A nonlinear viscoelasticity problem with memory in time

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Abstract: In this paper we consider a nonlinear hyperbolic equation with a memory term which can be used in mathematical models for viscoelasticity problems. The qualitative properties of the solution of the initial boundary value problem are studied. We propose numerical methods for the computation of approximations to the solution of the continuous problem and their stability properties are analysed. Finally, we include numerical experiments illustrating the performance of the proposed methods.

Key words: Viscoelasticity problem, qualitative properties, stability, numerical methods, implicit method, implicit-explicit method.

1 Introduction

Let us consider the hyperbolic equation

\[
\rho \frac{\partial^2 u}{\partial t^2}(x, t) + \alpha \frac{\partial u}{\partial t}(x, t) = \gamma \frac{\partial^2 u}{\partial x^2}(x, t) + \int_0^t k(t-s) \frac{\partial^2 u}{\partial x^2}(x, s) \, ds + f(x, t, u(x, t))
\]

where \( k(s) \) is a scalar function smooth enough and which will be specified later, with initial conditions

\[
\begin{align*}
    u(0, x) &= u_0(x), \quad x \in (a, b) \\
    \frac{\partial u}{\partial t}(0, x) &= u_1(x), \quad x \in (a, b)
\end{align*}
\]

and

\[
u(a, t) = u_a(t), \quad u(b, t) = u_b(t), \quad t > 0.
\]

Initial boundary value problem (IBVP) (1)-(3) arises from a variety of mathematical models in engineering and physical sciences. We mention, for instance, the theory of viscoelasticity (see for instance [6], [7], [8]). In this case \( u \) represents the displacement of a body with density \( \rho \), viscosity \( \alpha \) and under external force \( f \) depending of the displacement. The use of exponential kernels of type \( k(s) = -\frac{\sigma}{\tau} e^{-s/\tau} \) has been largely considered in the context of heat conduction problems with memory in time ([1], [5], [9]). Attending that the heat conduction problem can be seen has a singular perturbation when the density \( \rho \) is small,
in what follows we take the exponential kernel mentioned before.

Our aim in this paper is to study the qualitative behavior of the solution of (1)-(2)-(3) from the theoretical and numerical points of view. Estimates for the kinetic energy, potential energy and to the past in time of the gradient of the displacement are obtained and allow to conclude the stability of (1)-(2)-(3). From a numerical point of view and following the approach introduced in [3] for the generalized Fisher equation and in [4] for a linear viscoelasticity problem, we propose numerical methods which enable us to compute numerical approximations presenting the qualitative behavior of the continuous solution provided some conditions on the stepsize are imposed. Numerical results illustrating the behavior of the methods studied are also included.

The paper is organized as follows. In Section 2 the theoretical study is presented. Numerical methods are analysed in Section 3. Finally, in Section 4, the numerical simulation is included.

2 Continuous qualitative behavior

In Theorem 1 we establish an estimate to the kinetic and potential energies and to the past in time of the gradient of the displacement when \( \gamma \neq \sigma \). This result enables to conclude the stability of (1)-(2)-(3).

**Theorem 1** Let \( u \) be a solution of (1)-(2) with homogeneous boundary conditions. Let us suppose that

\[
\begin{align*}
    u(x, t) &\in [c, d], \quad x \in [a, b], \ t \in [0, T], \\
    \left( \frac{\partial u}{\partial t} \right) &\in [0, T], \text{ then, for each } t \in (0, T], \text{ holds}
\end{align*}
\]

\[
\begin{align*}
    \rho \left\| \frac{\partial u}{\partial t} (t) \right\|_{L^2}^2 &+ \left( \gamma - \sigma \right) \left\| \frac{\partial u}{\partial x} (t) \right\|_{L^2}^2 \\
    + \sigma &\left[ 1 - \frac{1}{\tau} \int_0^t e^{-\frac{s}{\tau}} \frac{\partial u}{\partial x} (s) \ ds + \frac{\partial u}{\partial x} (t) \right\|_{L^2}^2 \\
    &\leq e^{\max \left\{ 1, \frac{1}{\tau} \right\} \left( \frac{\sigma}{\rho + \gamma} \right) t} \left( \rho \left\| u_0 \right\|_{L^2}^2 + \left( \gamma - \sigma \right) \right) \\
    \end{align*}
\]

(6)

where \( f'_{\text{max}} = \max_{[a,b] \times [0,T] \times [c,d]} \frac{\partial f}{\partial z} \).

**Proof:** Multiplying each member of (1) by \( \frac{\partial u}{\partial t} \) with respect to the \( L^2 \) inner product and integrating by parts we obtain

\[
\begin{align*}
    \rho \left( \frac{\partial^2 u}{\partial t^2} (t), \frac{\partial u}{\partial t} (t) \right) + \alpha \left\| \frac{\partial u}{\partial t} (t) \right\|_{L^2}^2 \\
    = -\gamma \left( \frac{\partial u}{\partial x} (t), \frac{\partial^2 u}{\partial t \partial x} (t) \right) \\
    - \sigma \left( \int_0^t e^{-\frac{s}{\tau}} \frac{\partial u}{\partial x} (s) \ ds, \frac{\partial^2 u}{\partial x \partial t} (t) \right) + (f(u(t)), \frac{\partial u}{\partial t} (t)).
\end{align*}
\]

(7)

It can be shown that

\[
\begin{align*}
    \left( \frac{1}{\tau} \int_0^t e^{-\frac{s}{\tau}} \frac{\partial u}{\partial x} (s) \ ds, \frac{\partial^2 u}{\partial x \partial t} (t) \right) \\
    = \frac{1}{2} \frac{d}{dt} \int_0^t e^{-\frac{s}{\tau}} \frac{\partial u}{\partial x} (s) \ ds + \frac{\partial u}{\partial x} (t) \right\|_{L^2}^2 \\
    - \frac{1}{2} \frac{d}{dt} \left\| \frac{\partial u}{\partial x} (t) \right\|_{L^2}^2 - \frac{1}{\tau} \int_0^t e^{-\frac{s}{\tau}} \frac{\partial u}{\partial x} (s) \ ds \left\| \frac{\partial u}{\partial x} (t) \right\|_{L^2}^2 \\
    + \frac{1}{\tau} \int_0^t e^{-\frac{s}{\tau}} \frac{\partial u}{\partial x} (s) \ ds \left\| \frac{\partial u}{\partial x} (t) \right\|_{L^2}^2.
\end{align*}
\]

(8)

Due to the fact that \( f(x, t, 0) = 0 \), we have

\[
(f(u(t)), \frac{\partial u}{\partial t} (t)) \leq \frac{1}{4\eta^2} f'_{\text{max}}^2 \left\| u(t) \right\|_{L^2}^2 + \eta^2 \left\| \frac{\partial u}{\partial t} (t) \right\|_{L^2}^2
\]

(9)

for some positive constant \( \eta \).

Considering that

\[
\left( \frac{\partial^2 u}{\partial t^2} (t), \frac{\partial u}{\partial t} (t) \right) = \frac{1}{2} \frac{d}{dt} \left\| \frac{\partial u}{\partial t} (t) \right\|_{L^2}^2
\]

and

\[
\left( \frac{\partial u}{\partial x} (t), \frac{\partial^2 u}{\partial t \partial x} (t) \right) = \frac{1}{2} \frac{d}{dt} \left\| \frac{\partial u}{\partial x} (t) \right\|_{L^2}^2,
\]

for all \( t \in (0, T] \).
from (7), (8) and (9), we deduce the inequality
\[
\frac{d}{dt} \left( \rho \| \frac{\partial u}{\partial t} (t) \|^2_{L^2} + (\gamma - \sigma) \| \frac{\partial u}{\partial x} (t) \|^2_{L^2} \right) \\
+ \sigma \frac{1}{\tau} \int_0^t e^{\frac{-t-s}{\tau}} \frac{\partial u}{\partial x} (s) ds + \frac{\partial u}{\partial x} (t) \|^2_{L^2} \\
\leq 2(-\alpha + \eta^2) \| \frac{\partial u}{\partial t} (t) \|^2_{L^2} + 2\frac{\sigma}{\tau} \frac{\partial u}{\partial x} (t) \|^2_{L^2} \\
2\left( \frac{\sigma}{\tau} \right) + \frac{1}{2\eta^2} (f_{\text{max}}')^2 \| u(t) \|^2_{L^2}.
\]
Let \( \eta \) be defined by \( \eta^2 = \alpha + \rho/2 \). Using the Poincaré-Friedrichs inequality \( \| u(t) \|^2_{L^2} \leq (b-a)^2 \| \frac{\partial u}{\partial x} (t) \|^2_{L^2} \) in (10) we obtain the differential inequality
\[
\frac{d}{dt} \left( \rho \| \frac{\partial u}{\partial t} (t) \|^2_{L^2} + (\gamma - \sigma) \| \frac{\partial u}{\partial x} (t) \|^2_{L^2} \right) \\
+ \sigma \frac{1}{\tau} \int_0^t e^{\frac{-t-s}{\tau}} \frac{\partial u}{\partial x} (s) ds + \frac{\partial u}{\partial x} (t) \|^2_{L^2} \\
\leq \max \left\{ \frac{1}{1-\eta^2} \left( \frac{2\sigma}{\tau} + \frac{(f_{\text{max}}')^2 (b-a)^2}{\rho + 2\alpha} \right) \right\} \left( \rho \| \frac{\partial u}{\partial t} (t) \|^2_{L^2} + (\gamma - \sigma) \| \frac{\partial u}{\partial x} (t) \|^2_{L^2} \right) \\
- \frac{\sigma}{\tau} \frac{1}{\tau} \int_0^t e^{\frac{-t-s}{\tau}} \frac{\partial u}{\partial x} (s) ds \| u(t) \|^2_{L^2}
\]
which allows to conclude inequality (6).

Following the proof of Theorem 1 it can be shown the following stability result:

**Theorem 2** Let \( u \) and \( \tilde{u} \) be solutions of (1), (3) with initial conditions \( u_0, u_1 \) and \( \tilde{u}_0, \tilde{u}_1 \) respectively. Under the conditions of Theorem 1, for \( v = u - \tilde{u} \) holds the following
\[
\rho \| \frac{\partial v}{\partial t} (t) \|^2_{L^2} + (\gamma - \sigma) \| \frac{\partial v}{\partial x} (t) \|^2_{L^2} \\
+ \sigma \frac{1}{\tau} \int_0^t e^{\frac{-t-s}{\tau}} \frac{\partial v}{\partial x} (s) ds + \frac{\partial v}{\partial x} (t) \|^2_{L^2} \\
\leq e \max \left\{ \frac{1}{1-\eta^2} \left( \frac{2\sigma}{\tau} + \frac{(f_{\text{max}}')^2 (b-a)^2}{\rho + 2\alpha} \right) \right\} \left( \rho \| u_1 - \tilde{u}_1 \|^2_{L^2} + (\gamma - \sigma) \| u'_0 - \tilde{u}'_0 \|^2_{L^2} \right)
\]

From Theorem 2 we conclude the stability of the model with respect to perturbations of the initial velocity and displacement gradient.

### 3 A numerical method

Let us consider in \([a, b]\) a grid \( I_h = \{ x_j, j = 0, \ldots, N \} \) with \( x_0 = a, x_N = b \) and \( x_j - x_{j-1} = h \). In \([0, T] \) we consider the grid \( \{ t_n, n = 0, \ldots, M \} \) with \( t_0 = 0, t_M = T \) and \( t_{n+1} - t_n = \Delta t \).

We discretize the second partial derivative with respect to \( x \) in (1) using the second-order centered finite-difference operator \( \frac{\partial^2 u}{\partial x^2} \) defined by
\[
\frac{\partial u}{\partial x} (t) \approx \frac{u(x_{i+1}) - u(x_{i-1})}{2h}.
\]

By \( D_{2, x} \) we represent the second-order finite difference operator with respect to time levels,
\[
D_{2, t} \frac{\partial^2 u}{\partial t^2} (t) \approx \frac{u(t_{n+1}) - 2u(t_n) + u(t_{n-1})}{\Delta t^2}.
\]

In the stability and convergence analysis of the numerical methods studied in this paper we consider a discrete version of the \( L^2 \) norm that we present in what follows.

We denote by \( L^2 (I_h) \) the space of grid functions \( u_h \) defined in \( I_h \) such that \( u_h (x_0) = u_h (x_N) = 0 \). In \( L^2 (I_h) \) we consider the discrete inner product
\[
(v_h, w_h)_h = h \sum_{i=1}^{N-1} v_h (x_i) w_h (x_i), \quad v_h, w_h \in L^2 (I_h),
\]
and by \( \| . \|_{L^2 (I_h)} \) we denote the norm induced by the above inner product. For grid functions \( v_h \) and \( w_h \) defined in \( I_h \) we introduce the notations
\[
\frac{\partial u}{\partial x} (t) \approx \frac{u(x_{i+1}) - u(x_{i-1})}{2h}.
\]

By \( D_{2, t} \) and the memory term using a rectangular rule, we obtain a fully discrete approximation
Let this result we also conclude the stability of the behavior of the discrete version of Theorem 1 which allows to study the approximation defined by (14)-(15), a discrete
In what follows we establish for the numerical approximation defined by (14)-(15), a discrete version of Theorem 1 which allows to study the behavior of the discrete $L^2$ norm of the numerical gradients in time and space as well as the past in time of the numerical gradient in space. From this result we also conclude the stability of the method (14)-(15).

Theorem 3 Let $u^j_h$ be defined by (14)-(15) with $u_0(t) = u_b(t) = 0$, $t > 0$. Then

$$
\rho \|D_{-t}u^{n+1}_h\|_{L^2(I_h^t)}^2 + \|D_{-x}u^{n+1}_h\|_{L^2(I_h^x)}^2 \\
+ \sigma \frac{\Delta t}{\tau} \sum_{j=1}^{n+1} e^{-\frac{t}{\Delta t}j - 1} D_{-x}u^j_h + D_{-x}u^{n+1}_h \|_{L^2(I_h^x)}^2 \\
\leq S_p^n \left( \left( 2 + \sigma \left( \frac{\Delta t}{\tau} \right)^2 + 1 \right) \|D_{-x}u_0\|_{L^2(I_h^x)}^2 + \Delta t \left( 1 + \sigma \left( \frac{\Delta t}{\tau} \right)^2 \right) \|D_{-x}u_1\|_{L^2(I_h^x)}^2 + \|u_1\|_{L^2(I_h^t)}^2 \right)
$$

with

$$
S_p = \frac{\max_{\sigma, \gamma, \tau} \left( 1, \gamma + \sigma (3e^{\frac{\Delta t}{\tau}} + \frac{\Delta t}{\tau}) + 2e^{-\frac{\Delta t}{\tau}} (1 + \frac{\Delta t}{\tau}), \sigma \left( e^{-\frac{\Delta t}{\tau}} + 2e^{-\frac{\Delta t}{\tau}} (1 + \frac{\Delta t}{\tau}) \right) \right)}{1 - \Delta t \frac{\nu_{max}^2 - 2 \alpha}{\rho}},
$$

for $\Delta t$ such that

$$
\Delta t \leq \frac{\tau}{2\sigma} \left( \frac{\tau f_{max}^2}{\rho} - 2\sigma - \tau (b-a)^2 - \frac{2\alpha \tau}{\rho} \right) \\
+ \left( \left( \frac{\tau f_{max}^2}{\rho} - 2\sigma - \tau (b-a)^2 - \frac{2\alpha \tau}{\rho} \right)^2 \right)^{1/2} > 0
$$

provided that

$$
\frac{\tau f_{max}^2}{\rho} - 2\sigma - \tau (b-a)^2 - \frac{2\alpha \tau}{\rho} + 4\sigma (\gamma - \sigma - 1)^2 > 0
$$

holds.

Proof: Multiplying each member of (14) by $D_{-t}u^{n+1}_h$ with respect to the inner product $(.,.)_h$ and using summation by parts we obtain

$$
\rho(D_{2,t}u^n_h, D_{-t}u^{n+1}_h) + \alpha \|D_{-t}u^{n+1}_h\|^2_{L^2(I_h)} \\
= \gamma(D_{2,x}u^n_h, D_{-x}u^{n+1}_h) + (f_h(u^{n+1}_h), D_{-t}u^{n+1}_h)_h \\
- \sigma \left( \frac{\Delta t}{\tau} \sum_{j=1}^{n+1} e^{-\frac{t}{\Delta t}j - 1} D_{-x}u^j_h, D_{-x}D_{-t}u^{n+1}_h \right)_h,
$$

where $f_h(u^{n+1}_h)(x_j) = f(x_j, t_{n+1}, u^{n+1}_h(x_j))$.

We have

$$
(D_{2,t}u^n_h, D_{-t}u^{n+1}_h)_h \\
\geq \|D_{-t}u^{n+1}_h\|_{L^2(I_h)} - \|D_{-t}u^n_h\|_{L^2(I_h)},
$$

$$
(D_{2,x}u^n_h, D_{-x}u^{n+1}_h)_h \\
\leq \|D_{-x}u^n_h\|^2_{L^2(I_h^x)} - \|D_{-x}u^{n+1}_h\|^2_{L^2(I_h^x)}
$$

Attending that $f(x, t, 0) = 0$ we also have

$$
(f_h(u^{n+1}_h), D_{-t}u^{n+1}_h)_h \\
\leq \frac{\nu_{max}^2}{2} \|D_{-t}u^{n+1}_h\|^2_{L^2(I_h^t)} + \frac{1}{2} \|u^{n+1}_h\|^2_{L^2(I_h^x)} \\
- \frac{\nu_{max}^2}{2} \|D_{-x}u^{n+1}_h\|^2_{L^2(I_h^x)} + \frac{(b-a)^2}{2} \|D_{-x}u^{n+1}_h\|^2_{L^2(I_h^x)}.
$$
Using (22)-(24) in (21) we obtain

\[
\left(\frac{\rho}{2} + \Delta t (\alpha - \frac{f_{\text{max}}^2}{2})\right) \| D_{-t}u_h^{n+1} \|^2_{L^2(I^+_h)} + \left(\frac{\gamma}{2} - \frac{\Delta t (b - a)^2}{2}\right) \| D_{-x}u_h^{n+1} \|^2_{L^2(I^+_h)}
\leq \frac{\rho}{2} \| D_{-t}u_h^{n} \|^2_{L^2(I^+_h)} + \frac{\gamma}{2} \| D_{-x}u_h^{n} \|^2_{L^2(I^+_h)}
- \sigma \left(\frac{\Delta t^2}{\tau} \sum_{j=1}^{n+1} e^{-\frac{t_{n+1} - t_j}{\tau}} D_{-x}u_h^j, D_{-x} D_{-t}u_h^{n+1}\right)_{h,+}
\]

It can be shown that

\[
\left(\frac{\Delta t^2}{\tau} \sum_{j=1}^{n+1} e^{-\frac{t_{n+1} - t_j}{\tau}} D_{-x}u_h^j, D_{-x} D_{-t}u_h^{n+1}\right)_{h,+}
\]

can be estimated as follows

\[
- \left(\frac{\Delta t^2}{\tau} \sum_{j=1}^{n+1} e^{-\frac{t_{n+1} - t_j}{\tau}} D_{-x}u_h^j, D_{-x} D_{-t}u_h^{n+1}\right)_{h,+}
\leq - \frac{1}{2} \left(\frac{\Delta t^2}{\tau} \sum_{j=1}^{n+1} e^{-\frac{t_{n+1} - t_j}{\tau}} D_{-x}u_h^j + D_{-x} D_{-t}u_h^{n+1}\right)_{h,+}
\]

\[
+ \rho \| D_{-t}u_h^{n+1} \|^2_{L^2(I^+_h)} + \| D_{-x}u_h^{n+1} \|^2_{L^2(I^+_h)}
+ \sigma \left(\frac{\Delta t}{\tau} \sum_{j=1}^{n+1} e^{-\frac{t_{n+1} - t_j}{\tau}} D_{-x}u_h^j + D_{-x} D_{-t}u_h^{n+1}\right)_{h,+}
\]

Then, considering (26) in (25) we deduce

\[
\left(1 - \Delta t \left(\frac{f_{\text{max}}^2}{\rho} - 2\alpha\right)\right) \rho \| D_{-t}u_h^{n+1} \|^2_{L^2(I^+_h)}
+ \left(\gamma - 2\sigma \rho_1 - \Delta t (b - a)^2\right) \| D_{-x}u_h^{n+1} \|^2_{L^2(I^+_h)}
+ \sigma \left(\frac{\Delta t}{\tau} \sum_{j=1}^{n+1} e^{-\frac{t_{n+1} - t_j}{\tau}} D_{-x}u_h^j + D_{-x} D_{-t}u_h^{n+1}\right)_{h,+}
\leq \rho \| D_{-t}u_h^{n} \|^2_{L^2(I^+_h)} + \left(\gamma + 2\sigma \rho_2\right) \| D_{-x}u_h^{n} \|^2_{L^2(I^+_h)}
+ 2\sigma \rho_3 \left(\frac{\Delta t}{\tau} \sum_{j=1}^{n} e^{-\frac{t_n - t_j}{\tau}} D_{-x}u_h^j + D_{-x} D_{-t}u_h^{n}\right)_{h,+}
\]

Let \( \gamma, \sigma, \tau \) and \( f \) such that (20) holds. Then, for \( \Delta t \) satisfying (18) and (19), from (30) we establish

\[
\rho \| D_{-t}u_h^{n+1} \|^2_{L^2(I^+_h)} + \| D_{-x}u_h^{n+1} \|^2_{L^2(I^+_h)}
+ \sigma \left(\frac{\Delta t}{\tau} \sum_{j=1}^{n+1} e^{-\frac{t_{n+1} - t_j}{\tau}} D_{-x}u_h^j + D_{-x} D_{-t}u_h^{n+1}\right)_{h,+}
\leq S_p \left(\rho \| D_{-t}u_h^{n} \|^2_{L^2(I^+_h)} + \| D_{-x}u_h^{n} \|^2_{L^2(I^+_h)}\right)
\]

Using inequality (31) and attending that \( u_h^1(x_j) = u_0(x_j) + \Delta t u_1(x_j) \) we conclude the proof of (16).

As a corollary of Theorem 3 we conclude the following result.

**Corollary 1** Let \( u_h^j \) be defined by (14)-(15) with \( u_a(t) = u_b(t) = 0, t > 0 \). Under the assumption of Theorem 3, if

\[
\max_{\alpha, \gamma, \tau} \leq 1 + C\Delta t,
\]

then exists a positive time and space independent
shown that the global error satisfies the following
\[ e \leq \max_{i} \left( 2 + \sigma \left( \frac{\Delta t}{2} \right)^2 \right) \| D_{-x} u_0 \|_{L^2(I_h^+)} \]
where
\[ \rho = 1 + \sigma \left( \frac{\Delta t}{2} \right)^2 \| D_{-x} u_1 \|_{L^2(I_h^+)} \]
and
\[ e \leq \max_{i} \left( 2 + \sigma \left( \frac{\Delta t}{2} \right)^2 \right) \| D_{-x} u_{n+1} \|_{L^2(I_h^+)} \]

Let \( e_{n+1}^{h} \) be the global error defined by \( e_{n+1}^{h}(x_j) = u_{n+1}^{h}(x_j) - u(x_j, t_{n+1}) \). It can be shown that the global error satisfies the following
\[ \rho D_{x} e_{n+1}^{h}(x_i) + \alpha D_{-x} e_{n+1}^{h}(x_i) = \gamma D_{2,x} e_{n+1}^{h}(x_i) \]
and
\[ e_{n+1}^{h}(x_0) = e_{n+1}^{h}(x_N) = 0, \quad j = 1, \ldots, M - 1, \]
where \( T^j_h (x_i) \) represents the truncation error at time level \( t_j \) in \( x_i \).

Following the proof of Theorem 3, it can be shown that
\[ \rho \| D_{-x} e_{n+1}^{h} \|_{L^2(I_h^+)} \]
for \( \Delta t \) such that
\[ \Delta t \leq \frac{\tau}{2\sigma} \left( \frac{f_{\max}^2 + 1}{\rho} - 2\sigma - \sigma(b-a)^2 - \frac{2\alpha \tau}{\rho} \right) + \left( \left( \frac{f_{\max}^2 + 1}{\rho} - 2\sigma - \sigma(b-a)^2 - \frac{2\alpha \tau}{\rho} \right)^2 \right)^{1/2} \]
and
\[ 1 - \Delta t f_{\max}^2 + 1 - 2\alpha > 0 \]
provided that
\[ \frac{\tau}{2\sigma} \left( \frac{f_{\max}^2 + 1}{\rho} - 2\sigma - \sigma(b-a)^2 - \frac{2\alpha \tau}{\rho} \right) + \left( \left( \frac{f_{\max}^2 + 1}{\rho} - 2\sigma - \sigma(b-a)^2 - \frac{2\alpha \tau}{\rho} \right)^2 \right)^{1/2} > 0. \]

If (32) and the solution of (1)-(2)-(3) is smooth enough, from (36), we conclude the following
\[ \| D_{-x} e_{n+1}^{h} \|_{L^2(I_h^+)} \to 0, \]
and
\[ \| D_{-x} e_{n+1}^{h} \|_{L^2(I_h^+)} \to 0 \]
when \( h, \Delta t \to 0 \).

Conditions (18), (19), (20), for the time stepsize \( \Delta t \), allow to conclude the stability of the finite difference scheme (14)-(15). Those conditions depend on \( f_{\max}^2 \). We observe that if this quantity is very small , the upper bound in (18) can be negative. So, the sufficient condition - Corollary 1 - does not holds but we can not conclude the instability of the scheme. In this case, we should consider the following IMEX discretization:
\[ \rho D_{2,x} e_{n+1}^{h}(x_i) + \alpha D_{-x} e_{n+1}^{h}(x_i) = \gamma D_{2,x} e_{n+1}^{h}(x_i) \]
and
\[ e_{n+1}^{h}(x_0) = e_{n+1}^{h}(x_N) = 0, \quad j = 1, \ldots, M - 1, \]
where \( T^j_h (x_i) \) represents the truncation error at time level \( t_j \) in \( x_i \).
Theorem 4  Let \( u_h^j \) be defined by (15), (41) with \( u_q(t) = u_i(t) = 0, t > 0 \). Then holds (16) with

\[ S_p = \frac{\max_{\sigma, \alpha, \gamma} \max \{ \frac{1}{\Delta t}, \sigma \}}{2} , \]

for \( \Delta t \) such that

\[ \Delta t \leq \frac{\tau}{2\sigma} \left( (\tau - 2\sigma)^2 + 4\sigma(\gamma - \sigma - 1) \right)^{1/2} \]

provided that

\[ \tau - 2\sigma + \left( (\tau - 2\sigma)^2 + 4\sigma(\gamma - \sigma - 1) \right)^{1/2} > 0 \]

holds.

Proof: Attending that

\[
(f_h^u(u_h^h), D_t u_h^{n+1})_h \\
\leq \frac{f_{\max}^2}{4\eta^2} \| u_h^{n+1} \|_{L^2(I_h)}^2 + \eta^2 \| D_t u_h^{n+1} \|_{L^2(I_h)}^2 \\
\leq \frac{(b - a)^2 f_{\max}^2}{4\eta^2} \| D_x u_h^{n+1} \|_{L^2(I_h)}^2 + \eta^2 \| D_t u_h^{n+1} \|_{L^2(I_h)}^2 \\
\]

Following the proof of Theorem 3 it can be shown the following

\[
(\rho + 2\Delta t(\alpha - \eta^2)) \| D_t u_h^{n+1} \|_{L^2(I_h)}^2 + \left( \gamma - 2\sigma p_1 \right) \| D_x u_h^{n+1} \|_{L^2(I_h^*)}^2 \\
+ \sigma \| \Delta t \sum_{j=1}^{n+1} e^{\frac{t_h^{n+1} - t_j}{\tau}} D_x u_h^j + D_x u_h^{n+1} \|_{L^2(I_h^*)}^2 \]

\[
\leq \rho \| D_t u_h^{n+1} \|_{L^2(I_h)}^2 + \left( \gamma + 2\sigma p_2 \right) \| D_x u_h^{n+1} \|_{L^2(I_h^*)}^2 + \frac{(b - a)^2 f_{\max}^2}{2\eta^2} \| D_x u_h^{n+1} \|_{L^2(I_h^*)}^2 \\
+ 2\sigma p_3 \| \Delta t \sum_{j=1}^{n} e^{\frac{t_h^{n+1} - t_j}{\tau}} D_x u_h^j + D_x u_h^{n+1} \|_{L^2(I_h^*)}^2 \]

with \( p_1, p_2 \) and \( p_3 \) defined by (27), (28) and (29) respectively.

Then considering \( \eta^2 = \alpha + \frac{\rho}{2} \) we obtain

\[
\rho \| D_t u_h^{n+1} \|_{L^2(I_h)}^2 + \left( \gamma - 2\sigma p_1 \right) \| D_x u_h^{n+1} \|_{L^2(I_h^*)}^2 + \frac{(b - a)^2 f_{\max}^2}{2\eta^2} \| D_x u_h^{n+1} \|_{L^2(I_h^*)}^2 \\
+ \sigma \| \Delta t \sum_{j=1}^{n+1} e^{\frac{t_h^{n+1} - t_j}{\tau}} D_x u_h^j + D_x u_h^{n+1} \|_{L^2(I_h^*)}^2 \]

Finally, inequality (42) is easily obtained from (46).

For Theorem 4 holds a corollary analogous to Corollary 1. The convergence of the implicit-explicit method (41) with initial and boundary conditions (15) can be studied following the proof of the convergence of method (14)-(15).

4 Numerical Results

The methods studied in Section 3 are rewritten in a convenient way which makes easier their implementation. Method (14)-(15) is equivalent to
the following four level method
\[
\rho D_{2,t} u^n_h(x_i) + \alpha D_{-t} u^{n+1}_h(x_i) \\
-(\gamma + \frac{\Delta t}{\sigma}) D_{2,x} u^{n+1}_h(x_i) - f(x_i, t_{n+1}, u^{n+1}_h(x_i)) \\
e^{-\frac{\Delta t}{\tau}} \left( \rho D_{2,t} u^{n-1}_h(x_i) + \alpha D_{-t} u^n_h(x_i) \\
-\gamma D_{2,x} u^{n+1}_h(x_i) \right) + f(x_i, t_n, u^n_h(x_i)) \\
-e^{-\frac{\Delta t}{\tau}} f(x_i, t_n, u^{n-1}_h(x_i)),
\] (47)
for \(i = 1, \ldots, N-1, n = 2, \ldots, M\), and (14) for \(n = 1\).

IMEX method (41) is equivalent to the next method
\[
\rho D_{2,t} u^n_h(x_i) + \alpha D_{-t} u^{n+1}_h(x_i) \\
-(\gamma + \frac{\Delta t}{\sigma}) D_{2,x} u^{n+1}_h(x_i) \\
e^{-\frac{\Delta t}{\tau}} \left( \rho D_{2,t} u^{n-1}_h(x_i) + \alpha D_{-t} u^n_h(x_i) \right) \\
-\gamma D_{2,x} u^{n+1}_h(x_i) + f(x_i, t_n, u^n_h(x_i)) \\
-e^{-\frac{\Delta t}{\tau}} f(x_i, t_n, u^{n-1}_h(x_i)),
\] (48)
for \(i = 1, \ldots, N-1, n = 2, \ldots, M\). For \(n = 1\) we consider (41).

As in this paper the external force can be nonlinear in the displacement, a nonlinear system should be solved when in computation of the numerical approximation we use method (47). Otherwise, when method (48) is considered, the nonlinear system is replaced by a linear one.

The numerical results are obtained for \([a, b] = [0, 1], T = 1\), homogeneous boundary conditions, initial conditions \(u_1 = 0\) and
\[
u_0(x) = \begin{cases} 
0, & x \in [0, 0.35) \cup (0.65, 1] \\
1 + 10(x - 0.395), & x \in [0.35, 0.45) \\
1, & x \in [0.45, 0.55) \\
1 - 10(x - 0.65), & x \in (0.55, 0.65] 
\end{cases}
\]
We assume that the viscosity is one and the density \(\rho = 0.1\). We consider \(\sigma = \gamma = 0.01\), \(\tau = 1\).

In the numerical experiments we take \(\Delta t = h = 0.01\) and we consider different external forces. The numerical results obtained with method (41) for \(f(u) = ue^{-u}\) are plotted in Figure 1. The behavior of the displacement when the external force is given by \(f(u) = 0.5ue^u\) is illustrated in Figure 2. Method (41) fails on the evaluation of numerical approximations for the external force \(f(u) = 0.8ue^u\). In this case, using the implicit method (14) we obtain the numerical results plotted in Figure 3.

![Figure 1: Numerical solutions obtained with IMEX method and \(f(u) = ue^{-u}\).](image1.png)

![Figure 2: Numerical solutions obtained with IMEX method and \(f(u) = 0.5ue^u\).](image2.png)

References

Figure 3: Numerical solutions obtained with implicit method (14) and \( f(u) = 0.8ue^u \).


