Generalized Fourier Diffraction Theorem for Tomography

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Abstract: - In this paper, a generalized version of the Fourier diffraction theorem is presented and discussed. This version is applicable to Diffuse Photon Density waves, thermal waves as well as the conventional acoustic or electromagnetic waves.

Keywords: - Fourier diffraction theorem, Born approximation, tomography, thermal imaging, Diffuse Photon Density Waves

1. Introduction

Optical, and ultrasonic imaging techniques in various forms have been used for biomedical imaging and non-destructive testing for many years. More recently, there has been a great deal of interest in these approaches, with thermal imaging also experiencing rapid growth. While ultrasound has long been used for medical imaging [1], the use optical radiation via diffuse photon density waves for imaging inhomogeneities in turbid media is a newer development [2-6]. Photothermal tomographic imaging methods have also been used for non-destructive evaluation [7-9], and more recently for biomedical imaging [10]

One common mathematical element in all these modalities is that in moving to the (temporal) frequency domain, they all lead to the Helmholtz equation. With wave fields based on the wave equation (such as acoustic waves), the wave vector in the Helmholtz equation is real. However, with diffuse photon density and thermal waves, the wave vector becomes complex.

Most treatments of diffraction tomography assume that the unknown object to be imaged is embedded in a uniform background medium having known properties. It is also usually assumed that the object is weakly inhomogeneous. This permits the use of the Born approximation. With traditional diffraction tomography (based on the wave equation), it is known that under the preceding assumptions there is a relation between the spatial Fourier transform of the scattered field and the spatial Fourier transform of the object. This relationship - the Fourier Diffraction Theorem -has formed the basis for most reconstruction algorithms of diffraction tomography.

Since the resulting wave vectors of Diffuse Photon Density Wavefields (DPDW) and thermal wave fields are complex, the Fourier Diffraction Theorem as derived for a real wave vector is inapplicable without modification. The main goal of this article is to show how the Fourier diffraction theorem can be adapted to accommodate a general complex wave vector. We approach the solution of the forward problem using the Helmholtz equation from the point of view of diffraction tomography. Detection in a plane is assumed but our problem formulation but does not require plane wave illumination. The forward problem formulation is then specialized to the case of plane wave illumination in order to derive the generalized version of the Fourier diffraction theorem. We discuss the region of the three dimensional Fourier transform of the object that is obtained in the detection plane.

2. Equations

We consider acoustic pressure waves, diffuse photon density waves and thermal wave-fields.
If the Fourier transform of the DPDW, thermal and acoustic wave equations are taken, then this leads to a Helmholtz equation of the form

$$\nabla^2 u(\mathbf{r},\omega) + k^2 u(\mathbf{r},\omega) = -S\ (\mathbf{r},\omega)\ . \qquad (1)$$

The k and S terms take on the appropriate form for the modality in question. In particular, the subscripts will refer to the modality so that

$$k_p^2 = -\frac{\mu_a}{D} - \frac{i\omega}{cD} \qquad \text{DPDW}$$

$$k_t^2 = -\frac{i\omega}{\alpha} \qquad \text{Thermal} \qquad (2)$$

$$k_s^2 = \frac{\omega^2}{c_s^2} \qquad \text{Acoustic}$$

The variables in the above can be identified as follows: μ_a is the optical absorption coefficient (m^{-1}), c_s is the speed of sound (m/s), D is the optical diffusion coefficient (m), α is the thermal diffusivity (m^2/s) and c (m/s) is the speed of light in the medium. Only the acoustic wave number is real, with the thermal and photonic wave numbers being complex quantities. This is why the term 'pseudowave' equation is often used to describe the resulting Helmholtz equation.

3. Born Approximation

We will consider the wave field $u(\vec{r})$ to be the sum of two components, $u_0(\vec{r})$ and $u_s(\vec{r})$, that is

$$u(\vec{r}) = u_0(\vec{r}) + u_s(\vec{r}) . \qquad (3)$$

The component $u_0(\vec{r})$ is known as the incident field (or equivalently the illumination function) and is the field present without any inhomogeneities. It is thus given by the solution to

$$\left(\nabla^2 + k_0^2(\vec{r})\right) u_0(\vec{r}) = 0 . \qquad (4)$$

Here, k_0 denotes the inhomogeneity-free appropriate wave number. The component $u_s(\vec{r})$, known as the scattered field, will be that part of the total field that can be attributed solely to the inhomogeneities.

An approximate solution for $u_s(\vec{r})$ can be written using the Born approximation, which is valid for objects that are weakly inhomogeneous [1] where the scattering field is weak and far smaller than the total field. The result of the Born approximation is that the scattered field is found from

$$\left(\nabla^2 + k_0^2\right) u_s(\vec{r}) = -o(\vec{r}) u_0(\vec{r}) \qquad (5)$$

and by the use of Green's theorem:

$$u_s(\vec{r}) = \int g(\vec{r} - \vec{r}_0) o(\vec{r}_0) u_0(\vec{r}_0) d\vec{r}_0 . \qquad (6)$$

Here $g(\vec{r} - \vec{r}_0)$ is the appropriate Green's function and the integration must be taken over the domain of interest. The actual expressions for the Green's function and incident field will depend on the choice of geometry and background medium.

As for standard diffraction tomography theory, we now assume a background medium infinite in extent and an inhomogeneity structure of finite extent. The above equation for the scattered field is the most general form of the forward problem, valid for all points outside the inhomogeneity and for arbitrary source-detector configurations. Although the assumption of an infinite domain may not be the most physically realistic assumption, it is the simplest case for physical insight and can later be modified for different geometries.

4. Notation and Sign Conventions

From the preceding section, we see that the photonic, thermal and acoustic wave numbers are defined by the squares of their quantities. In the following, we will use the wave numbers themselves, namely k_p, k_t and k_s, which are the relevant square roots. We will write each k as the sum of a real and an imaginary part, so that $k_p = k_{pr} + ik_{pi}$ with k_{pr} denoting the real part of k_p and k_{pi} denoting the imaginary part of k_p. Similar notations will apply to the other k's. Since there are two square roots in the above equations, we will use the convention that a particular (photonic, thermal, acoustic) k is the square root of the corresponding k^2 such that the imaginary part of k is positive. Hence k_p is the square root of k_p^2 such that $k_{pi} > 0$.

The variables $(\omega_x, \omega_y, \omega_z)$ are the spatial frequency variables in the Fourier domain. We will also require use of a variable of the form $\gamma_\omega^2 = \omega_x^2 + \omega_y^2 - k^2$. The subscript on γ indicates the subscript of the corresponding k. Hence $\gamma_p^2 = \omega_x^2 + \omega_y^2 - k_p^2$ and so forth. We will write $\gamma = \gamma_r + i\,\gamma_i$ where γ is defined as the square root of γ^2 such that $\gamma_r > 0$. Similar to the convention for k, the subscripts for γ get carried over to its real and imaginary part so that, for example, we may write $\gamma_t = \gamma_{tr} + i\,\gamma_{ti}$.

5. Fourier Diffraction Theorem

We now proceed to derive the thermal Fourier diffraction theorem. We start with equation(5). We now take the Fourier transform of the equation where the transform is taken in terms of x and y only. The z variable is not transformed since we are interested in detecting the scattered wave in a z=constant plane. This leads to

$$\left(\frac{d^2}{dz^2}+\left(k_0^2-\omega_x^2-\omega_y^2\right)\right)U_s(\omega_x,\omega_y;z)= \\ =-O(\omega_x,\omega_y;z)U_0(\omega_x,\omega_y;z)$$
$$\left(\frac{d^2}{dz^2}+\left(i\gamma_\omega\right)^2\right)U_s(\omega_x,\omega_y;z)= \\ =-O(\omega_x,\omega_y;z)U_0(\omega_x,\omega_y;z) \quad (7)$$

The above equation is a simple ODE in the variable z with ω_x,ω_y as parameters. The Green's function for an infinite domain and complex wave number is known [11] so that we obtain

$$U_s(\omega_x,\omega_y;z)= \\ =\int_{-\infty}^{\infty}O(\omega_x,\omega_y;z')U_0(\omega_x,\omega_y;z')\frac{e^{-\gamma_\omega|z-z'|}}{2\gamma_\omega}dz' \quad (8)$$

We note that the 1D Green's function with wave number given by γ_ω is the Weyl Expansion for the 3D Green's function. This equation tells us that the scattered wave is a convolution of the Weyl expansion (which is an expression of the Green's function) with the product of the object and illumination functions. Let us refer to this product of the object function with the illumination function as the heterogeneity function. At any given spatial frequency ω_x,ω_y, the heterogeneity function can be thought of as the source for the scattered waves. The plane waves arising from different depths z' propagate along the z direction to the detection plane. During propagation, these waves experience different amplitude and phase variations which are given by $\frac{e^{-\gamma_\omega|z_d-z_j|}}{2\gamma_\omega}$ for a source located at $z=z_j$ and detected at $z=z_d$. The scattered wave detected at the plane $z=z_d$ is thus a sum (integral) of the plane waves originating from the heterogeneity functions at different depths, weighted by the amplitude attenuation and phase shift given by the Weyl expansion of the Green's function. The amplitude and phase of the Weyl expansion depend on the spatial frequencies ω_x,ω_y as well as the distance between the detection plane and the source term.

The integral over z' is an integral over the entire object function (which becomes zero where there is no inhomogeneity). We assume that the detection does not occur anywhere in the object so that in the above equation z will be either greater than z' (detection in transmission) or less than z' (reflection detection).

Thus for z > object we can write

$$U_s(\omega_x,\omega_y;z) \\ =\frac{e^{-\gamma_\omega z}}{2\gamma_\omega}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}o\cdot u_0(x',y',z')e^{+\gamma_{\omega r}z'}e^{-ix'\omega_x}e^{-iy'\omega_y}e^{+i\gamma_{\omega i}z'}dV' \\ =\frac{e^{-\gamma_\omega z}}{2\gamma_\omega}F_{3D}\left\{o(\vec{r})u_0(\vec{r})e^{+\gamma_{\omega r}z}\right\}\Big|_{\omega_z=-\gamma_{\omega i}} \quad z > \text{object} \quad (9)$$

Similarly for z < object, we can write

$$U_s(\omega_x,\omega_y;z)=\frac{e^{+\gamma_\omega z}}{2\gamma_\omega}F_{3D}\left\{o(\vec{r})u_0(\vec{r})e^{-\gamma_{\omega r}z}\right\}\Big|_{\omega_z=+\gamma_{\omega i}} \quad (10)$$

In the above equations, F_{3D} represents the full three-dimensional Fourier transform. The above says that the 2D Fourier transform of the scattered thermal wave detected in a plane is proportional to the 3D Fourier transform of the heterogeneity function multiplied by a decaying exponential term and evaluated at $\omega_z=-\gamma_{\omega i}$. The proportionality term is the Weyl expansion, depending on γ_ω, which in turn depends on the relative sizes of the spatial frequencies and wave number. This is the statement of the general Fourier diffraction theorem.

We note that the 3D Fourier region is evaluated at $\omega_z=-\gamma_{\omega i}$. From their definitions, $\gamma_{\omega i}$ will be zero for DPDW and thermal waves unless the temporal frequency terms start to dominate as they introduce the only possible source of imaginary numbers. By contrast, the acoustic version of $\gamma_{\omega i}$ is nonzero for smaller spatial frequencies and then becomes zero when the spatial frequencies start to dominate and become larger than the temporal frequency term.

Following the development for the standard diffraction theorem, we now further specialize to the case of plane wave illumination. For the case of plane wave illumination, this implies

$$U_0(\vec{r})=e^{\pm ik_0z}=e^{\pm i(k_r+ik_i)z}. \quad (11)$$

Note that the wave number can be complex so that care must be taken with the signs in order to express the wave attenuation with distance. In this instance, let us assume that the inhomogeneity is located somewhere such that z>0 and that the illumination is from the left so that we can assume $U_0(\vec{r})=e^{+ik_0z}=e^{+i(k_r+ik_i)z}$.

For z > object function, this leads to

$$U_s(\omega_x,\omega_y;z)=\frac{e^{-\gamma_\omega z}}{2\gamma_\omega}F_{3D}\left\{o(\vec{r})e^{(-k_i+\gamma_{\omega r})z}\right\}\Big|_{\omega_z=-k_r-\gamma_{\omega i}} \quad (12)$$

Similarly for z < object function, we can write

$$U_s(\omega_x,\omega_y;z)=\frac{e^{\gamma_\omega z}}{2\gamma_\omega}F_{3D}\left\{o(\vec{r})e^{(-k_i-\gamma_{\omega r})z}\right\}\Big|_{\omega_z=-k_r+\gamma_{\omega i}} \quad (13)$$

This is the generalized Fourier diffraction theorem specialized to plane wave illumination.

6. Discussion

Equations (9) and (10) are the most general form of the Fourier diffraction theorem. It can be readily observed that the triple integrals are scaled Fourier integrals evaluated at $\left(\omega_x,\omega_y,\pm\gamma_{\omega i}\right)$ of the heterogeneity function and a decaying exponential. The exponential decay is due to the attenuative property of thermal and/or DPDW waves. For acoustic traveling waves, this exponent is zero and thus this exponential term becomes unity.

Equations (12) and (13) are the statement of the Fourier diffraction theorem where the illumination function has been specialized to plane wave illumination. The triple integrals are scaled Fourier integrals evaluated at $\left(\omega_x,\omega_y,-k_r\pm\gamma_{\omega i}\right)$. The negative sign is taken for measurements in transmission and the positive sign for reflection measurements.

A two dimensional version of the standard Fourier diffraction theorem can be found in [1]. In order to compare reults, we rewrite their Fourier diffraction theorem in 3D and use the same notation as shown here. This gives

$$U_s(\omega_x,\omega_y;z)=\frac{e^{-i\gamma_{\omega i}z}}{2\gamma_\omega}F_{3D}\left\{o(\vec{r})\right\}\Big|_{\omega_z=-k_r\pm\gamma_{\omega i}} \quad (14)$$

When comparing the generalized version of the Fourier Diffraction Theorem with the standard version, we note that the generalized version differs primarily by additional exponential terms. There is an additional overall exponential attenuation term outside the integrals that is a function of the location of the plane of detection. The triple integrals are the Fourier transform of the object multiplied by a frequency-dependent attenuating exponential. This is entirely due to the complex nature of the wave vector k. If k were purely real, as for acoustic waves, then the attenuating exponentials become unity because both k_i and $\gamma_{\omega r}$ become zero and the generalized Fourier diffraction theorem reduces to the standard Fourier diffraction theorem.

It remains to determine which values of the 3D transform of the object are contained in the Fourier transform of the detected image. What we have on the plane of detection is a portion of a 3D Fourier transform on a subsurface defined by $\omega_z=\left(-k_r\pm\gamma_{\omega i}\right)$. This implies that data at different angles are needed to reconstruct a unique image of the object. Let us examine the shape of this subsurface. In the limiting case of the low frequency regime $\omega\rightarrow 0$, we note that the region is essentially the ω_z=-k_r plane for DPDW and thermal waves. In theory, this would permit easy multi-look Fourier plane reconstruction because no interpolation step is required to map the region into a rectilinear coordinate system. For low frequencies, the 3D Fourier region approaches a plane. For higher temporal frequencies, the region will be a partly curved region and will require interpolation for exact multi-look Fourier plane reconstruction in a rectilinear coordinate system. Figures 1 and 2 show a representative shape of the Fourier region for DPDW and thermal waves. The acoustic Fourier region is not illustrated as it is well known to be part of a sphere [1]. We note that for DPDW and thermal waves, it is indeed a curved region with a peak at the origin and then flattening out to a plane at areas far away from the origin. The size of the “hump” is dependent on the frequency of illumination. The higher the frequencies, then the wider and higher is the hump for DPDW and thermal waves. This is illustrated in Figures 4, 5 and 6 where the Fourier region is shown for ω_y =0 for clarity, with increasing frequencies. For a given value of the illumination (temporal) frequency, as both ω_x and ω_y approach infinity (that is to say, we move away from the origin), then $\gamma_{\omega i}\rightarrow 0$, $\omega_z\rightarrow -k_r$. So as we move away from the origin, the Fourier region approaches a ω_z constant plane. The smaller the illumination frequency, the more plane-like the Fourier region, reminiscent of the Fourier slice theorem. This is the opposite as for acoustic waves. It is known and shown in Figure 6, that for real wave vectors, as the frequency gets larger, the Fourier region flattens out into a plane, approaching the results of the Fourier Slice Theorem [1].

7. Summary

In summary, a generalized version of the Fourier Diffraction theorem has been presented. The theorem was presented for the case of a 3D infinite-space domain, in parallel with the traditional development of the standard Fourier diffraction theorem. The theorem was presented for the cases of both general illumination function and then further specialized to the case of plane wave illumination. The values of the 3D Fourier transform of the object (multiplied by an attenuating exponential) that are contained in the Fourier transform of the detection plane image were discussed and illustrated. Comparison with the traditional Fourier diffraction theorem was made. Data at different angles are necessary to uniquely reconstruct an object.

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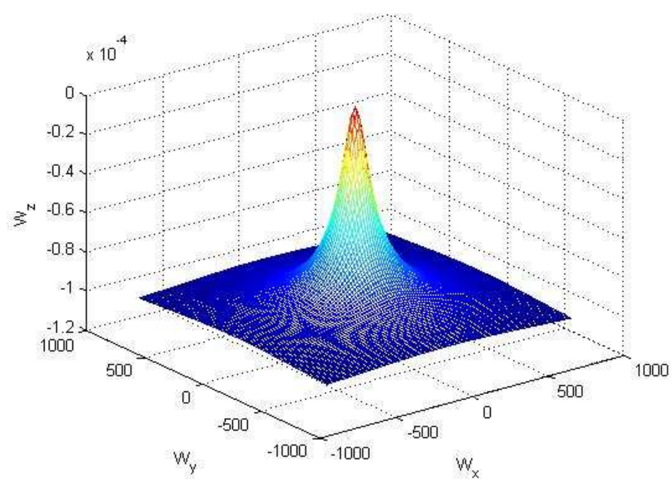

Figure 1: Fourier region for DPDW waves, measurements made in reflection

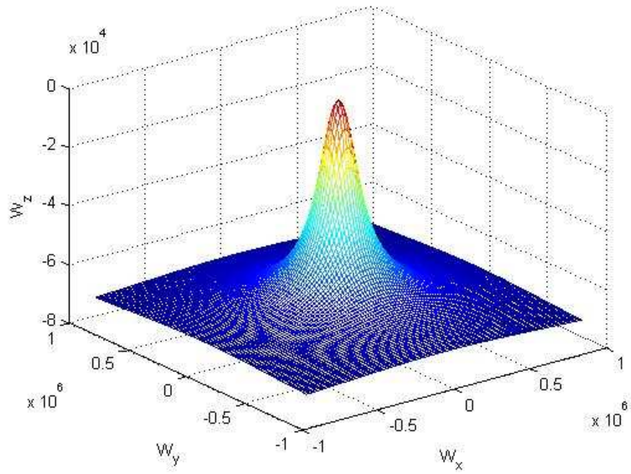

Figure 2: Fourier region for thermal waves

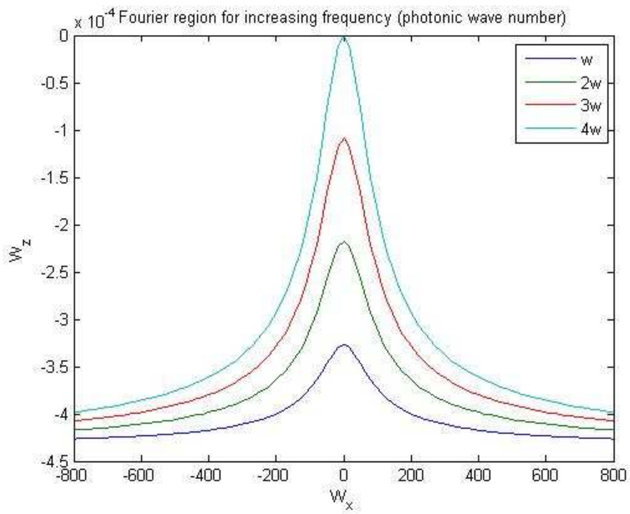

Figure 3: Fourier region for increasing frequency for DPDW waves

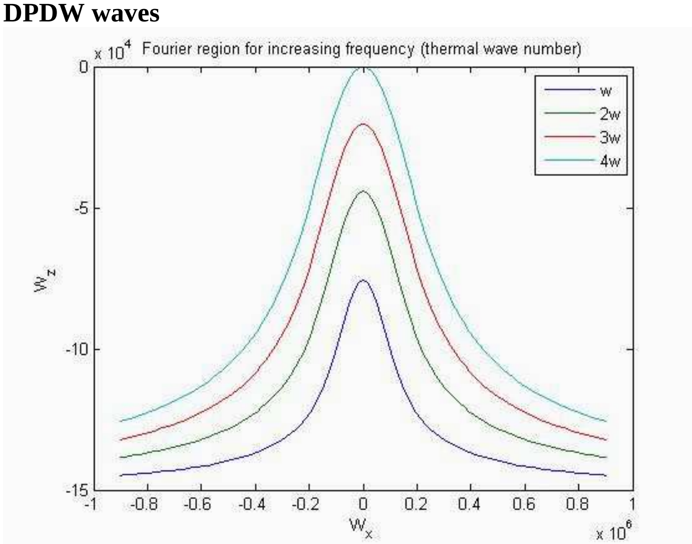

Figure 4: Fourier region as a function of increasing frequency for thermal waves

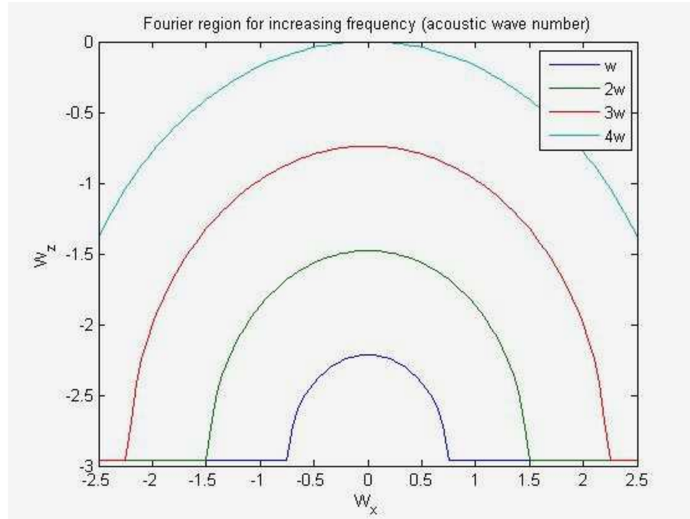

Figure 5: Fourier region as a function of increasing frequency for acoustic waves