A nonlinear viscoelasticity problem with memory in time

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Abstract:- In this paper we consider a nonlinear hyperbolic equation with a memory term which can be used in mathematical models for viscoleasticity problems. The qualitative properties of the solution of the initial boundary value problem are studied. We propose numerical methods for the computation of approximations to the solution of the continuous problem and their stability properties are analysed. Finally, we include numerical experiments illustrating the performance of the proposed methods.

Key words: Viscoelasticity problem, qualitative properties, stability, numerical methods, implicit method, implicit-explicit method.

1 Introduction

Let us consider the hyperbolic equation

$$\begin{aligned}&\rho\frac{\partial^2 u}{\partial t^2}(x,t)+\alpha\frac{\partial u}{\partial t}(x,t)=\gamma\frac{\partial^2 u}{\partial x^2}(x,t)\\&+\int_0^t k(t-s)\frac{\partial^2 u}{\partial x^2}(x,s)\,ds+f(x,t,u(x,t))\\&\qquad\qquad x\in(a,b),\ t>0,\end{aligned}\tag{1}$$

where $k(s)$ is a scalar function smooth enough and which will be specified later, with initial conditions

$$\begin{cases} u(0,x)=u_0(x),\ x\in(a,b)\\ \dfrac{\partial u}{\partial t}(0,x)=u_1(x), x\in(a,b)\end{cases}\tag{2}$$

and

$$u(a,t)=u_a(t),\ u(b,t)=u_b(t),\ t>0.\tag{3}$$

Initial boundary value problem (IBVP) (1)-(3) arises from a variety of mathematical models in engineering and physical sciences. We mention, for instance, the theory of viscoelasticity (see for instance [6], [7], [8]). In this case u represents the displacement of a body with density ρ, viscosity α and under external force f depending of the displacement. The use of exponential kernels of type $k(s)=-\frac{\sigma}{\tau}e^{-\frac{s}{\tau}}$ has been largely considered in the context of heat conduction problems with memory in time ([1], [5], [9]). Attending that the heat conduction problem can be seen has a singular perturbation when the density ρ is small, in what follows we take the exponential kernel mentioned before.

Our aim in this paper is to study the qualitative behavior of the solution of (1)-(2)- (3) from the theoretical and numerical points of view. Estimates for the kinetic energy, potential energy and to the past in time of the gradient of the displacement are obtained and allow to conclude the stability of (1)-(2)-(3). From a numerical point of view and following the approach introduced in [3] for the generalized Fisher equation and in [4] for a linear viscoelasticity problem, we propose numerical methods which enable us to compute numerical approximations presenting the qualitative behavior of the continuous solution provided some conditions on the stepsize are imposed. Numerical results illustrating the behavior of the methods studied are also included.

The paper is organized as follows. In Section 2 the theoretical study is presented. Numerical methods are analysed in Section 3. Finally, in Section 4, the numerical simulation is included.

2 Continuous qualitative behavior

In Theorem 1 we establish an estimate to the kinetic and potential energies and to the past in time of the gradient of the displacement when $\gamma \neq \sigma$. This result enables to conclude the stability of (1)-(2)-(3).

Theorem 1 *Let u be a solution of (1)-(2) with homogeneous boundary conditions. Let us suppose that*

$$u(x,t) \in [c,d],\ x \in [a,b], t \in [0,T], \tag{4}$$

with c, d constants, and

$$\frac{\partial^\ell u}{\partial t^\ell}(t),\ \int_0^t e^{-\frac{t-s}{\tau}} \frac{\partial^\ell u}{\partial x^\ell}(s)ds \in L^2[a,b], \tag{5}$$

for $\ell = 1,2, t \in (0,T]$.

If f is continuously differentiable in the third argument and $f(x,t,0) = 0$ for $x \in [a,b], t \in [0,T]$, then, for each $t \in (0,T]$, holds

$$\begin{aligned} &\rho\|\frac{\partial u}{\partial t}(t)\|^2_{L^2} + (\gamma-\sigma)\|\frac{\partial u}{\partial x}(t)\|^2_{L^2} \\ &+\sigma\|\frac{1}{\tau}\int_0^t e^{-\frac{t-s}{\tau}}\frac{\partial u}{\partial x}(s)\,ds + \frac{\partial u}{\partial x}(t)\|^2_{L^2} \\ &\leq e^{\max\left\{1, \frac{1}{\gamma-\sigma}\left(\frac{2}{\tau} + \frac{(f'_{\max})^2(b-a)^2}{\rho+2\alpha}\right)\right\}t}\Big(\rho\|u_1\|^2_{L^2} \\ &\qquad + (\gamma-\sigma)\|u_0'\|^2_{L^2}\Big) \end{aligned} \tag{6}$$

where $f'_{\max} = \max\limits_{[a,b]\times[0,T]\times[c,d]} \dfrac{\partial f}{\partial z}$.

Proof: Multiplying each member of (1) by $\frac{\partial u}{\partial t}$ with respect to the L^2 inner product and integrating by parts we obtain

$$\begin{aligned} &\rho(\frac{\partial^2 u}{\partial t^2}(t), \frac{\partial u}{\partial t}(t)) + \alpha\|\frac{\partial u}{\partial t}(t)\|^2_{L^2} \\ &= -\gamma(\frac{\partial u}{\partial x}(t), \frac{\partial^2 u}{\partial t \partial x}(t)) \\ &-\frac{\sigma}{\tau}(\int_0^t e^{-\frac{t-s}{\tau}}\frac{\partial u}{\partial x}(s)\,ds, \frac{\partial^2 u}{\partial x \partial t}(t)) + (f(u(t)), \frac{\partial u}{\partial t}(t))\,. \end{aligned} \tag{7}$$

It can be shown that

$$\begin{aligned} &(\frac{1}{\tau}\int_0^t e^{-\frac{t-s}{\tau}}\frac{\partial u}{\partial x}(s)\,ds, \frac{\partial^2 u}{\partial x \partial t}(t)) \\ &= \frac{1}{2}\frac{d}{dt}\|\frac{1}{\tau}\int_0^t e^{-\frac{t-s}{\tau}}\frac{\partial u}{\partial x}(s)\,ds + \frac{\partial u}{\partial x}(t)\|^2_{L^2} \\ &\quad -\frac{1}{2}\frac{d}{dt}\|\frac{\partial u}{\partial x}(t)\|^2_{L^2} - \frac{1}{\tau}\|\frac{\partial u}{\partial x}(t)\|^2_{L^2} \\ &\quad +\frac{1}{\tau}\|\frac{1}{\tau}\int_0^t e^{-\frac{t-s}{\tau}}\frac{\partial u}{\partial x}(s)\,ds\|^2_{L^2}. \end{aligned} \tag{8}$$

Due to the fact that $f(x,t,0) = 0$, we have

$$(f(u(t)), \frac{\partial u}{\partial t}(t)) \leq \frac{1}{4\eta^2}(f'_{\max})^2\|u(t)\|^2_{L^2} + \eta^2\|\frac{\partial u}{\partial t}(t)\|^2_{L^2} \tag{9}$$

for some positive constant η.

Considering that

$$(\frac{\partial^2 u}{\partial t^2}(t), \frac{\partial u}{\partial t}(t)) = \frac{1}{2}\frac{d}{dt}\|\frac{\partial u}{\partial t}(t)\|^2_{L^2}$$

and

$$(\frac{\partial u}{\partial x}(t), \frac{\partial^2 u}{\partial t \partial x}(t)) = \frac{1}{2}\frac{d}{dt}\|\frac{\partial u}{\partial x}(t)\|^2_{L^2},$$

from (7), (8) and (9), we deduce the inequality

$$\begin{aligned}
&\frac{d}{dt}\Big(\rho\|\frac{\partial u}{\partial t}(t)\|^2_{L^2} + (\gamma-\sigma)\|\frac{\partial u}{\partial x}(t)\|^2_{L^2}\\
&+\sigma\|\frac{1}{\tau}\int_0^t e^{-\frac{t-s}{\tau}}\frac{\partial u}{\partial x}(s)\,ds + \frac{\partial u}{\partial x}(t)\|^2_{L^2}\Big)\\
&\le 2(-\alpha+\eta^2)\|\frac{\partial u}{\partial t}(t)\|^2_{L^2} + \frac{2\sigma}{\tau}\|\frac{\partial u}{\partial x}\|^2_{L^2} \qquad (10)\\
&-\frac{2\sigma}{\tau}\|\frac{1}{\tau}\int_0^t e^{-\frac{t-s}{\tau}}\frac{\partial u}{\partial x}(s)\,ds\|^2_{L^2}\\
&\qquad + \frac{1}{2\eta^2}(f'_{\max})^2\|u(t)\|^2_{L^2}\,.
\end{aligned}$$

Let η be defined by $\eta^2 = \alpha + \rho/2$. Using the Poincaré-Friedrichs inequality $\|u(t)\|^2_{L^2} \le (b-a)^2\|\frac{\partial u}{\partial x}(t)\|^2_{L^2}$ in (10) we obtain the differential inequality

$$\begin{aligned}
&\frac{d}{dt}\Big(\rho\|\frac{\partial u}{\partial t}(t)\|^2_{L^2} + (\gamma-\sigma)\|\frac{\partial u}{\partial x}(t)\|^2_{L^2}\\
&+\sigma\|\frac{1}{\tau}\int_0^t e^{-\frac{t-s}{\tau}}\frac{\partial u}{\partial x}(s)\,ds + \frac{\partial u}{\partial x}(t)\|^2_{L^2}\\
&\le \max\Big\{1, \frac{1}{\gamma-\sigma}(\frac{2\sigma}{\tau} + \frac{(f'_{\max})^2(b-a)^2}{\rho+2\alpha})\Big\}\\
&\Big(\rho\|\frac{\partial u}{\partial t}(t)\|^2_{L^2} + (\gamma-\sigma)\|\frac{\partial u}{\partial x}(t)\|^2_{L^2}\Big)\\
&-\frac{\sigma}{\tau}\|\frac{1}{\tau}\int_0^t e^{-\frac{t-s}{\tau}}\frac{\partial u}{\partial x}(s)\,ds\|^2_{L^2}
\end{aligned} \qquad (11)$$

which allows to conclude inequality (6). ■

Following the proof of Theorem 1 it can be shown the following stability result:

Theorem 2 *Let u and $\tilde{u}$ be solutions of (1), (3) with initial conditions u_0, u_1 and $\tilde{u}_0, \tilde{u}_1$ respectively. Under the conditions of Theorem 1, for $v = u - \tilde{u}$ holds the following*

$$\begin{aligned}
&\rho\|\frac{\partial v}{\partial t}(t)\|^2_{L^2} + (\gamma-\sigma)\|\frac{\partial v}{\partial x}(t)\|^2_{L^2}\\
&+\sigma\|\frac{1}{\tau}\int_0^t e^{-\frac{t-s}{\tau}}\frac{\partial v}{\partial x}(s)\,ds + \frac{\partial v}{\partial x}(t)\|^2_{L^2}\\
&\le e^{\max\left\{1, \frac{1}{\gamma-\sigma}\left(\frac{2}{\tau} + \frac{(f'_{\max})^2(b-a)^2}{\rho+2\alpha}\right)\right\}t}\Big(\rho\|u_1-\tilde{u}_1\|^2_{L^2}\\
&\qquad + (\gamma-\sigma)\|u'_0 - \tilde{u}'_0\|^2_{L^2}\Big)
\end{aligned} \qquad (12)$$

From Theorem 2 we conclude the stability of the model with respect to perturbations of the initial velocity and displacement gradient.

3 A numerical method

Let us consider in $[a,b]$ a grid $I_h = \{x_j, j = 0,\ldots,N\}$ with $x_0 = a, x_N = b$ and $x_j - x_{j-1} = h$. In $[0,T]$ we consider the grid $\{t_n, n = 0,\ldots,M\}$ with $t_0 = 0, t_M = T$ and $t_{n+1} - t_n = \Delta t$.

We discretize the second partial derivative with respect to x in (1) using the second-order centered finite-difference operator $D_{2,x}$ defined by

$$D_{2,x}v_h^n(x_i) = \frac{v_h^n(x_{i+1}) - 2v_h^n(x_i) + v_h^n(x_{i-1})}{h^2}.$$

By $D_{2,t}$ we represent the second-order finite difference operator with respect to time levels,

$$D_{2,t}v_h^n(x_i) = \frac{v_h^{n+1}(x_i) - 2v_h^n(x_i) + v_h^{n-1}(x_i)}{\Delta t^2}.$$

In the stability and convergence analysis of the numerical methods studied in this paper we consider a discrete version of the L^2 norm that we present in what follows.

We denote by $L^2(I_h)$ the space of grid functions v_h defined in I_h such that $v_h(x_0) = v_h(x_N) = 0$. In $L^2(I_h)$ we consider the discrete inner product

$$(v_h, w_h)_h = h\sum_{i=1}^{N-1} v_h(x_i)w_h(x_i),\ v_h, w_h \in L^2(I_h), \qquad (13)$$

and by $\|.\|_{L^2(I_h)}$ we denote the norm induced by the above inner product. For grid functions w_h and v_h defined in I_h we introduce the notations

$$(v_h, w_h)_{h,+} = h\sum_{i=1}^{N} v_h(x_i)w_h(x_i)$$

$$\|w_h\|_{L^2(I_h^+)} = \left(h\sum_{i=1}^{N} w_h^2(x_i)\right)^{1/2}.$$

Discretizing the spatial derivatives using $D_{2,x}$ and $D_{2,t}$ and the memory term using a rectangular rule, we obtain a fully discrete approximation

u_h^n defined by

$$\begin{array}{l}\rho D_{2,t}u_h^n(x_i)+\alpha D_{-t}u_h^{n+1}(x_i)=\gamma D_{2,x}u_h^{n+1}(x_i)\\ +\dfrac{\sigma}{\tau}\Delta t\displaystyle\sum_{j=1}^{n+1}e^{-\frac{t_{n+1}-t_\ell}{\tau}}D_{2,x}u_h^j(x_i)\\ +f(x_i,t_{n+1},u_h^{n+1}(x_i)),\\ i=1,\ldots,N-1,\; n=1,\ldots,M-1,\end{array} \tag{14}$$

where

$$\begin{array}{l}u_h^j(x_0)=u_a(t_j),\; u_h^j(x_N)=u_b(t_j),\\ j=1,\ldots,M-1,\\ u_h^1(x_i)=u_0(x_i)+\Delta t u_1(x_i),\\ u_h^0(x_i)=u_0(x_i), i=1,\ldots,N-1.\end{array} \tag{15}$$

In what follows we establish for the numerical approximation defined by (14)-(15), a discrete version of Theorem 1 which allows to study the behavior of the discrete L^2 norm of the numerical gradients in time and space as well as the past in time of the numerical gradient in space. From this result we also conclude the stability of the method (14)-(15).

Theorem 3 *Let u_h^j be defined by (14)-(15) with $u_a(t)=u_b(t)=0, t>0$. Then*

$$\begin{array}{l}\rho\|D_{-t}u_h^{n+1}\|^2_{L^2(I_h)}+\|D_{-x}u_h^{n+1}\|^2_{L^2(I_h^+)}\\ +\sigma\|\dfrac{\Delta t}{\tau}\displaystyle\sum_{j=1}^{n+1}e^{-\frac{t_{n+1}-t_j}{\tau}}D_{-x}u_h^j+D_{-x}u_h^{n+1}\|^2_{L^2(I_h^+)}\\ \leq S_p^n\Big(\Big(2+\sigma((\dfrac{\Delta t}{\tau})^2+1)\Big)\|D_{-x}u_0\|^2_{L^2(I_h^+)}\\ +\Delta t\Big(1+\sigma((\dfrac{\Delta t}{\tau})^2+1)\Big)\|D_{-x}u_1\|^2_{L^2(I_h^+)}\\ \qquad +\|u_1\|^2_{L^2(I_h)}\Big)\end{array} \tag{16}$$

with

$$S_p=\frac{\max_{\sigma,\gamma,\tau}}{1-\Delta t\frac{f'^2_{max}-2\alpha}{\rho}}, \tag{17}$$

$$\begin{array}{l}\max\limits_{\sigma,\gamma,\tau}=\max\{1,\gamma+\sigma\big(3e^{-\frac{\Delta t}{\tau}}+\dfrac{\Delta t}{\tau}\\ +2e^{-2\frac{\Delta t}{\tau}}(1+\dfrac{\Delta t}{\tau})\big),\,\sigma\Big(e^{-\frac{\Delta t}{\tau}}+2e^{-2\frac{\Delta t}{\tau}}(1+\dfrac{\Delta t}{\tau})\Big)\}\end{array}$$

for Δt such that

$$\begin{array}{l}\Delta t\leq\dfrac{\tau}{2\sigma}\Big(\dfrac{\tau f'^2_{max}}{\rho}-2\sigma-\tau(b-a)^2-\dfrac{2\alpha\tau}{\rho}\\ +\Big(\Big(\dfrac{\tau f'^2_{max}}{\rho}-2\sigma-\tau(b-a)^2-\dfrac{2\alpha\tau}{\rho}\Big)^2\\ +4\sigma(\gamma-\sigma-1)\Big)^{1/2}\Big)\end{array} \tag{18}$$

and

$$1-\Delta t\frac{f'^2_{max}-2\alpha}{\rho}>0 \tag{19}$$

provided that

$$\begin{array}{l}\dfrac{\tau f'^2_{max}}{\rho}-2\sigma-\tau(b-a)^2-\dfrac{2\alpha\tau}{\rho}\\ +\Big(\Big(\dfrac{\tau f'^2_{max}}{\rho}-2\sigma-\tau(b-a)^2-\dfrac{2\alpha\tau}{\rho}\Big)^2\\ +4\sigma(\gamma-\sigma-1)\Big)^{1/2}>0\end{array} \tag{20}$$

holds.

Proof: Multiplying each member of (14) by $D_{-t}u_h^{n+1}$ with respect to the inner product $(.,.)_h$ and using summation by parts we obtain

$$\begin{array}{l}\rho(D_{2,t}u_h^n,D_{-t}u_h^{n+1})_h+\alpha\|D_{-t}u_h^{n+1}\|^2_{L^2(I_h)}\\ =\gamma(D_{2,x}u_h^{n+1},D_{-t}u_h^{n+1})_h+(f_h(u_h^{n+1}),D_{-t}u_h^{n+1})_h\\ -\sigma(\dfrac{\Delta t}{\tau}\displaystyle\sum_{j=1}^{n+1}e^{-\frac{t_{n+1}-t_j}{\tau}}D_{-x}u_h^j,D_{-x}D_{-t}u_h^{n+1})_{h,+},\end{array} \tag{21}$$

where $f_h(u_{n+1})(x_j)=f(x_j,t_{n+1},u_h^{n+1}(x_j))$.

We have

$$\begin{array}{l}(D_{2,t}u_h^n,D_{-t}u_h^{n+1})_h\\ \geq\dfrac{\|D_{-t}u_h^{n+1}\|^2_{L^2(I_h)}-\|D_{-t}u_h^n\|^2_{L^2(I_h)}}{2\Delta t},\end{array} \tag{22}$$

$$\begin{array}{l}(D_{2,x}u_h^{n+1},D_{-t}u_h^{n+1})_h\\ \leq\dfrac{\|D_{-x}u_h^n\|^2_{L^2(I_h^+)}-\|D_{-x}u_h^{n+1}\|^2_{L^2(I_h^+)}}{2\Delta t}.\end{array} \tag{23}$$

Attending that $f(x,t,0)=0$ we also have

$$\begin{array}{l}(f_h(u_h^{n+1}),D_{-t}u_h^{n+1})_h\\ \leq\dfrac{f'^2_{max}}{2}\|D_{-t}u_h^{n+1}\|^2_{L^2(I_h)}+\dfrac{1}{2}\|u_h^{n+1}\|^2_{L^2(I_h)},\\ \leq\dfrac{f'^2_{max}}{2}\|D_{-t}u_h^{n+1}\|^2_{L^2(I_h)}+\dfrac{(b-a)^2}{2}\|D_{-x}u_h^{n+1}\|^2_{L^2(I_h^+)}.\end{array} \tag{24}$$

Using (22)-(24) in (21) we obtain

$$
\begin{aligned}
&(\frac{\rho}{2}+\Delta t(\alpha-\frac{f'^2_{max}}{2}))\|D_{-t}u_h^{n+1}\|^2_{L^2(I_h)}\\
&+\Big(\frac{\gamma}{2}-\frac{\Delta t(b-a)^2}{2}\Big)\|D_{-x}u_h^{n+1}\|^2_{L^2(I_h^+)}\\
&\le \frac{\rho}{2}\|D_{-t}u_h^{n}\|^2_{L^2(I_h)}+\frac{\gamma}{2}\|D_{-x}u_h^{n}\|^2_{L^2(I_h^+)}\\
&-\sigma(\frac{\Delta t^2}{\tau}\sum_{j=1}^{n+1}e^{-\frac{t_{n+1}-t_j}{\tau}}D_{-x}u_h^j, D_{-x}D_{-t}u_h^{n+1})_{h,+}\,.
\end{aligned}
\tag{25}
$$

It can be shown that

$$
(\frac{\Delta t^2}{\tau}\sum_{j=1}^{n+1}e^{-\frac{t_{n+1}-t_j}{\tau}}D_{-x}u_h^j, D_{-x}D_{-t}u_h^{n+1})_{h,+}
$$

can be estimated as follows

$$
\begin{aligned}
&-(\frac{\Delta t^2}{\tau}\sum_{j=1}^{n+1}e^{-\frac{t_{n+1}-t_j}{\tau}}D_{-x}u_h^j, D_{-t}D_{-x}u_h^{n+1})_{h,+}\\
&\le -\frac{1}{2}\|\frac{\Delta t}{\tau}\sum_{j=1}^{n+1}e^{-\frac{t_{n+1}-t_j}{\tau}}D_{-x}u_h^j+D_{-x}u_h^{n+1}\|^2_{L^2(I_h^+)}\\
&+p_1\|\frac{\Delta t}{\tau}\sum_{j=1}^{n}e^{-\frac{t_{n}-t_j}{\tau}}D_{-x}u_h^j+D_{-x}u_h^{n}\|^2_{L^2(I_h^+)}\\
&+p_2\|D_{-x}u_h^n\|^2_{L^2(I_h^+)}+p_3\|D_{-x}u_h^{n+1}\|^2_{L^2(I_h^+)}
\end{aligned}
\tag{26}
$$

with

$$
p_1=\frac{e^{-\frac{\Delta t}{\tau}}}{2}+e^{-2\frac{\Delta t}{\tau}}(1+\frac{\Delta t}{\tau}), \tag{27}
$$

$$
p_2=\frac{1}{2}(3e^{-\frac{\Delta t}{\tau}}+\frac{\Delta t}{\tau})+e^{-2\frac{\Delta t}{\tau}}(1+\frac{\Delta t}{\tau}) \tag{28}
$$

and

$$
p_3=\frac{1}{2}\Big(1+2\frac{\Delta t}{\tau}+(\frac{\Delta t}{\tau})^2\Big). \tag{29}
$$

Then, considering (26) in (25) we deduce

$$
\begin{aligned}
&(1-\Delta t(\frac{f'^2_{max}-2\alpha}{\rho})\rho\|D_{-t}u_h^{n+1}\|^2_{L^2(I_h^+)}\\
&+\Big(\gamma-2\sigma p_1-\Delta t(b-a)^2\Big)\|D_{-x}u_h^{n+1}\|^2_{L^2(I_h^+)}\\
&+\sigma\|\frac{\Delta t}{\tau}\sum_{j=1}^{n+1}e^{-\frac{t_{n+1}-t_j}{\tau}}D_{-x}u_h^j+D_{-x}u_h^{n+1}\|^2_{L^2(I_h^+)}\\
&\le \rho\|D_{-t}u_h^{n}\|^2_{L^2(I_h^+)}+\Big(\gamma+2\sigma p_2\Big)\|D_{-x}u_h^{n}\|^2_{L^2(I_h^+)}\\
&+2\sigma p_3\|\frac{\Delta t}{\tau}\sum_{j=1}^{n}e^{-\frac{t_{n}-t_j}{\tau}}D_{-x}u_h^j+D_{-x}u_h^{n}\|^2_{L^2(I_h^+)}
\end{aligned}
\tag{30}
$$

Let γ,σ,τ and f such that (20) holds. Then, for Δt satisfying (18) and (19), from (30) we establish

$$
\begin{aligned}
&\rho\|D_{-t}u_h^{n+1}\|^2_{L^2(I_h^+)}+\|D_{-x}u_h^{n+1}\|^2_{L^2(I_h^+)}\\
&+\sigma\|\frac{\Delta t}{\tau}\sum_{j=1}^{n+1}e^{-\frac{t_{n+1}-t_j}{\tau}}D_{-x}u_h^j+D_{-x}u_h^{n+1}\|^2_{L^2(I_h^+)}\\
&\le S_p\Big(\rho\|D_{-t}u_h^{n}\|^2_{L^2(I_h^+)}+\|D_{-x}u_h^{n}\|^2_{L^2(I_h^+)}\\
&+\sigma\|\frac{\Delta t}{\tau}\sum_{j=1}^{n}e^{-\frac{t_{n}-t_j}{\tau}}D_{-x}u_h^j+D_{-x}u_h^{n}\|^2_{L^2(I_h^+)}\Big)
\end{aligned}
\tag{31}
$$

Using inequality (31) and attending that $u_h^1(x_j)=u_0(x_j)+\Delta t u_1(x_j)$ we conclude the proof of (16).

■

As a corollary of Theorem 3 we conclude the following result.

Corollary 1 *Let u_h^j be defined by (14)-(15) with $u_a(t)=u_b(t)=0, t>0$. Under the assumption of Theorem 3, if*

$$
\max_{\sigma,\gamma,\tau}\le 1+C\Delta t, \tag{32}
$$

then exists a positive time and space independent *constant $\mathcal{C}$ such that*

$$\rho\|D_{-t}u_h^{n+1}\|^2_{L^2(I_h)} + \|D_{-x}u_h^{n+1}\|^2_{L^2(I_h^+)} \\ +\sigma\|\frac{\Delta t}{\tau}\sum_{j=1}^{n+1} e^{-\frac{t_{n+1}-t_j}{\tau}} D_{-x}u_h^j + D_{-x}u_h^{n+1}\|^2_{L^2(I_h^+)} \\ \leq \mathcal{C}\Big(\Big(2+\sigma\big((\frac{\Delta t}{\tau})^2+1\big)\Big)\|D_{-x}u_0\|^2_{L^2(I_h^+)} \\ +\Delta t\Big(1+\sigma\big((\frac{\Delta t}{\tau})^2+1\big)\Big)\|D_{-x}u_1\|^2_{L^2(I_h^+)} \\ +\|u_1\|^2_{L^2(I_h)}\Big) \tag{33}$$

■

Let e_h^{n+1} be the global error defined by $e_h^{n+1}(x_j) = u_h^{n+1}(x_j) - u(x_j, t_{n+1})$. It can be shown that the global error satisfies the following

$$\rho D_{2,t}e_h^n(x_i) + \alpha D_{-t}e_h^{n+1}(x_i) = \gamma D_{2,x}e_h^{n+1}(x_i) \\ +\frac{\sigma}{\tau}\Delta t\sum_{j=1}^{n+1} e^{-\frac{t_{n+1}-t_\ell}{\tau}} D_{2,x}e_h^j(x_i) \\ +f(x_i,t_{n+1},u_h^{n+1}(x_i)) - f(x_i,t_{n+1},u(x_i,t_{n+1})) \\ +T_h^{n+1}(x_i) \\ i=1,\ldots,N-1,\ n=1,\ldots,M-1, \tag{34}$$

and

$$e_h^j(x_0) = e_h^j(x_N) = 0,\ j=1,\ldots,M-1, \\ e_h^1(x_i) = \Delta t T^1(x_i), \\ e_h^0(x_i) = 0, i=1,\ldots,N-1, \tag{35}$$

where $T_h^j(x_i)$ represents the truncation error at time level t_j in x_i.

Following the proof of Theorem 3, it can be shown that

$$\rho\|D_{-t}e_h^{n+1}\|^2_{L^2(I_h)} + \|D_{-x}e_h^{n+1}\|^2_{L^2(I_h^+)} \\ +\sigma\|\frac{\Delta t}{\tau}\sum_{j=1}^{n+1} e^{-\frac{t_{n+1}-t_j}{\tau}} D_{-x}e_h^j + D_{-x}e_h^{n+1}\|^2_{L^2(I_h^+)} \\ \leq S_p^{n+1}\Big((1+\sigma((\frac{\Delta t}{\tau})^2+1))\|D_{-x}T_h^1\|^2_{L^2(I_h^+)} \\ +\rho\|T_h^1\|^2_{L^2(I_h)}\Big) + \frac{\Delta t}{\max_{\sigma,\gamma,\tau}}\sum_{j=1}^{n} S_p^{n+1-j}\|T_h^{j+1}\|^2_{L^2(I_h)} \tag{36}$$

where

$$S_p = \frac{\max_{\sigma,\gamma,\tau}}{1-\Delta t\frac{f'^2_{max}+1-2\alpha}{\rho}}, \tag{37}$$

for Δt such that

$$\Delta t \leq \frac{\tau}{2\sigma}\Big(\frac{\tau(f'^2_{max}+1)}{\rho} - 2\sigma - \tau(b-a)^2 - \frac{2\alpha\tau}{\rho} \\ +\Big(\Big(\frac{\tau f'^2_{max}+1}{\rho} - 2\sigma - \tau(b-a)^2 - \frac{2\alpha\tau}{\rho}\Big)^2 \\ +4\sigma(\gamma-\sigma-1)\Big)^{1/2}\Big) \tag{38}$$

and

$$1 - \Delta t\frac{f'^2_{max}+1-2\alpha}{\rho} > 0 \tag{39}$$

provided that

$$\frac{\tau(f'^2_{max}+1)}{\rho} - 2\sigma - \tau(b-a)^2 - \frac{2\alpha\tau}{\rho} \\ +\Big(\Big(\frac{\tau(f'^2_{max}+1)}{\rho} - 2\sigma - \tau(b-a)^2 - \frac{2\alpha\tau}{\rho}\Big)^2 \\ +4\sigma(\gamma-\sigma-1)\Big)^{1/2} > 0. \tag{40}$$

If (32) and the solution of (1)-(2)-(3) is smooth enough, from (36), we conclude the following

$$\|D_{-t}e_h^{n+1}\|_{L^2(I_h)} \to 0, \\ \|D_{-x}e_h^{n+1}\|_{L^2(I_h^+)} \to 0$$

and

$$\|\frac{\Delta t}{\tau}\sum_{j=1}^{n+1} e^{-\frac{t_{n+1}-t_j}{\tau}} D_{-x}e_h^j + D_{-x}e_h^{n+1}\|_{L^2(I_h^+)} \to 0,$$

when $h, \Delta t \to 0$.

Conditions (18), (19), (20), for the time stepsize Δt, allow to conclude the stability of the finite difference scheme (14)-(15). Those conditions dependent on f'^2_{max}. We observe that if this quantity is very small , the upper bound in (18) can be negative. So, the sufficient condition - Corollary 1 - does not holds but we can not conclude the instability of the scheme. In this case, we should consider the following IMEX discretization:

$$\rho D_{2,t}u_h^n(x_i) + \alpha D_{-t}u_h^{n+1}(x_i) = \gamma D_{2,x}u_h^{n+1}(x_i) \\ +\frac{\sigma}{\tau}\Delta t\sum_{j=1}^{n+1} e^{-\frac{t_{n+1}-t_\ell}{\tau}} D_{2,x}u_h^j(x_i) + f(x_i,t_{n+1},u_h^n(x_i)), \\ i=1,\ldots,N-1,\ n=1,\ldots,M-1, \tag{41}$$

with the initial and boundary conditions (15).

The stability of the last scheme is established in the result.

Theorem 4 *Let u_h^j be defined by (15), (41) with $u_a(t) = u_b(t) = 0, t > 0$. Then holds (16) with*

$$S_p = \frac{\max_{\sigma,\gamma,\tau}}{1-\Delta t}\,, \tag{42}$$

$$\begin{aligned}\max_{\sigma,\gamma,\tau} &= \max\{1, \gamma + \sigma\big(3e^{-\frac{\Delta t}{\tau}} + \frac{\Delta t}{\tau} \\ &+2e^{-2\frac{\Delta t}{\tau}}(1+\frac{\Delta t}{\tau})\big) + \Delta t\frac{f_{max}'^2}{2\alpha+\rho}, \\ &\sigma\Big(e^{-\frac{\Delta t}{\tau}} + 2e^{-2\frac{\Delta t}{\tau}}(1+\tfrac{\Delta t}{\tau})\Big)\}\end{aligned}$$

for Δt such that

$$\begin{aligned}\Delta t \leq \frac{\tau}{2\sigma}\Big(\tau - 2\sigma + \Big(\Big(\tau-2\sigma\Big)^2 \\ +4\sigma(\gamma-\sigma-1)\Big)^{1/2}\Big)\end{aligned} \tag{43}$$

provided that

$$\begin{aligned}\tau - 2\sigma + \Big(\Big(\tau-2\sigma\Big)^2 \\ +4\sigma(\gamma-\sigma-1)\Big)^{1/2} > 0\end{aligned} \tag{44}$$

holds.

Proof: Attending that

$$\begin{aligned}&(f_h(u_h^n), D_{-t}u_h^{n+1})_h \\ &\leq \frac{f_{max}'^2}{4\eta^2}\|u_h^n\|^2_{L^2(I_h)} + \eta^2\|D_{-t}u_h^{n+1}\|^2_{L^2(I_h)} \\ &\leq \frac{(b-a)^2 f_{max}'^2}{4\eta^2}\|D_{-x}u_h^n\|^2_{L^2(I_h^+)} \\ &+\eta^2\|D_{-t}u_h^{n+1}\|^2_{L^2(I_h)}\,.\end{aligned}$$

Following the proof of Theorem 3 it can be shown the following

$$\begin{aligned}&\big(\rho + 2\Delta t(\alpha-\eta^2)\big)\|D_{-t}u_h^{n+1}\|^2_{L^2(I_h)} \\ &+\Big(\gamma - 2\sigma p_1\Big)\|D_{-x}u_h^{n+1}\|^2_{L^2(I_h^+)} \\ &+\sigma\|\frac{\Delta t}{\tau}\sum_{j=1}^{n+1} e^{-\frac{t_{n+1}-t_j}{\tau}} D_{-x}u_h^j + D_{-x}u_h^{n+1}\|^2_{L^2(I_h^+)} \\ &\leq \rho\|D_{-t}u_h^n\|^2_{L^2(I_h)} + \Big(\gamma + 2\sigma p_2 \\ &+\frac{(b-a)^2 f_{max}'^2}{2\eta^2}\Big)\|D_{-x}u_h^n\|^2_{L^2(I_h^+)} \\ &+2\sigma p_3\|\frac{\Delta t}{\tau}\sum_{j=1}^{n} e^{-\frac{t_n-t_j}{\tau}} D_{-x}u_h^j + D_{-x}u_h^n\|^2_{L^2(I_h^+)}\end{aligned} \tag{45}$$

with p_1, p_2 and p_3 defined by (27), (28) and (29) respectively.

Then considering $\eta^2 = \alpha + \frac{\rho}{2}$ we obtain

$$\begin{aligned}&\rho\|D_{-t}u_h^{n+1}\|^2_{L^2(I_h)} + \Big(\gamma - 2\sigma p_1\Big)\|D_{-x}u_h^{n+1}\|^2_{L^2(I_h^+)} \\ &+\sigma\|\frac{\Delta t}{\tau}\sum_{j=1}^{n+1} e^{-\frac{t_{n+1}-t_j}{\tau}} D_{-x}u_h^j + D_{-x}u_h^{n+1}\|^2_{L^2(I_h^+)} \\ &\leq \rho\|D_{-t}u_h^n\|^2_{L^2(I_h)} + \Big(\gamma + 2\sigma p_2 \\ &+\frac{(b-a)^2 f_{max}'^2}{2\alpha+\rho}\Big)\|D_{-x}u_h^n\|^2_{L^2(I_h^+)} \\ &+2\sigma p_3\|\frac{\Delta t}{\tau}\sum_{j=1}^{n} e^{-\frac{t_n-t_j}{\tau}} D_{-x}u_h^j + D_{-x}u_h^n\|^2_{L^2(I_h^+)}\end{aligned} \tag{46}$$

Finally, inequality (42) is easily obtained from (46).

■

For Theorem 4 holds a corollary analogous to Corollary 1. The convergence of the implicit-explicit method (41) with initial and boundary conditions (15) can be studied following the proof of the convergence of method (14)-(15).

4 Numerical Results

The methods studied in Section 3 are rewritten in a convenient way which makes easier their implementation. Method (14)-(15) is equivalent to the following four level method

$$\begin{aligned}&\rho D_{2,t}u_h^n(x_i)+\alpha D_{-t}u_h^{n+1}(x_i)\\&-(\gamma+\frac{\sigma}{\tau}\Delta t)D_{2,x}u_h^{n+1}(x_i)-f(x_i,t_{n+1},u_h^{n+1}(x_i))\\&=e^{-\frac{\Delta t}{\tau}}\Big(\rho D_{2,t}u_h^{n-1}(x_i)+\alpha D_{-t}u_h^n(x_i)\\&-\gamma D_{2,x}u_h^{n+1}(x_i)-f(x_i,t_n,u_h^n(x_i))\Big),\end{aligned}\tag{47}$$

for $i=1,\dots,N-1$, $n=2,\dots,M$, and (14) for $n=1$.

IMEX method (41) is equivalent to the next method

$$\begin{aligned}&\rho D_{2,t}u_h^n(x_i)+\alpha D_{-t}u_h^{n+1}(x_i)\\&-(\gamma+\frac{\sigma}{\tau}\Delta t)D_{2,x}u_h^{n+1}(x_i)\\&=e^{-\frac{\Delta t}{\tau}}\Big(\rho D_{2,t}u_h^{n-1}(x_i)+\alpha D_{-t}u_h^n(x_i)\\&-\gamma D_{2,x}u_h^{n+1}(x_i))\Big)+f(x_i,t_n,u_h^n(x_i))\\&-e^{-\frac{\Delta t}{\tau}}f(x_i,t_n,u_h^{n-1}(x_i)),\end{aligned}\tag{48}$$

for $i=1,\dots,N-1$, $n=2,\dots,M$. For $n=1$ we consider (41).

As in this paper the external force can be nonlinear in the displacement, a nonlinear system should be solved when in computation of the numerical approximation we use method (47). Otherwise, when method (48) is considered, the nonlinear system is replaced by a linear one.

The numerical results are obtained for $[a,b]=[0,1]$, $T=1$, homogeneous boundary conditions, initial conditions $u_1=0$ and

$$u_0(x)=\begin{cases}0,\ x\in[0,0.35)\cup(0.65,1]\\1+10(x-0.395),\ x\in[0.35,0.45)\\1,\ x\in[0.45,0.55]\\1-10(x-0.65),\ x\in(0.55,0.65].\end{cases}$$

We assume that the viscosity is one and the density $\rho=0.1$. We consider $\sigma=\gamma=0.01$, $\tau=1$.

In the numerical experiments we take $\Delta t=h=0.01$ and we consider different external forces. The numerical results obtained with method (41) for $f(u)=ue^{-u}$ are plotted in Figure 1. The behavior of the displacement when the external force is given by $f(u)=0.5ue^u$ is illustrated in Figure 2. Method (41) fails on the evaluation of numerical approximations for the external force $f(u)=0.8ue^u$. In this case, using the implicit method (14) we obtain the numerical results plotted in Figure 3.

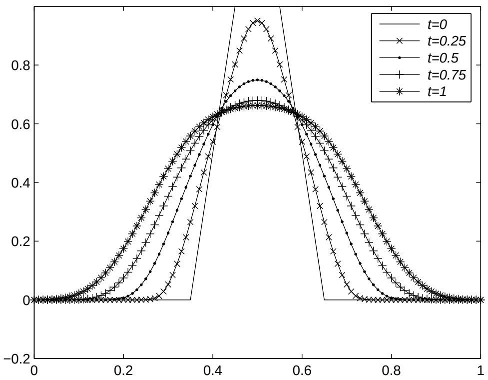

Figure 1: Numerical solutions obtained with IMEX method and $f(u)=ue^{-u}$.

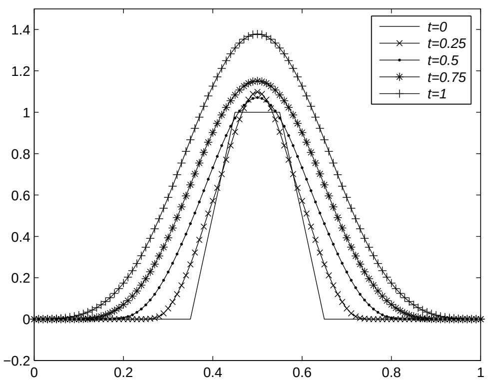

Figure 2: Numerical solutions obtained with IMEX method and $f(u)=0.5ue^u$.

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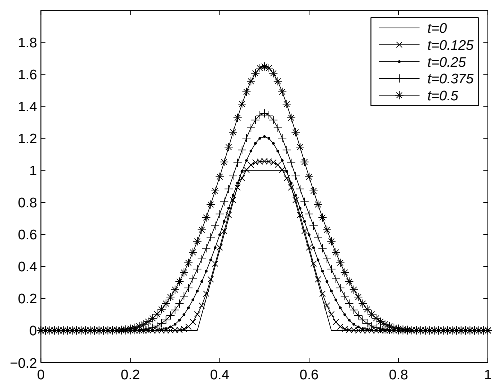

Figure 3: Numerical solutions obtained with implicit method (14) and $f(u) = 0.8ue^u$.

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