Water Resources Management of an Aquifer with Fuzzy Linear Programming

I. N. CHALKIDIS*, CH. D. TZIMOPOULOS*, CH. H. EVANGELIDES*, ST.YANNOPOULOS*,

*Department of Rural and Surveying Engineering,
Aristotle University of Thessaloniki
Aristotle University of Thessaloniki - 54124 Thessaloniki
GREECE

Abstract: Water resources management nowadays is a very important issue. Groundwater is the main fresh water supplier for drinking and irrigation purposes. In this paper the symmetric fuzzy linear programming (FLP) is applied to optimize the management of an aquifer. The solution achieved using FLP introduces with an acceptable cost the uncertainty that is imbedded in such problems. In this particular case FLP provided a number of alternative management solutions most of them milder than the ones provided by linear programming (LP). A sensitivity analysis of the violation degree (p_i) parameter is also performed.

Key-Words: Fuzzy linear programming, violation degree, groundwater management, aquifer.

1 Introduction

This paper shows an application of the symmetric fuzzy linear programming in optimizing the management of an aquifer. Also a way to determine the appropriate degree of violations for the problem constraints was utilized.

The first method for solving fuzzy linear programming problems was proposed by Zimmerman [5]. Zimmerman used Bellman and Zadeh's [1] interpretation that a fuzzy decision is a confluence of goals and constraints, denoted the max-min model because it considers that the best fuzzy decision is the union of the aggregated intersection of goal and constraints. Specifically if we have n fuzzy goals

$_1$ $\tilde{G}_n$

$$\left.\begin{array}{l}\mathbf{c}^T\mathbf{x} \gtrapprox \mathbf{z} \\ \mathbf{A}\mathbf{x} \lessapprox \mathbf{b} \\ \mathbf{x} \geq \mathbf{0}\end{array}\right\} \quad (4)$$

The symbol $\lessapprox$ denotes the fuzzified version of $\leq$ and has the linguistic interpretation "essentially smaller than or equal to". Respectively the symbol $\gtrapprox$ denotes the fuzzified version of $\geq$ and has the linguistic interpretation "essentially grater than or equal to".

Using matrix formulation and substituting: $\begin{pmatrix} -c \\ \mathbf{A} \end{pmatrix} = B$ and $\begin{pmatrix} -z \\ b \end{pmatrix} = d$ the model becomes:

$$\left.\begin{array}{l}\mathbf{Bx} \lessapprox \mathbf{d} \\ \mathbf{x} \geq \mathbf{0}\end{array}\right\} \quad (5)$$

In this formulation it is obvious that there are no differences in the expression of the objective function and of the constraints, so that the model is fully symmetric.

Each of the (m+1) rows of model (5) is represented by a fuzzy set with membership function $\mu_i(x)$. According to equation (2) the membership function is:

$$\mu_{\tilde{D}}(x) = \min_i \{\mu_i(x)\} \quad (6)$$

Where $\mu_i(x)$ can be interpreted as the degree to which x satisfies the fuzzy inequality $B_i x \leq d_i$ (where B_i is the i^{th} row of B).

If a crisp optimal solution is required the maximizing solution of the equation (6) can be selected:

$$\max_{x\geq 0} \min_i \{\mu_i(x)\} = \max_{x \geq 0} \mu_{\tilde{D}}(x) \quad (7)$$

The membership function $\mu_i(x)$ should be 0 if the objective function or one of the constraints are strongly violated and 1 if they are very satisfied. Also $\mu_i(x)$ should be increased monotonously from 0 to 1. Assuming that the membership function $\mu_i(x)$ increases linearly then:

$$\mu_i(x) = \begin{cases} 1 & \text{if} \quad B_i x \leq d_i \\ 1 - \dfrac{B_i x - d_i}{p_i} & \text{if} \quad d_i < B_i x \leq d_i + p_i \\ 0 & \text{if} \quad B_i x > d_i + p_i \end{cases} \quad (8)$$

where $(i = 1, \ldots, m+1)$, p_i is the tolerance interval which gives the admissible violations of the constraints and the objective function and is subjectively chosen. Substitution of the equation (8) into (7) yields the following:

$$\max_{x \geq 0} \min_i \left(1 - \frac{B_i x - d_i}{p_i}\right) \quad (9)$$

Substitution of the membership function $\mu_D(x)$ by a new variable "λ" and using equations (6) (7) and (9) then:

$$\lambda = \mu_D(x) = \min_i \left(1 - \frac{B_i x - d_i}{p_i}\right)$$

or $\lambda \leq \left(1 - \dfrac{B_i x - d_i}{p_i}\right)$

or $\lambda p_i + B_i x \leq d_i + p_i$ (10)

$(i = 1, \ldots, m+1)$

According to equations (9) and (10):

$$\left.\begin{array}{lc} \text{maximize} & \lambda \\ \text{such us} & \lambda p_i + B_i x \leq d_i + p_i \\ \text{and} & x \geq 0 \\ (i = 1, \ldots, m+1) & \end{array}\right\} \quad (11)$$

If the vector (λ, x_0) is the optimal solution to formulation (11) then x_0 is the maximizing solution (7) of the model with membership function as specified in (8).

2 Problem Formulation

The main problem in the management of an aquifer is the determination of the quantity of the available water from each well in order to minimize the pumping cost and the sustainability of the aquifer. In linear programming the values of some parameters are considered as defined such as the hydraulic conductivity K in every direction or the specific storage of the aquifer or the variation of the water discharge as a function of the head elevation. Also small changes may occur in an aquifer from time to time. So the reality is uncertain. In this case an effort is made to include all the uncertainty in the constraints and this is the reason that Fuzzy linear programming is used [3].

2.1 Optmization

For the implementation of fuzzy linear programming the symmetric fuzzy linear programming method was used as described in the previous section 1.1. First the optimization problem was solved using linear programming [2],[4] without any uncertainty on the cost coefficients and so the objective function and the constraints are formulated [3].

Table 1 shows the names of the wells of the aquifer, the initial elevation head, the final elevation head at the end of the irrigation period, the head difference, the required manometric head (H_{man}), the percentage of the fluctuation of the elevation head to the manometric head and finally the cost coefficients.

Table 1: cost coefficients C_i

well code x_i	Initial head elevation	Head elevation at the and of the irrigation period	Δh2	Hman	$\frac{\Delta h_2}{H_{man}}\%$	C_i
G-142	585	579	6	151	3.97	6,13
G-116	585	589	-4	91	-4.40	3,69
G-150	580	580	0	130	0.00	5,28
G-122	615	612	3	113	2.65	4,59
G-107	605	603	2	77	2.60	3,13
G-92	565	571	-6	79	-7.59	3,21
G-109	585	584	1	61	1.64	2,48
G-119	610	609	1	96	1.04	3,90
G-133	670	668	2	102	1.96	4,14
G-136	705	702	3	93	3.23	3,78
G-129	590	590	0	125	0.00	5,08
G-137	575	576	-1	139	-0.72	5,64
G-127	575	577	-2	118	-1.69	4,79
G-104	610	606	4	74	5.41	3,00
G-115	585	587	-2	93	-2.15	3,78
G-100	580	580	0	120	0.00	4,87
G-103	600	598	2	67	2.99	2,72
G-99	580	579	1	101	0.99	4,10
G-94	550	551	-1	89	-1.12	3,61
G-155	590	589	1	66	1.52	2,68
G-95	555	558	-3	87	-3.45	3,53

Since the fluctuation of the elevation head during the irrigation period has very small influence on the manometric head, the cost coefficients are considered constant during the irrigation period. Thus the objective function becomes:

$$C_1 Q_{G142} + C_2 Q_{G116} + \ldots + C_{21} Q_{G95} =$$

$$= \sum_{i=1}^{21} C_i x_i = \mathbf{Cx} \quad (12)$$

where **C** is the matrix of the cost coefficients as shown on table 1

$$\mathbf{C} = \begin{bmatrix} 6,13 \\ 3,69 \\ \vdots \\ 3,53 \end{bmatrix} \quad (13)$$

and **x** is the matrix with the unknown values x: $\mathbf{x} = [x_1 \quad x_2 \quad \ldots \quad x_{21}]$ which are the desired pumping discharges (m^3/d) of each well in order to minimize the pumping cost considering all the constraints.

The constraints are placed on the discharge quantity of the wells regarding the daily pumped discharge from the aquifer. For simplicity the discharge quantities are considered as positive so the problem is transposed into a minimization problem, and consequently constraints of non negativity of the variables are posed:

$$x_1 + x_2 + \ldots + x_{21} \geq 16500 \quad m^3/d \quad (14)$$

and also on the allowable daily discharge (m^3/d) of each well:

$$\left.\begin{array}{llll} x_1 \leqslant 792 & x_7 \leqslant 581 & x_{13} \leqslant 634 & x_{19} \leqslant 924 \\ x_2 \leqslant 792 & x_8 \leqslant 871 & x_{14} \leqslant 660 & x_{20} \leqslant 792 \\ x_3 \leqslant 792 & x_9 \leqslant 1003 & x_{15} \leqslant 924 & x_{21} \leqslant 845 \\ x_4 \leqslant 924 & x_{10} \leqslant 792 & x_{16} \leqslant 845 & \\ x_5 \leqslant 792 & x_{11} \leqslant 897 & x_{17} \leqslant 1083 & \\ x_6 \leqslant 1056 & x_{12} \leqslant 792 & x_{18} \leqslant 1056 & \end{array}\right\} \quad (15)$$

The constraints in matrix form are

$$Ax \leqslant b \quad (16)$$

Where **A** is the matrix of the coefficients and b is the matrix of the constraints. Finally the problem under solution using linear programming in matrix form is formulated:

$$\left.\begin{array}{l} \text{Minimize} \quad \mathbf{f(x)} = \mathbf{Cx} \\ \text{Under the constraints:} \quad \mathbf{Ax} \leq \mathbf{b} \text{ and } \mathbf{x} > 0, \end{array}\right\} \quad (17)$$

The value of the objective function that results from linear programming is 63.424 €/irrigation period.

3 Problem Solution

In order to introduce the uncertainty that incur for the wells of the aquifer, the acceptable degree of violation or the tolerance interval is introduced p_i ($i= 1,2,\ldots,22+1$) of the 22 constraints and the objective function as described in section 1.1.

The selection of violation coefficients p_i is subjective depending on the user, according to Zimmermann [5]. Rules for assigning values to coefficient p_i.were not traced in international literature. In this paper for the acceptable degree of violation a percentage (%) of the constrain coefficients and of the objective function is considered. Finally the influence of coefficient p_i on the results is investigated. An example of calculations for the degree of violation is given in table (2).

Particularly the violation p_1 of the objective function from 12,5% to 0,5% is considered. Similarly the violations $p_{2\text{-}23}$ for the constraints from 12,5% to 0,5% are considered. All the possible combinations for p_1 and $p_{2\text{-}23}$ are taken into consideration for the following percentages: 12,5%, 10%, 7,5%, 7%, 6,5%, 5%, 2,5% and 0,5%, resulting in 64 combinations. The procedure for p_1 and $p_{2\text{-}23}$ equal to 2,5% is described in details.

The initial calculations are shown in Table 2.

Table 2: Values of violation coefficients p_i (degree of violation).

	description	i	Value b_i	Value p_i	p_i (%)
Objective function	Minimum cost of pumping water per irrigation period (€)	**1**	63.424	1.585,6	2,5
Total pumping water discharge	**Allowed pumping discharge (m^3/day)**	**2**	16.500	412,5	2,5
well G-142		**3**	792	19,8	2,5
well G-116		**4**	792	19,8	2,5
well G-150		**5**	792	19,8	2,5
well G-122		**6**	924	23,1	2,5
well G-107		**7**	792	19,8	2,5
well G-92		**8**	1056	26,4	2,5
well G-109		**9**	581	14,5	2,5
well G-119		**10**	871	21,8	2,5
well G-133		**11**	1003	25,1	2,5
well G-136		**12**	792	19,8	2,5
well G-129		**13**	897	22,4	2,5
well G-137		**14**	792	19,8	2,5
well G-127		**15**	634	15,9	2,5
well G-104		**16**	660	16,5	2,5
well G-115		**17**	924	23,1	2,5
well G-100		**18**	845	21,1	2,5
well G-103		**19**	1083	27,1	2,5
well G-99		**20**	1056	26,4	2,5
well G-94		**21**	924	23,1	2,5
well G-155		**22**	792	19,8	2,5
well G-95		**23**	845	21,1	2,5

After the selection of coefficients p_i and according to the fuzzy linear programming theory, the formulation of the problem is:

$$\left.\begin{array}{l} \mathbf{c}^T\mathbf{x} \lesssim \mathbf{z} \\ \mathbf{A}\mathbf{x} \lesssim \mathbf{b} \\ \mathbf{x} \geq \mathbf{0} \end{array}\right\} \quad (18)$$

where $\mathbf{c}^T\mathbf{x}$ is the objective function of the problem and $\mathbf{c}^T = \mathbf{C}$ which is the cost coefficient matrix so:

$$\mathbf{Cx} = \sum_{i=1}^{21} C_i x_i = C_1 Q_{G142} + C_2 Q_{G116} + \ldots + C_{21} Q_{G95} \quad (19)$$

In this particular case the cost coefficients utilized are from Table 1.

$$\mathbf{C} = \begin{bmatrix} 6{,}13 \\ 3{,}69 \\ \vdots \\ 3{,}53 \end{bmatrix} \text{ and } \mathbf{x} = \left[x_1 \quad x_2 \quad \ldots \quad x_{21}\right].$$

The value of aspiration level z used, was taken from the linear programming results as 63.424 €/irrigation period. **A** and **b** are the coefficient and constrain limit matrixes correspondingly as given previously in Table 1.

Finally the problem formulation for fuzzy linear programming in matrix form becomes:

Minimization of $\mathbf{f(x)} = \mathbf{Cx} \lesssim \mathbf{z}$

Under the constraints: $\mathbf{Ax} \lesssim \mathbf{b}$ and $\mathbf{x} \geq \mathbf{0}$, (20)

Following the description of section 1.1 we arrive at:

$$\left.\begin{array}{lc} \text{maximize} & \lambda \\ \text{such us} & \lambda p_i + B_i x \leq d_i + p_i \\ \text{and} & x \geq 0 \\ (i = 1,\ldots,m+1) & \end{array}\right\} \qquad (11)$$

If the optimal solution to problem (11) is the vector (λ, x_0), then x_0 is the maximizing solution (7) of the model with membership function as specified in (8).

3.1 Results

The results for linear and fuzzy linear programming are shown in Table 3 [3].

Table 3: Results for linear and fuzzy linear programming for $p_1 = p_{2\text{-}23}$ =2,5%.

Objective function variables	Wells	Max. allowed pumping water discharge (m^3/d)	LP, (m^3/d)	FLP, (m^3/d)	FLP-LP (m^3/d)
X1	G-142	792	0	0	0
X2	G-116	792	792	797	5
X3	G-150	792	792	797	5
X4	G-122	924	924	930	6
X5	G-107	792	792	797	5
X6	G-92	1056	1056	1062	6
X7	G-109	581	581	585	4
X8	G-119	871	871	876	5
X9	G-133	1003	1003	1009	6
X10	G-136	792	792	797	5
X11	G-129	897	897	903	6
X12	G-137	792	237	448	211
X13	G-127	634	634	638	4
X14	G-104	660	660	664	4
X15	G-115	924	924	930	6
X16	G-100	845	845	850	5
X17	G-103	1083	1083	1090	7
X18	G-99	1056	1056	1062	6
X19	G-94	924	924	930	6
X20	G-155	792	792	797	5
X21	G-95	845	845	850	5
X22=λ	-	-	-	0.754	-
Totals	-	17847	16500	16812	312
Costs (€)	-		63424	65010	1586

The higher variations from the maximum allowed values of the daily pumping water discharge are observed on parameters x_1 and x_{12} both on linear programming and fuzzy linear programming solution. Three dimensional figures for all 64 combinations of p_1 and $p_{2\text{-}23}$ on parameters x_1 and x_{12} are shown at fig. 1 and 2.

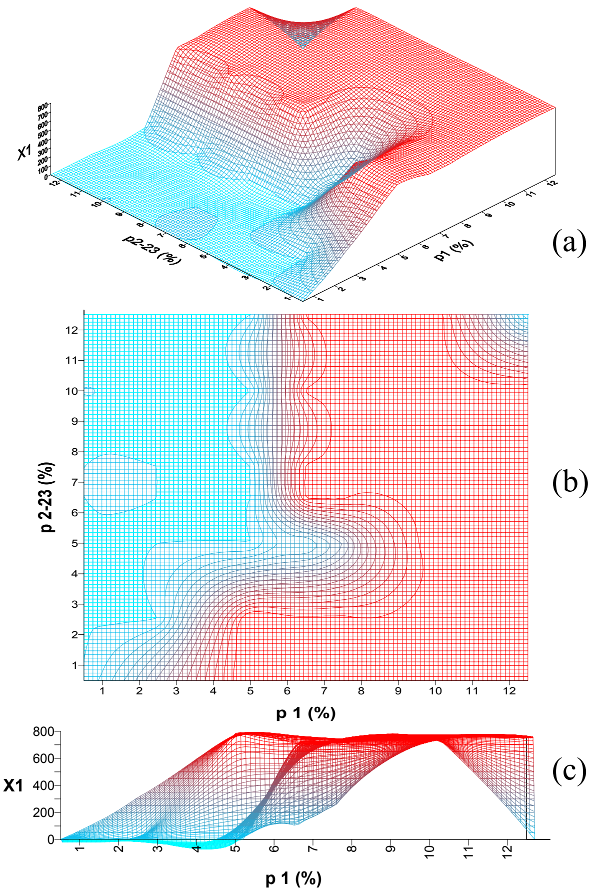

Fig. 1. Three dimensional presentation of the variation of the parameter x1 as a function of the violation coefficients p_1 and $p_{2\text{-}23}$. a) 3D perspective projection, b) vertical profile, c) horizontal profile.

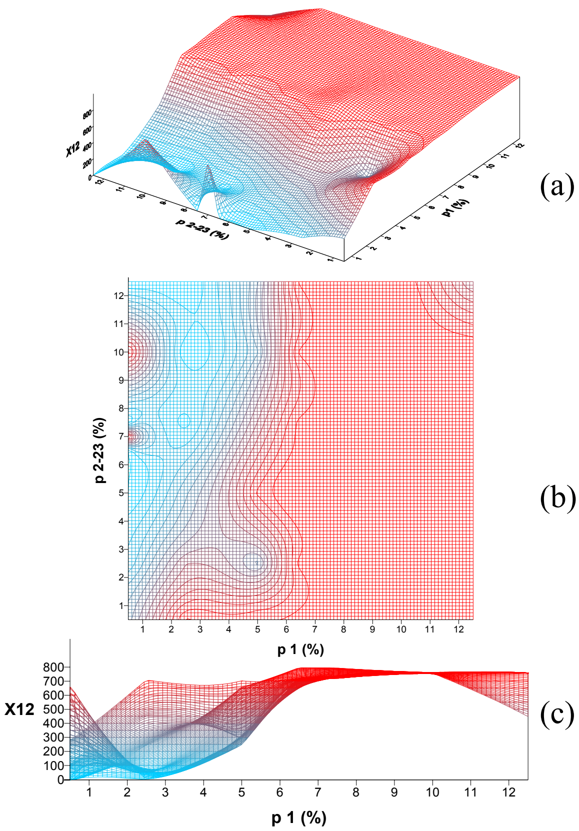

Fig. 2. Three dimensional presentation of the variation of the parameter x_{12} as a function of the violation coefficients p_1 and p_{2-23}. a) 3D perspective projection, b) vertical profile, c) horizontal profile.

4 Conclusions

The influence of the violation p1 of the objective function on the final result, as shown in Fig. 1 and 2, is significant for parameter x1 when p1 is up to 7% and for parameter x_{12} when it is up to 6%. On the other hand the influence of violations p_{2-23} of the constraints on the final result is strong for all the range of the trials for parameters x_1 and $x_{12.}$

The solution achieved using fuzzy linear programming introduces with an acceptable cost as shown in Table 3 the uncertainty that is imbedded in such problems.

Finally in this particular case FLP provided a number of alternative management solutions most of them milder than the ones provided by LP.

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