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Symbolic Computation of Transient flow with memory inside a movable tube

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Abstract: - Certain work about symbolic computational hydrodynamics is realized. Using computer algebra, the explicit solution for the Navier-Stokes equation corresponding to the transient flow within a movable tube, for certain fluid with hydro-dynamical memory and susceptibility to amplify the hydrodynamic perturbations, is derived. The method of solution is the Laplace Transform Technique with inverse by residues. The solution is obtained by means of certain algorithm for computer algebra. As a result we obtain certain threshold phenomena for the triggering of the global amplification of the hydrodynamic perturbations.

Key-Words: - Navier-Stokes equation, Residue Theorem, Computer Algebra, Memory, Amplification, Basic Multiplicative Number, Threshold, Bessel Functions.

1. Introduction

We consider the transitory and laminar flow inside an infinitely long and movable cylindrical tube, for the case of a fluid with viscous-elastic properties which are described by a hydrodynamic memory and with a susceptibility-resistivity to the amplification of hydrodynamic perturbations. The mathematical model that is used, is the linear Navier-Stokes equation with memory term, sources and drains of fields of speed and external forces possibly of electromagnetic origin; complemented with a condition of movable boundary. The effective transport equation that is considered here, is an integral-differential equation of the type reaction-diffusion with memory and external sources. Such equation in spite of being linear, is quite complex and although it can be solved analytically of an explicit way, the determination of this solution is difficult to obtain using pencil and paper and the method of separation of variables cannot be used. Here we show that the solution of our model can be obtained symbolically using computer algebra by means of an appropriate algorithm. This algorithm is based on the technique of the Laplace transform, together with an realization of the inverse transformation by means of the application of the theorem of the residues of the theory of the functions of complex variable. From the solution that is obtained, the basic multiplicative number for the amplification of hydrodynamic perturbations on the considered fluid, is derived.

2 The Mathematical Problem

The Navier-Stokes equations have the form [1, 2, 3]

$$\rho(\frac{\partial}{\partial t}\vec{v} + (\vec{v}.\nabla)\vec{v}) = \eta\nabla^2\vec{v} - \nabla P + \vec{f} \qquad (1)$$

where ρ is the density, η is the dynamical viscosity, $\vec{v}$ is the velocity field, ∇P is the pressure gradient, $\vec{f}$ is the external force, and ∇^2 is the laplacian operator.

Here, we consider the case of axially symmetric flow inside a movable infinitely long circular tube of radius *a*. Using cylindrical coordinates (r,Θ,z), we assume that the form of the velocity field of the fluid is given by

$$\vec{v} = v(r,t)\vec{e}_z \qquad (2)$$

where $\vec{e}_z$ is an unitary vector at the direction of the axis of tube that coincides with the z axis.

Using (2) we have

$$(\vec{v}.\nabla)\vec{v} = v(r,t)\frac{\partial}{\partial z}(v(r,t)\vec{e}_z) = v(r,t)(\frac{\partial v(r,t)}{\partial z})\vec{e}_z = 0\,. \qquad (3)$$

The substitution of (2) and (3) in (1), gives

$$\rho\frac{\partial v(r,t)}{\partial t} = \eta\nabla^2 v(r,t) - \frac{\partial P}{\partial z} + f_z \qquad (4)$$

where f_z is the axial component of the external force, which is considered as the resultant of three forces, to know: a) certain force that comes from the susceptibility of fluid to amplify the hydrodynamic perturbations, b) other force that comes from the

effects of hydrodynamic memory and c)-certain electromagnetically induced force.
We postulate that the form of the driven force is

$$f_z = \rho(kv(r,t) + M(r,t) + c_1 + c_2 r^2) \qquad (5)$$

where k is the effective constant of susceptibility of amplification of hydrodynamic perturbations, M(r,t) is the memory function and c_1 and c_2 are two constants with electromagnetic origin.
The substitution of (5) en (4) gives

$$\frac{\partial v(r,t)}{\partial t} = \mu \nabla^2 v(r,t) + kv(r,t) + M(r,t) + c_1 - \frac{1}{\rho}\frac{\partial P}{\partial z} + c_2 r^2 \qquad (6)$$

where μ is the cinematic viscosity, defined as

$$\mu = \frac{\eta}{\rho} \ , \qquad (7)$$

and the laplacian operator takes the form

$$\nabla^2 v(r,t) = \frac{\partial^2 v(r,t)}{\partial r^2} + \frac{1}{r}\frac{\partial v(r,t)}{\partial r} \ . \qquad (8)$$

The constant of susceptibility denoted k, has the following structure

$$\kappa = \beta_1 S_0 - \gamma_1 \qquad (9)$$

where β_1 is the intensity of hydrodynamic perturbations, S_0 is the susceptibility to amplification, and γ_1 is the coefficient of attenuation of perturbations.
At concomitance with (9) the structure of the hydrodynamic memory function is

$$\mathrm{M}(r, t) = M_1(r, t) - M_2(r, t) \qquad (10)$$

where $M_1(r,t)$ is the memory function for the susceptibility of amplification with the form

$$M_1(r, t) = \beta_2 S_0 \int_0^t \mathrm{v}(r, \tau)\, \mathrm{e}^{(-\varepsilon_1 (t-\tau))}\, d\tau \qquad (11)$$

being β_2 the constant of susceptibility memory and being $\in_1$ the factor of attenuation of the susceptibility memory.
From the other side, $M_2(r,t)$ is the memory function for the attenuation of perturbations, with the form

$$M_2(r, t) = \gamma_2 \int_0^t \mathrm{v}(r, \tau)\, \mathrm{e}^{(-\varepsilon_2 (t-\tau))}\, d\tau \qquad (12)$$

being γ_2 the constant of attenuation memory and being $\in_2$ the factor of decay of the attenuation memory.

We assume that initially the fluid is at rest, it is to say, we take the following initial condition

$$\mathrm{v}(r, 0) = 0 \ . \qquad (13)$$

We consider the case of movable infinitely long tube, with the following boundary condition at the wall of the tube

$$\mathrm{v}(a, t) = v_b\, \mathrm{e}^{(-\delta t)} \qquad (14)$$

where v_b is the velocity of fluid at the wall, and δ is the factor of attenuation of such velocity.
The question here, is to obtain the explicit analytical solution of the equation (6) with (8)-(12) and subjected to the initial condition (13) and the boundary condition (14); and to derive from the obtained solution some interesting characteristic of the fluid under consideration.

3 Problem Solution

The equation (6) is linear but can not be solved by means of the method of separation of variables. It is necessary to apply the Laplace Transform method making the inverse transform by means of the theorem of residues [4]. Since the necessary manipulations to solve the equations (6) are too voluminous as for to be realized by hand with pencil and paper, it is necessary to apply some type of system of computer algebra that allows symbolic computation [5,6].

3.1. Method of Solution

Figure 1 shows a sketch of the algorithm that we have used to solve (6) with (8)-(14). As it is observed, the inputs of the algorithm are: Eq, that represents the equation (6); I.C. that represents the initial conditions (13); B.C. that represents the boundary condition (14); and F.C. that represents a certain finitude condition for the solutions.
The output for the algorithm is the explicit solution of (6)-(14) and certain threshold parameter, denoted R_0, which will be explained more later.
The algorithm operates as it follows.
The inputs Eq., I.C., and B.C, by means of a Laplace Transformer are turned into a transformed equation denoted T.Eq. and a transformed boundary condition denoted T.B.C. Then, T.Eq, T.B.C. and F.C. are processed by a certain Dsolver that generates the transformed solution denoted Tsol.

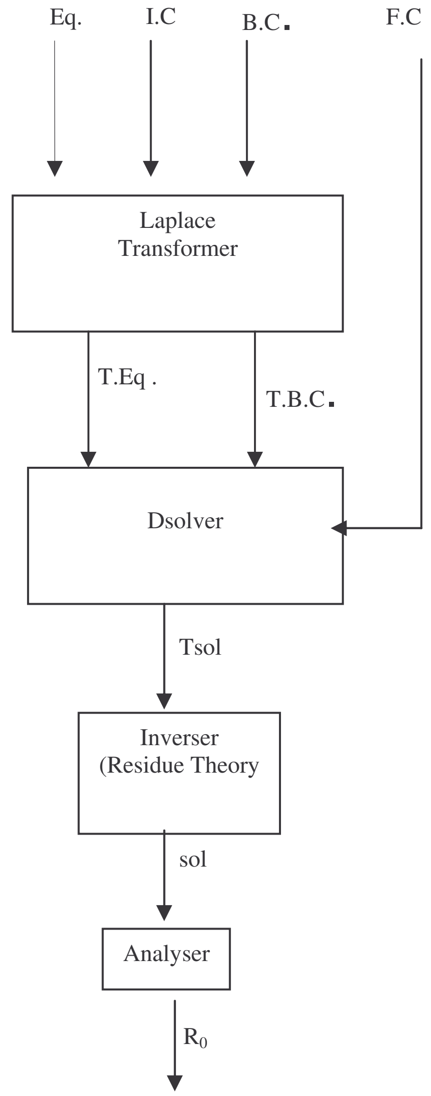

Figure 1. Sketch of the algorithm

Next, Tsol is processed by means of an inverser with residue theorem, and we obtain the explicit form of the solution, denote sol. Finally using a stability analyzer, we deduce the explicit form of R_0 and the algorithm is finished.

3.2. Results of Computations

The solution of the equation (6) with (8)-(14) that is obtained using our algorithm of computer algebra is given at Figure 2. and has the form

$$\mathrm{v}(r, t) = v_1(r, t) + v_2(r) + v_3(r, t) \tag{15}$$

where

$$v_1(r, t) = \frac{J_0(\sqrt{\lambda(-\delta)}\, r)\, v_b\, \mathrm{e}^{(-\delta\, t)}}{J_0(\sqrt{\lambda(-\delta)}\, a)} \tag{16}$$

$$v_2(r) = v_{2,1}(r) + v_{2,2}(r) \tag{17}$$

$$v_{2,1}(r) = -\frac{-4\, c_2 + \lambda(0)\, c_2\, r^2 + c_3\, \lambda(0)}{\lambda(0)^2\, \mu} \tag{18}$$

$$v_{2,2}(r) = -\frac{(4 c_2 - c_3 \lambda(0) - \lambda(0) c_2 a^2) J_0\left(\lambda(0)^{\left(\frac{1}{2}\right)} r\right)}{J_0\left(\lambda(0)^{\left(\frac{1}{2}\right)} a\right) \lambda(0)^2 \mu} \tag{19}$$

$$v_3(r, t) = \sum_{i=1}^{3} \left(\sum_{n=1}^{\infty} G_{i,n}(r, t) \right) \tag{20}$$

$$G_{i,n}(r, t) = \frac{G_{1,i,n}(r, t)}{G_{2,i,n}(r, t)} \tag{21}$$

$$G_{1,i,n} = -2\, \mathrm{e}^{(S_{i,n}\, t)}\, J_0\left(\frac{\alpha_n r}{a}\right) (K_1 c_2 + K_2 c_3 + K_3) \tag{22}$$

$$K_1 = a^4\, (\alpha_n - 2)\, (\alpha_n + 2)\, (S_{i,n} + \delta) \tag{23}$$

$$K_2 = a^2\, \alpha_n^{\ 2}\, (S_{i,n} + \delta) \tag{24}$$

$$K_3 = v_b\, \alpha_n^{\ 4}\, \mu\, S_{i,n} \tag{25}$$

$$G_{2,i,n}(r, t) = a^2 \alpha_n^{\ 3} \mu S_{i,n} (S_{i,n} + \delta) J_1(\alpha_n) \left(\lim_{s \to S_{i,n}} \frac{d}{ds} \lambda(s) \right) \tag{26}$$

$$c_3 = c_1 - \frac{\frac{\partial}{\partial z} P}{\rho} \tag{27}$$

with

$$\lambda(s) = \frac{\mathrm{L}(s)}{\mu\, (s + \varepsilon_1)\, (s + \varepsilon_2)} \tag{28}$$

$$\mathrm{L}(s) = -s^3 + A_1\, s^2 + A_2\, s + A_3 \tag{29}$$

$$A_1 = \beta_1\, S_0 - \gamma_1 - \varepsilon_2 - \varepsilon_1 \tag{30}$$

$$A_2 = -\gamma_1 \varepsilon_2 + \beta_2 S_0 - \gamma_1 \varepsilon_1 - \gamma_2 + \beta_1 S_0 \varepsilon_2 + \beta_1 S_0 \varepsilon_1 - \varepsilon_1 \varepsilon_2 \tag{31}$$

$$A_3 = \beta_2\, S_0\, \varepsilon_2 - \gamma_1\, \varepsilon_1\, \varepsilon_2 - \gamma_2\, \varepsilon_1 + \beta_1\, S_0\, \varepsilon_1\, \varepsilon_2 \tag{32}$$

and being $S_{i,n}$ the roots of the equation of third degree

$$s^3 + B_1\, s^2 + B_2\, s + B_3 = 0 \tag{33}$$

with

$$B_1 = -\beta_1 S_0 + \gamma_1 + \frac{\alpha_n^2 \mu}{a^2} + \varepsilon_2 + \varepsilon_1 \tag{34}$$

$$B_2 = B_{2,1} + B_{2,2} \tag{35}$$

$$B_{2,1} = \gamma_1 \varepsilon_2 - \beta_2 S_0 + \gamma_1 \varepsilon_1 - \beta_1 S_0 \varepsilon_2 - \beta_1 S_0 \varepsilon_1 + \varepsilon_1 \varepsilon_2 \tag{36}$$

$$B_{2,2} = \frac{\alpha_n^2 \mu \varepsilon_2}{a^2} + \frac{\varepsilon_1 \alpha_n^2 \mu}{a^2} + \gamma_2 \tag{37}$$

$$B_3 = -\beta_2 S_0 \varepsilon_2 + \varepsilon_1 \gamma_1 \varepsilon_2 + \varepsilon_1 \gamma_2 + \frac{\varepsilon_1 \alpha_n^2 \mu \varepsilon_2}{a^2} - \beta_1 S_0 \varepsilon_1 \varepsilon_2 \tag{38}$$

Finally, $J_m(x)$ is the Bessel function of order m of the first kind, a is the radius of the circular tube and α_n are the zeroes of J_0, namely [7]

$$J_0(\alpha_n) = 0 \tag{39}$$

3.3 Analysis of Results

It is evident from the equations (15)-(39) that the solution of (6) is very formidable as for to be manipulated by hand using pen and paper. The great advantage of computer algebra is prominent.

As it is observed at the equation (15), the solution of (6), consists of three summands. The first one is given at (16), the second is given at (17)-(19) and the third is given at (20)-(26) with the specifications (27)-(39). For $\delta > 0$, the first summand (16), decays exponentially with the time and it does not represent any amplification of the hydrodynamic perturbations.

The second summand (17)-(19) does not depend on time and represents the stationary flow that is established within the tube and it does not give any kind of amplification of perturbations.

The third summand (20)-(26) is also a sum of terms with exponential dependence on time and some of such terms can be exponentially increasing with the time and then an amplification of the hydrodynamic perturbations would be had. The condition so that a such amplification is generated is that (33) admits a positive real solution, it is to say that $B_3 < 0$, where B_3 is given at the equation (38). This last condition can be rewritten as the threshold condition for the triggered of the amplification of hydrodynamic perturbations, namely, $R_{0,n} > 1$, where

$$R_{0,n} = \frac{(\varepsilon_2 \beta_2 + \varepsilon_2 \beta_1 \varepsilon_1) S_0}{\varepsilon_1 \gamma_1 \varepsilon_2 + \varepsilon_1 \gamma_2 + \frac{\varepsilon_1 \alpha_n^2 \mu \varepsilon_2}{a^2}} \tag{40}$$

The critical parameter $R_{0,,n}$ will be named the basic multiplicative number and measures the effective tendency of fluid to amplify the perturbations. The fundamental or ground value of the multiplicative number (40), corresponds to n=1, with α_1=2.405, namely

$$R_{0,1} = \frac{\varepsilon_2 (\beta_2 + \beta_1 \varepsilon_1) S_0 a^2}{(\gamma_1 a^2 \varepsilon_2 + \gamma_2 a^2 + 5.784025 \mu \varepsilon_2) \varepsilon_1} \tag{41}$$

The equation (41) can be rewritten as

$$R_{0,1} = \frac{R_0}{1 + \frac{5.784025 \mu \varepsilon_2}{a^2 (\gamma_1 \varepsilon_2 + \gamma_2)}} \tag{42}$$

where R_0 is the basic multiplicative number when $a \longrightarrow \infty$, and is given by

$$R_0 = \frac{S_0 \varepsilon_2 (\beta_2 + \beta_1 \varepsilon_1)}{\varepsilon_1 (\gamma_1 \varepsilon_2 + \gamma_2)} \tag{43}$$

Now, for the case when $\delta = 0$, the equation (16) lost its dependence on time and can be considered as certain stationary flow inside the tube jointly with the terms (17)-(19). The condition for (16)-(19) are really genuine stationary flows is

$$0 < \lambda(0) \tag{44}$$

which is equivalent to L(0)> 0 with L(s) being given at (29), and this last is equivalent to $A_3 > 0$, where A_3 is given at the equation (32). This condition can be rewritten as $R_0 > 1$, where R_0 is given at (43). Finally, when $\delta < 0$, then (16) has exponential increasing with time and corresponds to the irruption of the global amplification of perturbations without necessity of any kind of hydrodynamic threshold

We can see from (42) and (43) that

$$R_{0,1} < R_0 \tag{45}$$

Now, from the perspective of the possibility of to control the flows, we have that the percentage of decreasing of the susceptibility of the fluid S_0, both to prevent the stationary flows (16)-(19) as to obstruct the exponential amplification of the perturbations (20)-(26), are given, respectively by

$$P_{st} = 1 - \frac{1}{R_0} \tag{46}$$

$$P_A = 1 - \frac{1}{R_{0,\,1}} \tag{47}$$

From (45)-(47) we deduce that

$$P_A < P_{st} \tag{48}$$

it is to say the percentage of decreasing of susceptibility to prevent the stationary flows, denote P_{st} is greater than the percentage of decreasing the susceptibility for to prevent the global amplification of perturbations, denoted P_A. In other words is more expedite to avoid the amplification of the perturbations that to avoid the presence of stationary flows. .

Finally, from the solution that was obtained we can to derive the formula for the total flow across the section of the tube, which is given at the Fig. 3, and it is defined as

$$\mathrm{Q}(t) = 2\,\pi \int_0^a \mathrm{v}(r,\,t)\; r\; dr \tag{49}$$

4 Conclusion

The problem that was considered here, is a linear problem, whose solution can be obtained symbolically using certain algorithm and then the basic multiplicative number for the amplification of hydrodynamic perturbations on the fluid, can be derived. Our principal contribution is the equation (42).

The algorithm that was used can be applied for others more complex linear problems with boundary conditions. It is evident from this work that computer algebra is very useful to study those problems on fluid dynamics that demands the analytical solution of the Navier-Stokes equation. The method that was proposed here, can be applied also for linear boundary problems of transport phenomena such as heat and mass transfer..

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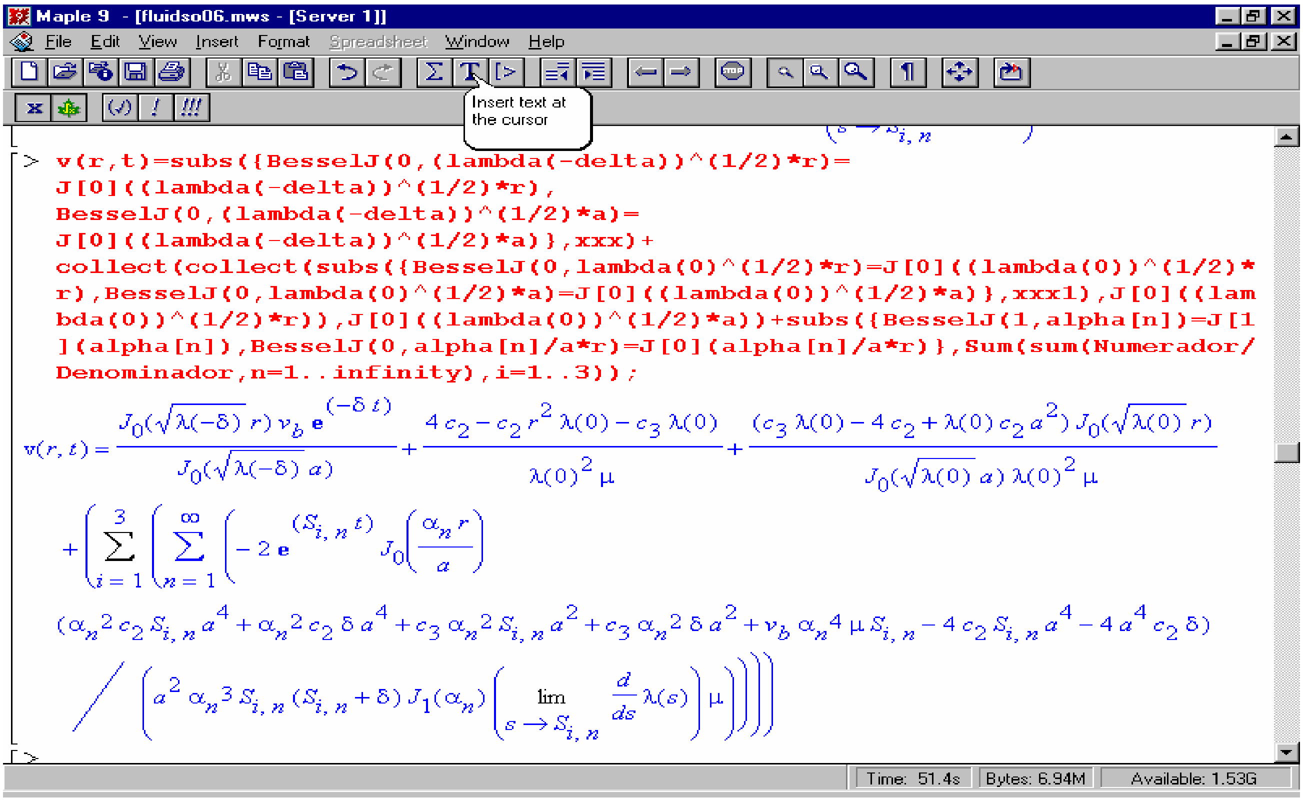

Figure 2. Full analytical solution of the problem (6)-(14)

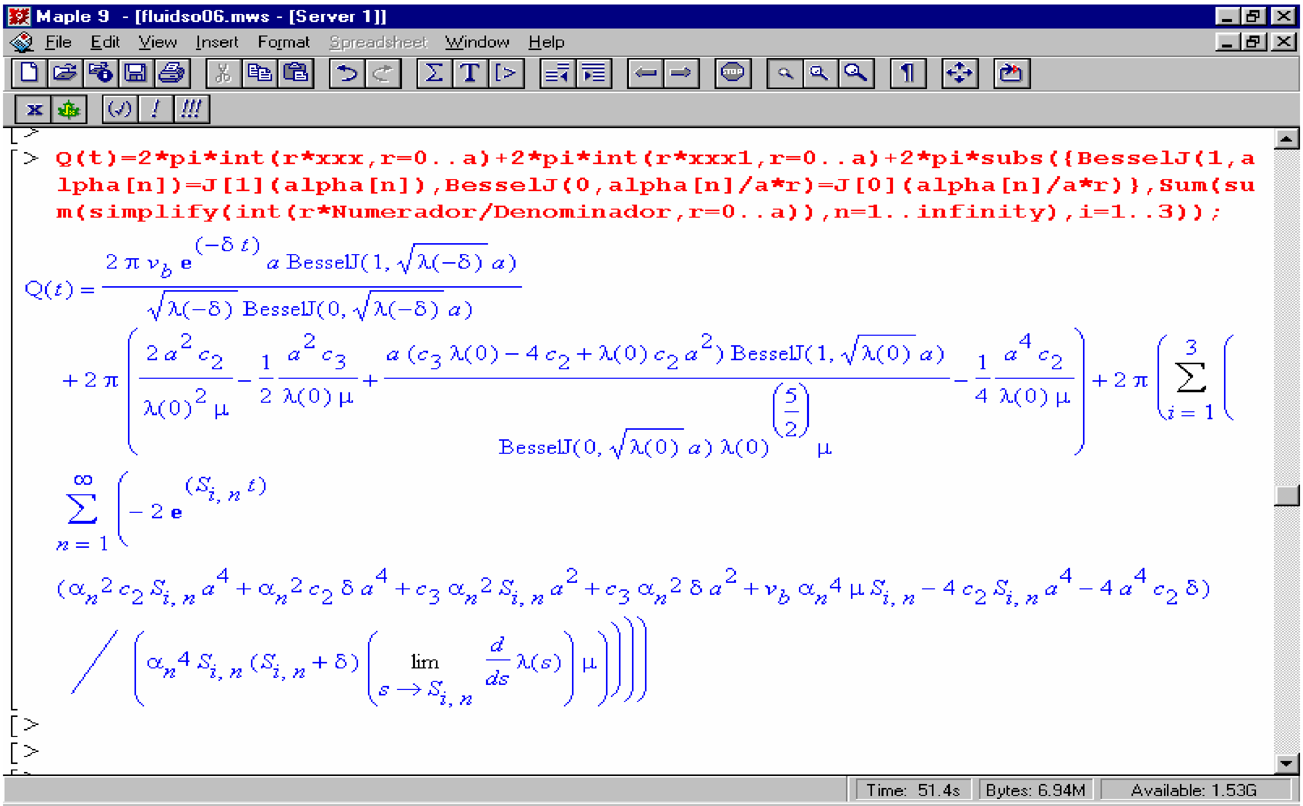

Figure 3. Explicit form of the equation (49)