Applications of the Malliavin Calculus of Bismut type without probability

REMI LEANDRE
Université de Bourgogne
Institut de Mathématiques
21000, Dijon
FRANCE

Abstract: We give a survey about the translation of Malliavin Calculus of Bismut type in semi-group theory and about some applications to heat-kernels where the Bismut condition plays a big role

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1 Introduction

This review paper is dedicated to Bismut's book "Large deviations and the Malliavin Calculus" ([7]) from where main novelties of the Malliavin Calculus for densities come. Namely there are two parts in the Malliavin Calculus:

-)A part involved with functional analysis. If the functional analysis part of the Malliavin Calculus has a lot of precursors (See for instance work of Albeverio-Hoegh-Krohn [1], Hida [8], Fomin [2], Berezanskii [5]...), the main novelty of the Malliavin Calculus was to complete the classical differential operations on the Wiener space in all the L^p such that the test functional space of the Malliavin Calculus is an algebra of functional almost surely defined, because there is no Sobolev imbedding in infinite dimension. The main interest of the Malliavin Calculus is that it can be applied to diffusions.

-)A part involved with the regularity of the law of diffusions. The main theorem of Malliavin in this part is the following: if the Malliavin matrix associated to a functional has an inverse which belongs to all the L^p, the law of this R^d valued variable has got a smooth density with respect to the Lebesgue measure on R^d. Bismut ([6]), in the case of diffusions, has got another way to get Malliavin's theorem, the role of the infinite dimensional Sobolev Calculus being replaced by the stochastic flow theorem.

In the first part of this review paper, we remark that in Bismut's approach of the Malliavin Calculus all quantities involved satisfy convenient stochastic differential equations. So the formulas of [6] can be interpreted in semi-group theory: it is the purpose of our work [19]. We briefly describe in this part the main ingredient of [19], thas is the Cameron-Martin-Maruyama-Girsanov formula in semi-group theory as well as the elementary integration by part formula, which are in stochastic analysis the basical ingredient of Bismut's way of the Malliavin Calculus for densities. We give afterwards the results of [19].

In the second part of this paper, we translate in semi-group theory our proof of Varadhan's estimates for hypoelliptic heat kernels, lower bound which were got by ourself in [11] and [12]. We remark in the considerations of [11] and [12], only stochastic differential equation appear, which can be interpreted in semi-group theory. The analytical tool is the Malliavin Calculus depending from a parameter of Bismut's type, which was introduced in [11] and in [12], which is translated in semi-group theory in [20]. The second main ingredient is that the Bismutian's distance associated to the degenerated operator is equal to the traditional subriemannian metric: this allows to understand in semi-group theory the role of Bismut's condition ([6]) in the asymptotics. We describe briefly later a situation where the two distances are not equal.

In the last part of this paper, we translate in semi-group theory a result of Ben-Arous and us [4] giving a condition in order a heat-kernel is strictly positive, which was using Bismut's procedure ([7]), that is a kind of implicit theorem for a functional (the diffusion) almost surely defined, which allows to get an integrated expression for the density of the diffusion. Our translation ([21]) is based upon a translation of the Wong-Zakai approximation of a diffusion in semi-group theory, which allows to get a translation of discrete Bell's approach of the Malliavin Calculus ([3]), where we pass to the infinite dimension via this analytical Wong-Zakai approximation instead of using the theory of stochastic differential equations and the martingale inequalities. Bismut's procedure is therefore reduced to the use of the classical implicit function theorem with some uniform integrability conditions provided by a discrete version of the classical Stroock-Varadhan support theorem for diffusions.

For application of the Malliavin Calculus to heat kernel, we refer to the survey of Kusuoka ([9]), Watanabe ([25]) and various surveys of Léandre ([14], [15], [16], [17], [18]).

2 The theorem of Malliavin without probability

Let us consider some vector fields X_i $i = 0, .., m$ on R^d with bounded derivatives of all order. Let L be the Hoermander type operator

$$Lf = X_0 + 1/2 \sum_{i>0} X_i^2 \tag{1}$$

acting on smooth bounded functions on R^d. It can be written as

$$Lf =< X_0, Df > +1/2 \sum_{i>0} < DX_i X_i, Df > + 1/2 \sum_{i>0} < X_i, D^2 f, X_i > \tag{2}$$

In (1), vector fields are consider as first order differential operators and in (2) vectors fields are considered as smooth applications from R^d into R^d. Let us consider the generator

$$L^h = L + \sum_{i>0} h_t^i X_i \tag{3}$$

where $t \to h_t^i$ are smooth bounded functions which don't depend on x. L^h generates a inhomogeneous Markov semi-group P^h acting on bounded continuous functions on R^d.

Let us consider on R^{d+1} some vector fields

$$\tilde{X}_i^t = (X_i, h_t^i u) \tag{4}$$

and the generator on R^{d+1} acting on smooth functions on $\tilde{f}$ on R^{d+1}:

$$\tilde{L}^h(\tilde{f}) =< X_0, \tilde{D}\tilde{f} > + 1/2 \sum_{i>0} < DX_i X_i, \tilde{D}\tilde{f} > + 1/2 \sum_{i>0} < \tilde{X}_i^t, \tilde{D}^2 \tilde{f}, \tilde{X}_i^t > \tag{5}$$

It generates a semi-group $\tilde{P}^h$ operating on the bounded continuous functions on R^{d+1}. In the sequel, for the integrability conditions, we refer to the appendix.

Theorem 1 *(Quasi-invariance)*

$$P_t^h f(x) = \tilde{P}_t^h [uf](x, 1) \tag{6}$$

Proof: We have, since the vector fiels $\tilde{X}_i^t$ are linear in u:

$$\tilde{P}^h [uf](x, u_0) = \tilde{P}^h [uf](x, 1) u_0 \tag{7}$$

for any bounded continuous f on R^d such that:

$$\tilde{L}^h \tilde{P}^h [uf](.,.)|(x, 1) = L^h \tilde{P}^h [uf](., 1)|(x) \tag{8}$$

Therefore the result arises by using uniqueness of the solution of the parabolic equation associated to L^h. ♢

Let us consider the vector field $\overline{X}_i^h = (X_i, h_t^i)$ and the generator on R^{d+1} acting on smooth functions $\tilde{f}$on R^{d+1}

$$\overline{L}^h \tilde{f} =< X_0, \tilde{D}f > +1/2 \sum_{i>1} < DX_i X_i, \tilde{D}\tilde{f} > + 1/2 \sum_{i>0} < \overline{X}_i^h, \tilde{D}^2 \tilde{f}, \overline{X}_i^h > \tag{9}$$

It generates a semi-group $\overline{P}^h$ acting on the bounded continuous functions on R^{d+1}.

Theorem 2 *(Elementary integration by parts formula):*

$$\int_0^t P_{t-s} \sum_{i>0} h_s^i X_i [P_s f]] ds = \overline{P}_t^h [uf](x, 0) \tag{10}$$

Proof: We have $\overline{P}_t^h [uf](x, u_0) = A_t(x) u_0 + B_t(x)$ because $\frac{\partial}{\partial u}$ commute with $\overline{L}^h$. Therefore,

$$\overline{P}_t^h [uf](x, u_0) = P_t[f](x) u_0 + \overline{P}_t^h [uf](x, 0) \tag{11}$$

such that

$$\frac{\partial}{\partial t} \overline{P}^h [uf](.,.)|(x, 0) = L\overline{P}^h [uf](., 0)|(x) + \sum_{i>1} h_t^i < X_i, P_t[f](x) > \tag{12}$$

with starting condition 0.

On the other hand $F(t,x) = \int_0^t P_{t-s}[\sum_{i>0} h_u^i X_i[P_s[f]]]ds$ is solution of the parabolic equation:

$$\frac{\partial}{\partial t}F(t,x) = LF(t,x) + \sum_{i>0} h_t^i X_i P_t[f](x) \quad (13)$$

which is the same than (12) with the same initial condition.

$\diamondsuit$

Let us consider the vector fields on $R^d \times Gl(R^d) \times M^d = V^d$ where $Gl(R^d)$ is the space of invertible matrices on R^d, and M^d the space of matrices on R^d:

$$\hat{X}_i = (X_i, DX_i U, 0) \quad (14)$$

and

$$\hat{X} = (0, 0, \sum_{i>0} < U^{-1}X_i, . >^2) \quad (15)$$

Let us consider the semi-group operating on continuous bounded functionals on V^d generated by $\hat{L}$:

$$\hat{L}\hat{f} = 1/2 \sum_{i>0} < \hat{X}_i, \hat{D}^2\hat{f}, \hat{X}_i > + 1/2 \sum_{i>0} < D\hat{X}_i\hat{X}_i, \hat{D}\hat{f} > + < \hat{X}_0, \hat{D}\hat{f} > + < \hat{X}, \hat{D}\hat{f} > \quad (16)$$

We consider $\hat{P}_t$ the semi-group associated to $\hat{L}$.

Theorem 3 *(Malliavin): If $\hat{P}_t[V^{-p}](x, I, 0) < \infty$ for all p, we have that $P_t f(x) = \int_{R^d} p_t(x,y) f(y) dy$ where $y \to p_t(x,y)$ is smooth positive.*

Scheme of the proof: The proof is based upon the following considerations: under Malliavin's non-degeneracy assumption, we can improve the elementary integration by parts formula (10) in order to get

$$P_t[D^r f](x) = Q_t[fR](x, I, 0) \quad (17)$$

for a convenient enlarged semi-group Q_t where the inverse of the Malliavin matrix with big power appear. We consider Fourier exponential and we deduce from (17) that the Fourier transform of $P_t(x)(dy)$ has a quick decay.

$\diamondsuit$

Remark: V is called the Malliavin matrix associated to the diffusion.

3 Varadhan estimates without probability

Let us consider $m+1$ vector fields on R^d with derivatives at each order bounded. Let us consider the vector spaces defined inductively by:

$$E_0(x) = (X_1(x), .., X_m(x)) \quad (18)$$

$$E_{l+1}(x) = E_l(x) \cup [E_l, (X_1, .., X_m)](x) \quad (19)$$

We suppose that in the starting point x there exists an l such that $E_l(x)$ spanns R^d (Strong Hoermander's hypothesis).

We consider an Hoermander's type operator:

$$L = X_0 + 1/2 \sum_{i\geq 1} X_i^2 \quad (20)$$

The heat semi-group associated to L has a smooth density $p_t(x,y)$. It is the purposed of Hoermander's theorem [22]. We can use to show that the Malliavin Calculus without probability of Bismut type. To inverse the Malliavin matrix, there are a lot of methods in stochastic analysis which use the full path measure. But the method of Léandre ([10]) which was used to get a generalization of Hoermander's theorem for jump process, which uses basically the Markov property, that is the semi-group property can be used in this set-up where we translate the Malliavin Calculus in semi-group theory.

We are concerned in this paper by the behaviour of $p_t(x,y)$ when $t \to 0$. We introduce as it is classical the Carnot-Caratheory distance defined as follows: let $t \to h_t^i$ m elements of $L^2([0,1])$. We consider the horizontal curve:

$$dx_t(h) = \sum_{i\geq 1} X_i(x_t(h)) h_t^i dt \quad (21)$$

We define $d^2(x,y)$ as the infimum of $\sum \int_0^1 |h_t^i|^2 dt = \|h\|^2$ when $x_0(h) = x$ and $x_1(h) = y$. In the sequel we will do the following assumption: $d(x,y) < \infty$ for all y in R^d.

Our result is the following:

Theorem 4 *When $t \to 0$*

$$\underline{\lim} 2t \log p_t(x,y) \geq -d^2(x,y) \quad (22)$$

Scheme of the proof: us consider the Hilbert space H of L^2 maps from $[0,1]$ into R^m. We can consider according Bismut ([6]) the elements h where $h \to x_1(h)$ is a submersion.

The main remark of Léandre ([11], [12]) is that

$$d^2(x,y) = d_B^2(x,y) \quad (23)$$

where in $d_B^2(x,y)$ (so-called Bismutian distance) we take the infimum of $\|h\|^2$ where $x_1(h)(x) = y$, $x_0(h) = x$ and $h \to x_1(h)$ is a submersion in h. In order to see that, we remark that in some sense due to the non-degenaracy condition of Malliavin, $h \to x_1(h)$ is "almost surely" a submersion: so we can go through a small path from x and since (21) is autonomous, we can come back to x and go to y by a minimizing curve (It was called in [12] little loop principle). Afterwards since we want only to get a lower bound, we analyze the diffusion along a curve which is almost minimizing and where Bismut's condition is checked: the fluctuation around this curve is a non-degenerated Gaussian variable which has therefore a strictly positive density. Let us recall that Bismut pointed out ([6]) if we were analysing the diffusion along a minimizing curve as it was done for instance a long time ago by Molchanov ([23]), the first fluctuation is in general a degenerated Gaussian measure, which has no density,, because Bismut condition is not in general satisfied for minimizing curves between x and y. We remark that in [11] [12] only stochastic differential equations appear, which can be translated in semi-group theory in [20]. In order to pass to density, we translate in [20] in semi-group theory, the Malliavin Calculus of Bismut type depending of a parameter of Léandre [11], [12].

$\diamondsuit$

Remark: We can follow this proof in order to show without using probabilities the following result which was shown by Ben Arous-Léandre ([4]) by using probabilities. Let L_ϵ be the generator $X_0 + \frac{\epsilon^2}{2}\sum_{i>0} X_i^2$ where the involved vector fields satisfy still the strong Hoemander's hypothesis. The associated semi-group under assumption similar to the assumptions of the introduction has an heat-kernel $p_t^\epsilon(x,y)$. Instead of considering (21), we consider:

$$dx_t(h) = \sum_{i>0} X_i(x_t(h))h_t^i dt + X_0(x_t(h))dt \quad (24)$$

We introduce the quantity $d_B^2(x,y)$ (so called pseudo Bismutian distance) which is the infimum of $\|h\|^2$ such that $x_0(h) = x$, $x_1(h) = y$ and $h' \to x_1(h')$ is a submersion in h. We have only in this case $d_B^2(x,y) \geq d^2(x,y)$ because the equation (24) is not autonomous. We get when $\epsilon \to 0$

$$\underline{\lim} 2\epsilon^2 \log p_1^\epsilon(x,y) \geq -d_B^2(x,y) \quad (25)$$

4 Positivity theorem in semi-group theory

Let us consider the equation starting from x:

$$dx_t^r(z)(x) = rX_0(x_t^r(z)(x))dt + \sum_{i>0} X_i(x_t^r(z)(x))z^i dt \quad (26)$$

where the z^i follows a Gaussian law on R^m with covariance rI and average 0. We consider the kernel operating on smooth function with bounded derivatives at all order

$$Q_r f(x) = E[f(x_r^r(.)(x))] \quad (27)$$

An elementary computation show if L is defined as in (20) and f is such a function that when $r \to 0$:

$$Q_r f(x) = f(x) + rLf(x) + O(r^2) \quad (28)$$

We take $r = 1/n$ for a given integer and we iterate $Q_{1/n}$ k-times. We get a kernel $Q_{1/n}^k$ which satisfies when k varies a difference equation which is a good discrete approximation by (28) of the parabolic equation satisfies by the semi-group P_t associated to L. We get

Theorem 5 *(Wong-Zakai) When $n \to \infty$, we get if f is a smooth function with bounded derivatives at each order that $Q_{1/n}^n f(x) \to P_1 f(x)$*

Wong-Zakai approximation is in stochastic analysis the approximation of the Stratonovitch differential equation issued of x

$$dx_t(x) = X_0(x_t(x))dt + \sum_{i>0} X_i(x_t(x))dw_t^i \quad (29)$$

where the w_t^i are independent Brownian motions. In the Wong-Zakai approximation, we replace formally in (29) the leading Brownian motions by their polygonal approximation $(w_.^i)^n$. We get therefore a random ordinary differential equation $x_t^n(x)$ and we get

$$Q_{1/n}^n f(x) = E[f(x_1^n(x))] \quad (30)$$

We consider the map $(w_.^i)^n \to x_1^n(x)$ from the finite dimensional Gaussian space described by the polygonal approximation of the leading Brownian motion into R^d. We can compute the derivative of the flow $D_x x_1^n(x)$ associated to this random ordinary differential equation as well as the Gram matrix V_n associated to the map from this Gaussian finite dimensional equation into R^d. The both quantities satisfies random ordinary equation, and for these we can improve a little bit the Wong-Zakai approximation of Theorem

5, by introducing a convenient kernel $\hat{Q}_{1/n}\hat{f}(x,U,V)$ where x belongs to R^d, U belongs to Gl_d and V is a matrix on R^d. We can show that

$$\hat{Q}^n_{1/n}\hat{f}(x,U,V) \to \hat{P}_1\hat{f}(x,U,V) \tag{31}$$

for a convenient set of test function $\hat{f}$ for the semi-group defined in (16).

On the finite dimensional Gaussian space, we can integrate by parts for $x^n_1(x)$ and pass to the limit through the analytical Wong-Zakai approximation in order to arrive to integration by parts formula of the type (17). We get the following theorem originally proved by Bell ([3]) by using stochastic integrals and martingales inequalities:

Theorem 6 *(Bell)Let g be a smooth function with bounded derivatives at each order on R^+ equal to 0 in a neighborhood of 0. Let μ^n the measure $f \to \hat{Q}^n_{1/n}[fg(detV)](x,I,0)$ and let μ be the measure $f \to \hat{P}_1[fg(detV)](x,I,0)$. The both measures have smooth densities p^n and p and when $n \to \infty$ p^n tends to p uniformly.*

This theorem allows us to prove the next theorem, originally proved in [4] by using Bismut's procedure, a kind of implicit function theorem for the almost surely defined solution of (29):

Theorem 7 *(Ben Arous-Léandre) The measure $P_1(x)(dy)$ is bounded below by a measure having a strictly positive density in y_0 as soon as the Bismutian pseudo-distance $d_B(x,y_0)$ is finite.*

Scheme of the proof: We can look by the previous theorem to the finite dimensional situation, where we can apply the classical finite dimensional implicit theorem. But we have to apply uniformly this implicit function theorem for big n in μ^n: it is possible by a discrete version of the classical Stroock-Varadhan support theorem for diffusions.
$\diamondsuit$

5 Conclusion

We have reviewed some applications of the Malliavin Calculus of Bismut type without probability to the study of some heat kernels, where probabilistic tools were before used: the main remark is that in the probabilist treatment of these, only suitable stochastic differential equation appear, which lead to considerations which can be interpreted in semi-group theory.

References:

[1] Albeverio S. Hoegh-Krohn R.: Dirichlet forms and diffusion processes on rigged Hilbert spaces.*Z.W.* 40 (1977), 1-57.

[2] Averbuh V.I., Smolyanov O.G., Fomin S.V.: Generalized functions and differential equations in linear spaces. I. Differentiable measures. *Trudy Moskow Math. Ob.* 24, 1971, 133-174.

[3] Bell D.: *The Malliavin Calculus,* Pitman Monograph 34. Wiley, New-York 1987.

[4] Ben Arous G. Léandre R.: Décroissance exponentielle du noyau de la chaleur sur la diagonale (II). *P.T.R.F.* 90, 1991, pp. 372-402.

[5] Berezanskii Y.M.: The self-adjointness of elliptic operators with infinite number of variables. *Ukrain. Math. J.* 27, 1975, pp. 729-742.

[6] Bismut J.M.: Martingales, the Malliavin Calculus and hypoellipticity under general Hoermander's condition. *Z.W* 56, 1981, pp.469-505.

[7] Bismut J.M: *Large deviations and the Malliavin Calculus,* Prog. Math. 45. Birkhauser, Basel 1984.

[8] Hida T.: *Analysis of Brownian functionals,* Carleton Mathematical Lecture Notes 13, Ottawa, 1975.

[9] Kusuoka S.: More recent theory of Malliavin Calculus. *Sugaku* 5, 1992, pp.155-173.

[10] Léandre R.: *Extension du théoreme de Hoermander a divers processus de sauts,* These. Université de Franche-Comté, 1984.

[11] Léandre R.: Estimation en temps petit de la densité d'une diffusion dégénérée. *C.R.A.S.* T 301., 1985, pp. 801-804.

[12] Léandre R.: Minoration en temps petit de la densité d'une diffusion dgénérée. *J.F.A.* 74, 1987, pp.399-414.

[13] Léandre R.: Intégration dans la fibre associée a une diffusion dégénérée. *P.T.R.F.* 76, 1987, 341-358.

[14] Léandre R.: Applications quantitatives et qualitatives du Calcul de Malliavin. In *Séminaire Franco-Japonais* (Paris. E.N.S.) M. Métivier S. Watanabe eds. L.N.M. 1322, 1988, pp.109-134. English translation: In *Geometry of Random motion* (Cornell) R. Durrett, M. Pinsky eds. Cont. Maths. 73 ,1988, pp.173-196.

[15] Léandre R.: Strange behaviour of the heat-kernel, on the diagonal. In *Stochastic processes,Physics and Geometry*(Ascona). S. Albeverio edt. World Scientific, Singapore, 1990, pp. 516-527.

[16] Léandre R.: Malliavin Calculus for a general manifold. In *Séminaire X. E.D.P* J.M. Bony, J.Y. Chemin, J. Sjoestrand eds. Exposé XXV. Publ. Ecole Polytechnique ,2003.

[17] Léandre R.: Stochastic mollifier and Nash inequality. In *Recent developments in stochastic analysis and related topics*. (Beijing) Ma Z.M., Roeckner M. eds. World Scientific, Singapore, 2004, pp.246-257.

[18] Léandre R.: Positivity theorem for a general manifold. *S.O.R.T.* 29, 2005, pp. 11-26.

[19] Léandre R.: Malliavin Calculus of Bismut type without probability. To appear Festschrift in honour of K. Sinha. A.M. Boutet de Monvel edt.

[20] Léandre R.: Varadhan estimates without probability. To appear *Mathematical methods in engineerings* (Ankara). D. Baleanu edt.

[21] Léandre R.: Positivity theorem in semi-group theory. Preprint (2006).

[22] Malliavin P.: Stochastic calculus of variations and hypoelliptic operators. In *Proceedings of the International Symposium on Stochastic differential equations* (Kyoto) K. Itô edt. Wiley, New-York, 1978, pp.195-263.

[23] Molchanov S.: Diffusion processes and Riemannian geometry. *Rus. Math. Surv.* 30, 1975, pp.1-63.

[24] Nualart D. :*The Malliavin Calculus and related topics*, Springer, Heidelberg 1995.

[25] Watanabe S.: Stochastic analysis and its application. *Sugaku* 5, 1992, pp.51-72.