Design of Two-Dimensional Signal adapted filter bank from One Dimensional filters

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Abstract : - We present the design of two dimensional finite impulse response(FIR) signal adapted filter banks for an input signal whose power spectral density is separable. One dimensional FIR filters are designed for each of the separable components using iterative greedy algorithm. The first and last filters in this 2D filter bank are found to be separable and matching closely with ideal ones .The remaining filters are nonseparable and time varying. Since most of the input signal energy is packed in the first subband this design method is useful in optimal representation of images.

Keywords: - Principal component filter banks, Two-dimensional filter bank, Separable filters, Orthonormal filter bank, Finite impulse response filter bank.

1. Introduction

The optimal multiresolution analysis of signals and images can be done effectively using principal component filter banks (PCFB). A PCFB is a type of filter bank that minimizes the mean squared error between the input and any low resolution approximation of it. They are known to be optimal for data compression. In addition to this, they offer optimal solutions to many problems in communication and signal processing under certain theoretical assumptions. The uniform 1D PCFB was introduced by M Tsatsatsanis et al[1]. Its existence, design and optimality results for various objectives in communication were studied by P P Vaidyanathan et al[2].

One Dimensional PCFBs was generalized to higher dimensions by B.Xuan and Robert H Bamberger[3]. They had also developed 2D FIR PCFB using Roesser state-space representation of a paraunitary (PU) FIR polyphase matrix[4]. According to Farshid Delgosha et.al. the drawback of this representation is the resultant complicated algorithm and lack of a ready-to-use realization [5]. Traditional design methods for 1D compaction filters[6] cannot be extended to higher dimensions directly due to the lack of multidimensional (MD) spectral factorization theorem.

The use of nonseparable filter banks in image processing applications is due to the fact that it would capture the geometrical features in the image such as lines, edges and textures. The optimization algorithm for 1D [7] can be extended to 2D nonseparable filter banks, but the factorization for 2D PU matrix does not cover the whole family of such matrices[8]. Furthermore, the problem with nonseparable 2D filterbank is mainly the design complexity.

In this paper we design 2D FIR signal adapted filter bank using iterative greedy algorithm for 1D signals which is of less computational complexity. If the power spectral density (psd) of a 2D signal $S_{xx}(\omega_1,\omega_2)$ can be approximated as the product of two separable components, 1D FIR filter bank can be designed for each separable component and then form 2D FIR filter bank from them. The first and last subband filters in this 2D filter bank are found to be separable, others nonseparable and time varying. The main advantage of this approach, is the reduced computation work load associated with optimization and low implementation cost. The filter bank in this case is assumed to be maximally decimated and orthonormal i.e., polyphase matrix E(**z**) satisfies

$$E(\mathbf{z})^{\dagger}E(\mathbf{z})=I \quad (1)$$

The paper is organized as follows. Notion of one dimensional and two dimensional uniform PCFBs are outlined in section **2**. Section **3** describes the design of 2D PU signal adapted filter banks from 1D filter bank using iterative greedy algorithm. Section **4** gives simulation results.

2. Principal Component filter banks

2.1 One dimensional uniform PCFB

A filter bank in a class of uniform orthonormal M- channel filter banks is said to be a PCFB for the given input psd, if its subband variance vector majorizes [9] the subband variance vector of all other filter banks in that class. They are optimal if the minimisation objective is a concave function of the subband variance vector. The existence of PCFB depends both on the class and the input psd [2].

The PCFB for a given input signal x(n) can be constructed by comparing the values of its psd $S_{xx}(e^{j\omega})$ at M alias frequencies, $\omega_k=\omega+2\pi k/M$, where M is the number of channels in the filter bank and $0 \le k \le M-1$[10]. The polyphase matrix of the optimal filter bank diagonalizes the M fold blocked psd matrix of the input signal given by

$$\mathbf{S_{XX}}(e^{j\omega})= \frac{1}{M}\sum_{k=0}^{M-1} S_{xx}(e^{\frac{j(\omega-2\pi k)}{M}})\, V_k(e^{j\omega})\, V_k^{\dagger}(e^{j\omega}) \quad (2)$$

where

$$V_k(e^{j\omega}) = \begin{bmatrix} 1 \\ e^{-j\frac{(\omega-2\pi k)}{M}} \\ e^{-j\frac{(\omega-2\pi k)2}{M}} \\ \vdots \\ e^{-j\frac{(\omega-2\pi k)(M-1)}{M}} \end{bmatrix} \quad (3)$$

Since the majorization property is a necessary condition for optimality, the eigen values of the blocked matrix are arranged in descending order for each ω. Hence the polyphase matrix of a PCFB can be computed from the corresponding eigen vectors $V_k(e^{j\omega})$ of the blocked psd matrix. From (2) it can be seen that the shape of the input spectrum affects only the eigen values of $\mathbf{S_{XX}}(e^{j\omega})$, and not the eigen vectors. The eigen values of the blocked psd matrix at any frequency ω are the values of $S_{xx}(e^{j(\omega-2\pi k)/M})$, where $0 \le k \le M-1$

2.2 Two dimensional uniform PCFB

Most of the results on 1D filter banks can be generalized in a straight forward manner to 2D systems. Let the column vectors $\mathbf{n}=[n_1\ n_2]^T$, $\mathbf{z}=[z_1,z_2]^T$, $\boldsymbol{\omega}=[\omega_1,\omega_2]^T$ denote 2D variables. If x(**n**) is a wide sense stationary(WSS) process with psd $S_{xx}(\boldsymbol{\omega})$ and M is a 2×2 nonsingular integer matrix referred to as resampling matrix, 2D PCFB can be constructed by comparing the values of $S_{xx}(\boldsymbol{\omega})$ in the set $\left\{ \begin{matrix} S_{xx}(\boldsymbol{\omega}-2\pi M^{-T}u_i),\ u_i \in N(M^T), \\ i=0,\cdots,P-1 \end{matrix} \right\}$ where P=J(M)=|det(M)| is the resampling density and *N*(M) is the set of all integer vectors in the fundamental parallelepiped (FPD) generated by M, i.e. $N(M)=\{k_M \mid k_M \in Mx\}$ where $x=[x_0\ x_1]^T, x_i \in [0,1), i=0,1$.

Similar to one dimensional PCFB the spectral density matrix can be written as

$$\mathbf{S_{XX}}(\boldsymbol{\omega}) = \frac{1}{P}\sum_{u_i} S_{xx}(M^{-T}(\boldsymbol{\omega}-2\pi u_i))V_{u_i}(\boldsymbol{\omega})V_{u_i}^{\dagger}(\boldsymbol{\omega}) \quad (4)$$

where

$$V_{u_i}(\boldsymbol{\omega})=\begin{bmatrix} e^{-j(\boldsymbol{\omega}-2\pi u_i)^T M^{-1}k_0} \\ e^{-j(\boldsymbol{\omega}-2\pi u_i)^T M^{-1}k_1} \\ \vdots \\ e^{-j(\boldsymbol{\omega}-2\pi u_i)^T M^{-1}k_{P-1}} \end{bmatrix} \quad (5)$$

The eigen values of the blocked psd matrix at any $\boldsymbol{\omega}$ are the values of $S_{xx}(M^{-T}(\boldsymbol{\omega}-2\pi u_i)), u_i \in N(M^T)$. The eigen vectors of the blocked psd matrix can be used to compute the polyphase matrix of PCFB since it diagonalises the psd matrix.

Let the input psd $S_{xx}(\boldsymbol{\omega})$ be separable i.e.,

$$S_{xx}(\boldsymbol{\omega})=S_1(\omega_1)S_2(\omega_2). \quad (6)$$

Then the power spectrum matrix can be written as the kronecker product of 1D psd matrices.

$$\mathbf{S_{XX}}(\boldsymbol{\omega})=\mathbf{S_2}(\omega_2)\otimes\mathbf{S_1}(\omega_1) \quad (7)$$

Since the spectral density matrix is positive definite and Hermitian, it has a unitary diagonalization[9] given by

$$\mathbf{S_1}(\omega_1)=t_1\lambda_1 t_1^{\dagger}. \quad (8)$$

where λ_1 consists of positive eigen values and t_1 consists of normalized eigen vectors. Similarly

$$\mathbf{S_2}(\omega_2)=t_2\lambda_2 t_2^{\dagger} \quad (9)$$

Substituting (8) and (9) in (7) gives

$$\mathbf{S_{XX}}(\boldsymbol{\omega})=(t_2\otimes t_1)(\lambda_2\otimes\lambda_1)(t_2\otimes t_1)^{\dagger} \quad (10)$$

$\lambda = \lambda_2 \otimes \lambda_1$ results in a diagonal matrix with diagonal values as the product of eigen values of the matrices $\mathbf{S_2}(\omega_2)$ and $\mathbf{S_1}(\omega_1)$ which in turn depends on the values of, $S_2(\omega_2)$ and $S_1(\omega_1)$. Hence the diagonal values of λ, except the first and last, need not be arranged in descending order at all frequency points, even though diagonal values of λ_1 and λ_2 are arranged in descending order. Therefore $t_2 \otimes t_1$ diagonalizes $\mathbf{S_{XX}}(\boldsymbol{\omega})$, but it will not act as the 2D polyphase matrix of PCFB for any general psd. If the psd of input signals are such that diag(λ) are arranged in descending order at all frequency points, $t_2 \otimes t_1$ will be the polyphase matrix of PCFB. Then all the filters will be separable. For a general psd, only the first and the last filters in the PCFB will be separable. The other filters $H_k(\mathbf{z})$, $1 \le k \le P-2$ can be obtained from 1D PCFBs derived from $\mathbf{S_2}(\omega_2)$ and $\mathbf{S_1}(\omega_1)$, whose eigen values form the k^{th} larger eigen value of $\mathbf{S_{XX}}(\boldsymbol{\omega})$. They will be time varying, but at each frequency point they will be separable[11].

3. Design of 2D FIR PU signal adapted filter banks

We design 2D FIR signal adapted filter banks by designing 1D filters for the two separable components of 2D psd, $\mathbf{S_{XX}}(\boldsymbol{\omega})$. The resampling is separable if M is diagonal. The resampling matrix used in 2D system is M = diag(M_1,M_2) .The 1D FIR filter banks are designed for decimation factors M_1 and M_2, using iterative greedy algorithm [7].

3.1 Design of 1D FIR PU signal adapted filter banks

Let $D(\omega)$ be the analysis polyphase matrix of the infinite order 1D PCFB for the blocked psd, $\mathbf{S_1}(\omega_1)$ and $E(\omega)$ be the polyphase matrix of the 1D FIR PU filter bank to be designed for a decimation factor M_1. $E(\omega)$ is approximated with $D(\omega)$ by minimizing the weighted mean-squared Frobenius norm error between $D(\omega)$ and $E(\omega)$ given by

$$\varepsilon = \frac{1}{2\pi}\int_0^{2\pi} W(\omega)\left\|D(\omega)-E(\omega)\right\|_F^2 d\omega \quad (11)$$

To solve this optimization problem , with the PU constraint , the FIR polyphase matrix $E(\omega)$ is parameterized using the Householder –like building blocks.

$$E(z)=V(z)E_0 \quad (12)$$

where $V(z)$ is a $M_1\times M_1$ PU matrix consisting of N-1 degree one Householder –like building blocks of the form

$$V(z)=\prod_{i=N-1}^{1}V_i(z) \quad (13)$$

$$V_i(z)=I-v_i v_i^{\dagger}+z^{-1}v_i v_i^{\dagger} \quad 1\le i\le N-1 \quad (14)$$

where v_i are unit norm vectors. E_0 is a $M_1\times M_1$ unitary matrix. Each parameter is individually optimized at each iteration holding all other parameters fixed.

Since the desired response $D(\omega)$ suffers from a phase ambiguity, a modification to this iterative algorithm named as phase feedback modification[7] is applied in this design. The phase of each column of the FIR PU polyphase matrix $E(\omega)$ is fed back to that of the desired response in order to minimize the mean squared error. The mean squared error is found to be small with this modification and the FIR response becomes closer to the ideal one.

3.2 2D FIR filters from 1D filters

The 2D filters $H_k(\mathbf{z})$, k = 0, P-1 can be calculated from 1D filters as

$$H_0(\mathbf{z}) = H_{00}(z_1)H_{10}(z_2) \qquad (15)$$

$$H_{P-1}(\mathbf{z}) = H_{0M_1-1}(z_1)H_{1M_2-1}(z_2) \qquad (16)$$

where $H_{00}(z_1), H_{0M_1-1}(z_1)$ are the 1D filters designed for an input psd $S_1(\omega_1)$ and $H_{10}(z_2)$, $H_{1M_2-1}(z_2)$ are the 1D filters for $S_2(\omega_2)$ respectively. For a general psd, the remaining filters $H_k(\mathbf{z})$, $1 \le k \le P-2$ can also be calculated from 1D filters as

$$H_k(\mathbf{z}) = H_{0m_1}(z_1)H_{1m_2}(z_2), \quad 0 \le m_1 \le M_1 - 1, 0 \le m_2 \le M_2 - 1 \qquad (17)$$

The filters H_{0m_1}, H_{1m_2}, are selected depending on the eigen values of $\mathbf{S_1}(\omega_1)$ and $\mathbf{S_2}(\omega_2)$ which form the k^{th} larger eigen value of $\mathbf{S_{XX}}(\boldsymbol{\omega})$ at each frequency point. So these filters are found to be time varying. This is the disadvantage of this approach. But this design may be useful in the optimal representation of images by lower resolution versions of them due to the energy compaction even in the first band.

In image processing, the images are generally modelled as a 2D random stationary field. A random field with nonzero mean can always be transformed to a zero mean random field by subtracting the mean from it. The autocorrelation function of a random field is called separable when it can be expressed as a product of one dimensional autocorrelation functions. A separable stationary autocorrelation function often used in image processing [12] is

$$r(m,n) = \sigma^2 \rho_1^{|m|} \rho_2^{|n|} \qquad |\rho_1| < 1, \quad |\rho_2| < 1 \qquad (18)$$

Here σ^2 represents the variance of the random field and $\rho_1 = R(1,0)/\sigma^2, \rho_2 = R(0,1)/\sigma^2$ are the one-step correlations in the m and n directions, respectively.The corresponding spectral density function of a stationary random field can be written as

$$S_{xx}(\omega_1,\omega_2) = \sum_{n_1=-\infty}^{\infty} \sum_{n_2=-\infty}^{\infty} r(n_1,n_2) e^{-j(\omega_1 n_1 + \omega_2 n_2)} \qquad (19)$$

substituting for $r(n_1,n_2)$ from (18)

$$S_{xx}(\omega_1,\omega_2) = \frac{\sigma^2(1-\rho_1^2)(1-\rho_2^2)}{(1+\rho_1^2 - 2\rho_1\cos\omega_1)(1+\rho_2^2 - 2\cos\omega_2)}$$

$$= \sigma^2 S_1(\omega_1)\, S_2(\omega_2) \qquad (20)$$

where $S_1(\omega_1) = \dfrac{1-\rho_1^2}{1+\rho_1^2 - 2\rho_1\cos\omega_1}$,

$S_2(\omega_2) = \dfrac{1-\rho_2^2}{1+\rho_2^2 - 2\rho_2\cos\omega_2}$

Now the one dimensional filters $H_{0m_1}(z_1), 0 \le m_1 \le M_1 - 1$, $H_{1m_2}(z_2), 0 \le m_2 \le M_2 - 1$ are designed for $S_1(\omega_1)$ and $S_2(\omega_2)$ respectively, using the algorithm described in section 3.1

4. Simulation results

We design FIR PU 2D PCFB-like filter bank for a real WSS (mean zero) input signal x(n) with psd $S_{xx}(\boldsymbol{\omega})$ as shown in Fig 1., which is separable. The

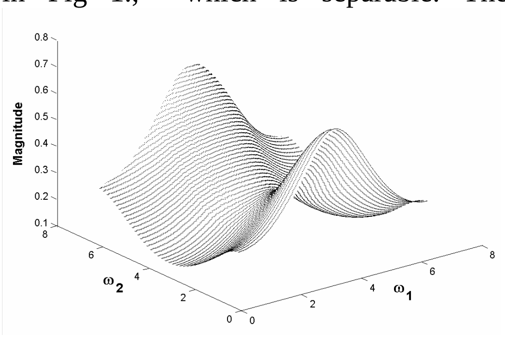
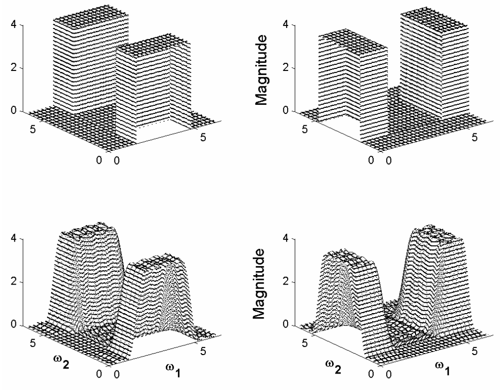


Fig.1 Input psd $S_{xx}(\boldsymbol{\omega})$

resampling matrix used is M=diag{2,2}.

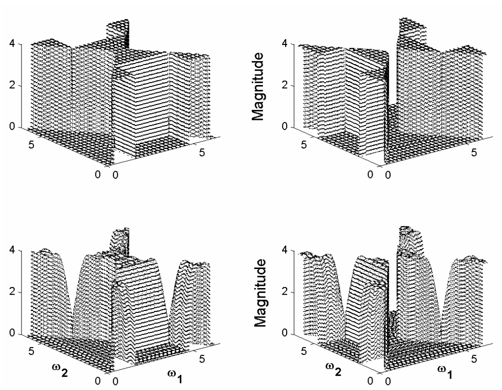
Each filter of the filter bank is designed for a length of 8×8. Fig. 2 and Fig.3. show the magnitude squared response of the subband filters of the designed FIR PU PCFB-like filter bank and that of the PCFB for P = |det(M)| = 4. The input signal energy compacted in the subband filters of FIR filter bank is calculated and it is shown in Table1.

98.54% of the energy stored in the first subband filter of PCFB is packed in that of



(a) (b)

Fig.2(a) magnitude squared responses of the first subband filter of PCFB and designed FIR filter bank (b) responses of fourth subband filter



(a) (b)

Fig.3(a) magnitude squared responses of the second subband filter of PCFB and designed FIR filter bank (b) responses of third subband filter

Fig.4(a) original image (b) reconstructed image

the FIR filter bank. As the order of the filter is increased, its response approaches closer to that of PCFB. We applied this algorithm to a test image "Lena" and a four channel FIR optimal filter bank is designed for a filter length of 8×8. Using the first filter of the optimal filter bank, the subband image is generated which is then used to reconstruct the original image. The reconstructed image along with the original image is shown in Fig. 4

Index for channel number i	Subband variances σ_i^2	
	FIR PU Filter bank	PCFB
0	0.5066	0.5141
1	0.3177	0.3638
2	0.2472	0.2788
3	0.2055	0.2007

Table 1 subband variance of FIR filter bank and PCFB for an input psd of 0.3394

The psd of a 2D signal can be approximated very closely as the product of separable components. If this is possible, this design procedure may be adopted in image analysis for getting a low resolution version of images from large databases.

5. Conclusion

We have designed 2D FIR PU, PCFB-like filter banks for a 2D psd, which is separable, using iterative greedy algorithm. The first and last filters in this filter bank are found to be separable and matching closely with the ideal ones. The other filters are nonseparable and time varying for a general psd. But they can be time invariant also, depending on the input psd. The main advantage of this approach is the reduced design complexity compared to the design of nonseparable filter banks.

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