{z}\|_{L^4} \\ &\leq c_5 \, \|\boldsymbol{g}^*\|_1^2 \, |||\boldsymbol{z}|||, \end{aligned} \quad (28)$$

$$\begin{aligned} |a_2(\boldsymbol{z}, \boldsymbol{g}^*, \boldsymbol{z})| &= \left| \int_\Omega \omega(\boldsymbol{z}) \, \boldsymbol{g}^{*\perp} \cdot \boldsymbol{z} \, \mathrm{d}\boldsymbol{x} \right| \\ &\leq \|\omega(\boldsymbol{z})\|_0 \, \|\boldsymbol{g}^*\|_{L^4} \, \|\boldsymbol{z}\|_{L^4} \\ &\leq c_6 \, |||\boldsymbol{z}|||^2 \, \|\boldsymbol{g}^*\|_1, \end{aligned} \quad (29)$$

Substituting (26)–(29) into (25) and using (19), (22), we get

$$\begin{aligned} a(\boldsymbol{g}^* + \boldsymbol{z}, \boldsymbol{z}) &\geq \nu \, |||\boldsymbol{z}|||^2 - \nu \, \|\boldsymbol{g}^*\|_1 \, |||\boldsymbol{z}||| \\ &\quad - c_5 \, \|\boldsymbol{g}^*\|_1^2 \, |||\boldsymbol{z}||| - c_6 \, \|\boldsymbol{g}^*\|_1 \, |||\boldsymbol{z}|||^2 \\ &\geq |||\boldsymbol{z}||| \Big(\nu \, |||\boldsymbol{z}||| - \nu \, c_1 \, c_2 \, \|\boldsymbol{g}\|_{s;\Gamma_i} - c_5 \, c_1^2 \, c_2^2 \, \|\boldsymbol{g}\|_{s;\Gamma_i}^2 \\ &\quad - c_6 \, c_1 \, c_2 \, \|\boldsymbol{g}\|_{s;\Gamma_i} \, |||\boldsymbol{z}||| \Big). \end{aligned} \quad (30)$$

This completes the proof. □

Lemma 5 *There exists $\epsilon > 0$ such that if*

$$\|\boldsymbol{g}\|_{s;\Gamma_i} < \epsilon, \quad (31)$$

then the form $a(\boldsymbol{g}^* + \boldsymbol{z}, \boldsymbol{z})$ is coercive on the space V. It means that

$$\lim_{|||\boldsymbol{z}||| \to +\infty} a(\boldsymbol{g}^* + \boldsymbol{z}, \boldsymbol{z}) = +\infty. \quad (32)$$

Proof. Lemma 4 implies that it is sufficient to choose $\epsilon = \nu / c_4$. □

4 Construction of approximations

The existence of a weak solution will be proven by the Galerkin method.

The space V is a separable Hilbert space. By analogy with [2] or [17], we shall use a basis $\{\mathbf{e}_i\}_{i=1}^\infty$ in V which consists of elements from $\mathcal{V}$ and which is orthonormal with respect to the scalar product $(.\,,\,.)_V$. For any $n \in \mathbb{N}$ we set

$$V_n := \mathcal{L}\{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\},$$

i.e. the linear hull of the functions $\mathbf{e}_1$, $\mathbf{e}_2$, …, $\mathbf{e}_n$. The approximate solution of problem (23) will be constructed as an element of V_n:

$$\boldsymbol{z}_n = \sum_{k=1}^n \vartheta_k \, \mathbf{e}_k \,. \quad (33)$$

If we set $\vartheta = (\vartheta_1, \ldots, \vartheta_n)$ and

$$|\vartheta| = \left(\sum_{k=1}^n \vartheta_k^2 \right)^{1/2},$$

then

$$|||\boldsymbol{z}_n||| = \left(\sum_{k,l=1}^n \vartheta_k \, \vartheta_l \, (\mathbf{e}_k, \mathbf{e}_l)_V \right)^{1/2} = |\vartheta|. \quad (34)$$

The approximate solution $\boldsymbol{z}_n$ can now be searched for as an element of V_n which satisfies

$$a(\boldsymbol{g}^* + \boldsymbol{z}_n, \boldsymbol{v}) = (\boldsymbol{f}, \boldsymbol{v})_0 + b(\boldsymbol{h}, \boldsymbol{v}) \quad (35)$$

for all $\boldsymbol{v} \in V_n$. The problem (35) is equivalent to the system

$$a(\boldsymbol{g}^* + \boldsymbol{z}_n, \mathbf{e}_k) = (\boldsymbol{f}, \mathbf{e}_k)_0 + b(\boldsymbol{h}, \mathbf{e}_k) \quad (36)$$

for $k = 1, 2, \ldots, n$. Expressing the form a by means of a_1 and a_2, (36) can be rewritten in the form

$$\begin{aligned} a_1(\boldsymbol{g}^* + \boldsymbol{z}_n, \mathbf{e}_k) + a_2(\boldsymbol{g}^* + \boldsymbol{z}_n, \boldsymbol{g}^* + \boldsymbol{z}_n, \mathbf{e}_k) \\ = (\boldsymbol{f}, \mathbf{e}_k)_0 + b(\boldsymbol{h}, \mathbf{e}_k) \end{aligned} \quad (37)$$

for $k = 1, 2, \ldots, n$. Substituting (33) into (37), we obtain

$$\begin{aligned} a_1(\boldsymbol{g}^*, \mathbf{e}_k) + \sum_{l=1}^n \vartheta_l \, a_1(\mathbf{e}_l, \mathbf{e}_k) + a_2(\boldsymbol{g}^*, \boldsymbol{g}^*, \mathbf{e}_k) \\ + \sum_{l=1}^n \vartheta_l \, [a_2(\boldsymbol{g}^*, \mathbf{e}_l, \mathbf{e}_k) + a_2(\mathbf{e}_l, \boldsymbol{g}^*, \mathbf{e}_k)] \\ + \sum_{l,m=1}^n \vartheta_l \vartheta_m \, a_2(\mathbf{e}_l, \mathbf{e}_m, \mathbf{e}_k) - (\boldsymbol{f}, \mathbf{e}_k)_0 \\ - b(\boldsymbol{h}, \mathbf{e}_k) = 0 \end{aligned} \quad (38)$$

for $k = 1, 2, \ldots, n$. This is a system of n quadratic equations for the unknowns $\vartheta_1, \vartheta_2, \ldots, \vartheta_n$. If we denote by $A_k(\vartheta)$ the left–hand side of the k–th equation and set

$$A(\vartheta) \;=\; (A_1(\vartheta),\, \ldots,\, A_n(\vartheta)), \tag{39}$$

the system (38) can be written as one equation

$$A(\vartheta) \;=\; O \tag{40}$$

in $\mathbb{R}^n$. (O denotes the zero element of $\mathbb{R}^n$.) We shall further use the next lemma.

Lemma 6 *Let A be a continuous mapping of $\mathbb{R}^n$ into $\mathbb{R}^n$. If there exists $R > 0$ such that*

$$A(\vartheta) \cdot \vartheta \;\geq\; 0 \tag{41}$$

for all $\vartheta \in \mathbb{R}^n$ in the sphere with the radius R, then the equation

$$A(\vartheta) \;=\; O \tag{42}$$

has at least one solution $\vartheta \in \mathbb{R}^n$ such that $|\vartheta| \leq R$.

The proof of the lemma can be found in [2] (Lemma 4.1.53) or in [17] (Lemma II.1.4).

Obviously, our mapping A, defined by (39), maps continuously $\mathbb{R}^n$ into $\mathbb{R}^n$. Using Lemma 4 and the theorem on traces, we can successively express and estimate the scalar product $A(\vartheta) \cdot \vartheta$ in this way:

$$\begin{aligned}
A(\vartheta) \cdot \vartheta \;=\; & \sum_{k=1}^{n} A_k(\vartheta)\, \vartheta_k \;=\; \sum_{k=1}^{n} \vartheta_k\, a_1(\boldsymbol{g}^*, \mathbf{e}_k) \\
& + \sum_{k,l=1}^{n} \vartheta_k\, \vartheta_l\, a_1(\mathbf{e}_l, \mathbf{e}_k) + \sum_{k=1}^{n} \vartheta_k\, a_2(\boldsymbol{g}^*, \boldsymbol{g}^*, \mathbf{e}_k) \\
& + \sum_{k,l=1}^{n} \vartheta_k\, \vartheta_l\, [a_2(\boldsymbol{g}^*, \mathbf{e}_l, \mathbf{e}_k) + a_2(\mathbf{e}_l, \boldsymbol{g}^*, \mathbf{e}_k)] \\
& + \sum_{k,l,m=1}^{n} \vartheta_k\, \vartheta_l\, \vartheta_m\, a_2(\mathbf{e}_l, \mathbf{e}_m, \mathbf{e}_k) \\
& - \sum_{k=1}^{n} \vartheta_k\, (\mathbf{f}, \mathbf{e}_k)_0 - \sum_{k=1}^{n} \vartheta_k\, b(\mathbf{h}, \mathbf{e}_k) \\
= \;\; & a_1(\boldsymbol{g}^*, \boldsymbol{z}_n) + a_1(\boldsymbol{z}_n, \boldsymbol{z}_n) + a_2(\boldsymbol{g}^*, \boldsymbol{g}^*, \boldsymbol{z}_n) \\
& + a_2(\boldsymbol{z}_n, \boldsymbol{g}^*, \boldsymbol{z}_n) - (\boldsymbol{f}, \boldsymbol{z}_n)_0 - b(\boldsymbol{h}, \boldsymbol{z}_n) \\
= \;\; & a(\boldsymbol{g}^* + \boldsymbol{z}_n, \boldsymbol{z}_n) - (\boldsymbol{f}, \boldsymbol{z}_n)_0 - b(\boldsymbol{h}, \boldsymbol{z}_n) \\
\geq \;\; & |||\boldsymbol{z}_n|||\, \Big(\nu\, |||\boldsymbol{z}_n||| - \nu\, c_2\, c_1\, \|\boldsymbol{g}\|_{s;\Gamma_i} \\
& - c_3\, \|\boldsymbol{g}\|_{s;\Gamma_i}^2 - c_4\, \|\boldsymbol{g}\|_{s;\Gamma_i}\, |||\boldsymbol{z}_n||| \Big) \\
& - c_7\, \|\boldsymbol{f}\|_0\, |||\boldsymbol{z}_n||| - c_8\, \|\boldsymbol{h}\|_{0;\Gamma_o}\, |||\boldsymbol{z}_n|||.
\end{aligned} \tag{43}$$

Now, if $\boldsymbol{g}$ is so small that it satisfies (31) with $\epsilon = \nu/c_4$, then the right hand side of (43) is greater than or equal to zero for sufficiently large $|\vartheta|$, namely, for $|\vartheta| = R_0$, where

$$\begin{aligned}
R_0 \;=\; & \Big[\nu\, c_2\, c_1\, \|\boldsymbol{g}\|_{s;\Gamma_i} + c_3\, \|\boldsymbol{g}\|_{s;\Gamma_i}^2 + c_7\, \|\boldsymbol{f}\|_0^2 \\
& + c_8\, \|\boldsymbol{h}\|_{0;\Gamma_o} \Big] \Big/ \Big[\nu - c_4\, \|\boldsymbol{g}\|_{s;\Gamma_i} \Big].
\end{aligned} \tag{44}$$

Lemma 6 implies that equation (42) has at least one solution ϑ with $|\vartheta| \leq R_0$. The function $\boldsymbol{z}_n$ given by (33) is the sought approximate solution of the problem (23). Due to (34),

$$|||\boldsymbol{z}_n||| \;\leq\; R_0. \tag{45}$$

Thus, we have proven the following lemma.

Lemma 7 *Let us assume that $\boldsymbol{g}$ is so small that it satisfies (31) with $\epsilon = \nu/c_4$. Let $n \in \mathbb{N}$. Then the problem (35) has a solution $\boldsymbol{z}_n \in V_n$ satisfying the estimate (45).*

We shall further always assume that $\epsilon = \nu/c_4$ and $\boldsymbol{g}$ satisfies (31).

5 Convergence of approximate solutions

Since the space V is reflexive, the boundedness of the sequence $\{\boldsymbol{z}_n\}$ implies that there exists a subsequence (for the sake of simplicity denoted again by $\{\boldsymbol{z}_n\}$) and an element $\boldsymbol{z} \in V$ such that

$$\boldsymbol{z}_n \;\longrightarrow\; \boldsymbol{z} \quad \text{weakly in } V. \tag{46}$$

The norm in V is equivalent to the norm of the space $H^1(\Omega)^2$, hence

$$\boldsymbol{z}_n \;\longrightarrow\; \boldsymbol{z} \quad \text{weakly in } H^1(\Omega)^2. \tag{47}$$

The space $H^1(\Omega)^2$ is compactly imbedded into $L^q(\Omega)^2$ for every $q \geq 1$. This implies that

$$\boldsymbol{z}_n \;\longrightarrow\; \boldsymbol{z} \quad \text{strongly in } L^q(\Omega)^2 \tag{48}$$

for every $q \geq 1$.

6 The limit transition

The equation (35) can be written in the form

$$\begin{aligned}
& a_1(\boldsymbol{g}^*, \boldsymbol{v}) + a_1(\boldsymbol{z}_n, \boldsymbol{v}) + a_2(\boldsymbol{g}^*, \boldsymbol{g}^*, \boldsymbol{v}) \\
& \quad + a_2(\boldsymbol{z}_n, \boldsymbol{g}^*, \boldsymbol{v}) + a_2(\boldsymbol{g}^*, \boldsymbol{z}_n, \boldsymbol{v}) \\
& \quad + a_2(\boldsymbol{z}_n, \boldsymbol{z}_n, \boldsymbol{v}) \;=\; (\boldsymbol{f}, \boldsymbol{v})_0 + b(\boldsymbol{h}, \boldsymbol{v}).
\end{aligned} \tag{49}$$

Let $\boldsymbol{v} \in V_m$ for fixed $m \in \mathbb{N}$. Then $\boldsymbol{v}$ is infinitely differentiable in $\overline{\Omega}$. Further, let $n \in \mathbb{N}$, $n \geq m$. We shall write $\boldsymbol{z}_n = (z_{n1}, z_{n2})$, $\boldsymbol{z} = (z_1, z_2)$, $\boldsymbol{g^*} = (g_1^*, g_2^*)$ a $\boldsymbol{v} = (v_1, v_2)$. By (47), we have

$$a_1(\boldsymbol{z}_n, \boldsymbol{v}) \longrightarrow a_1(\boldsymbol{z}, \boldsymbol{v}). \tag{50}$$

Further, in view of (47), we get

$$\begin{aligned} &a_2(\boldsymbol{z}_n, \boldsymbol{g^*}, \boldsymbol{v}) - a_2(\boldsymbol{z}, \boldsymbol{g^*}, \boldsymbol{v}) \\ &\quad = \int_\Omega \omega(\boldsymbol{z}_n - \boldsymbol{z})\, \boldsymbol{g^*}^\perp \cdot \boldsymbol{v}\, \mathrm{d}\boldsymbol{x} \longrightarrow 0. \end{aligned} \tag{51}$$

The next term satisfies

$$\begin{aligned} &a_2(\boldsymbol{g^*}, \boldsymbol{z}_n, \boldsymbol{v}) - a_2(\boldsymbol{g^*}, \boldsymbol{z}, \boldsymbol{v}) \\ &\quad = \int_\Omega \omega(\boldsymbol{g^*})\, (\boldsymbol{z}_n^\perp - \boldsymbol{z}^\perp) \cdot \boldsymbol{v}\, \mathrm{d}\boldsymbol{x} \longrightarrow 0 \end{aligned} \tag{52}$$

due to the strong convergence of the sequence $\{\boldsymbol{z}_n\}$ to $\boldsymbol{z}$ in $L^2(\Omega)^2$, see (48). The possibility of the limit transition in the term $a_2(\boldsymbol{z}_n, \boldsymbol{z}_n, \boldsymbol{v})$ can be proven as follows:

$$\begin{aligned} &|a_2(\boldsymbol{z}_n, \boldsymbol{z}_n, \boldsymbol{v}) - a_2(\boldsymbol{z}, \boldsymbol{z}, \boldsymbol{v})| \\ &\quad \leq |a_2(\boldsymbol{z}_n, \boldsymbol{z}_n, \boldsymbol{v}) - a_2(\boldsymbol{z}_n, \boldsymbol{z}, \boldsymbol{v})| \\ &\qquad + |a_2(\boldsymbol{z}_n, \boldsymbol{z}, \boldsymbol{v}) - a_2(\boldsymbol{z}, \boldsymbol{z}, \boldsymbol{v})| \\ &\quad = \left| \int_\Omega \omega(\boldsymbol{z}_n)\, (\boldsymbol{z}_n^\perp - \boldsymbol{z}^\perp) \cdot \boldsymbol{v}\, \mathrm{d}\boldsymbol{x} \right| \\ &\qquad + \left| \int_\Omega \omega(\boldsymbol{z}_n - \boldsymbol{z})\, \boldsymbol{z}^\perp \cdot \boldsymbol{v}\, \mathrm{d}\boldsymbol{x} \right| \\ &\quad \leq c_9\, \|\boldsymbol{z}_n\|_1\, \|\boldsymbol{z}_n - \boldsymbol{z}\|_{L^4}\, \|\boldsymbol{v}\|_{L^4} \\ &\qquad + \left| \int_\Omega \omega(\boldsymbol{z}_n - \boldsymbol{z})\, \boldsymbol{z}^\perp \cdot \boldsymbol{v}\, \mathrm{d}\boldsymbol{x} \right| \end{aligned}$$

where c_9 is a constant independent of n. The first term on the right–hand side tends to zero as $n \to +\infty$ due to the strong convergence of the sequence $\{\boldsymbol{z}_n\}$ to $\boldsymbol{z}$ in $L^q(\Omega)^2$, see (48). The second term tends to zero for $n \to +\infty$ due to the weak convergence of $\{\boldsymbol{z}_n\}$ to $\boldsymbol{z}$ in $H^1(\Omega)^2$, see (47). Thus,

$$a_2(\boldsymbol{z}_n, \boldsymbol{z}_n, \boldsymbol{v}) \longrightarrow a_2(\boldsymbol{z}, \boldsymbol{z}, \boldsymbol{v}). \tag{53}$$

The limits in (50)–(53) finally imply the possibility of the limit transition in (49), which means that the limit function $\boldsymbol{z}$ satisfies (23) for all $\boldsymbol{v} \in V_m$. Since the natural number m was chosen arbitrarily, (23) is satisfied for all $\boldsymbol{v} \in \cup_{m=1}^{+\infty} V_m$. This union is dense in V (because it obviously contains all elements of the basis $\mathbf{e}_1$, $\mathbf{e}_2, \ldots$) and each term in (23) depends continuously on $\boldsymbol{v}$ in the norm of V. This consideration enables us to conclude that (23) holds for all $\boldsymbol{v} \in V$. Consequently, the function $\boldsymbol{u}$ defined by the identity $\boldsymbol{u} = \boldsymbol{g^*} + \boldsymbol{z}$ is a weak solution of the problem defined in Definition 1. This result is formulated as the following theorem.

Theorem 8 (on the existence of a weak solution) *Let the norm $\|\boldsymbol{g}\|_{s;\,\Gamma_i}$ satisfy (31) with $\epsilon = \nu/c_4$. Then there exists a weak solution $\boldsymbol{u}$ of the cascade flow problem defined in Definition 1.*

7 Conclusion

As we have already mentioned in Section 1, if the weak solution $\boldsymbol{u}$ given by Theorem 8 is "smooth", we can integrate back by parts from (31). We can use the Helmholtz decomposition of $L^2(\Omega)^2$ into the direct sum of two closed orthogonal subspaces, one of them being the space of solenoidal functions with the normal component equal to zero on the boundary (in the sense of traces) and the second of them being the space of functions of the type $\nabla\phi$. Thus, we arrive at the existence of a function q which satisfies (15) on the outflow Γ_o. The original pressure p is then given by (4). The question up to which rate is the boundary condition (15) physically relevant, can only be judged from the comparison of numerical results with experimentally obtained data. However, the form (8) of the Navier–Stokes equation is very advantageous from the analytic point of view, because the nonlinear term $\omega(\boldsymbol{u})\, \boldsymbol{u}^\perp$ is pointwise perpendicular to the velocity $\boldsymbol{u}$. Thus, if we test the equation (8) by $\boldsymbol{u}$ (i.e. we multiply it by $\boldsymbol{u}$), the nonlinear term disappears pointwise and it is not necessary to simplify it or even remove it by means of the integration by parts. This enables to derive simply the necessary estimates without using artificial additional terms like e.g. $\frac{1}{2}\,(\boldsymbol{u} \cdot \boldsymbol{n})^- \boldsymbol{u}$ in (2).

In the monographs [6], [13], [17] and [2], the existence of a weak solution of the boundary value problem for the Navier–Stokes equations in a bounded domain Ω with the Dirichlet boundary condition

$$\boldsymbol{u}\Big|_{\partial\Omega} = \boldsymbol{g}, \tag{54}$$

prescribed on the whole boundary $\partial\Omega$ is established under the assumption that $\boldsymbol{g} \in H^{1/2}(\partial\Omega)^k$ (where $k = 2$ or $k = 3$) and under the assumption that

$$\int_\Gamma \boldsymbol{g} \cdot \boldsymbol{n}\, \mathrm{d}S = 0 \tag{55}$$

for each component Γ of the boundary $\partial\Omega$. In this case, it is not necessary to assume the smallness of the function $\boldsymbol{g}$ in a suitable norm, because the extension $\boldsymbol{g^*}$ of $\boldsymbol{g}$ into Ω can be constructed so that the norm $\|\boldsymbol{g}^*\|_1$ is arbitrarily small. The general problem, when the boundary of Ω has several components, the Dirichlet boundary condition (54) is prescribed on $\partial\Omega$ and the function $\mathbf{g}$ satisfies the condition

$$\int_{\partial\Omega} \boldsymbol{g} \cdot \boldsymbol{n} \, \mathrm{d}S \, = \, 0 \tag{56}$$

in the whole, but the integrals of $\boldsymbol{g} \cdot \boldsymbol{n}$ over the individual components of the boundary are nonzero, is solved in [5]. However, in this case, the proof of the existence of a weak solution requires the assumption that the flows between the different components of $\partial\Omega$ are sufficiently small. The question of existence of a solution without this assumption on smallness represents a challenging open problem. Our problem has a similar character, the role of different components of the boundary with a "big" possible flow between them now play Γ_i and Γ_o. Thus, we have proved the existence of a weak solution under the restrictive assumption (31).

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