Optimization Methods for Optimal Control of Nonlinear Elliptic Systems

JOHN COLETSOS and BASIL KOKKINIS
Department of Mathematics, School of Applied Mathematics and Physics
National Technical University of Athens
Zografou Campus, 15780 Athens, Greece

Abstract: - We consider an optimal control problem for systems governed by a highly nonlinear second order elliptic partial differential equation, with control and state constraints. The problem is formulated in the classical and in the relaxed form, and various necessary conditions for optimality are given. For the numerical solution of these problems, we propose a penalized gradient projection method generating classical controls and a penalized conditional descent method generating relaxed controls. We study the behavior in the limit of sequences constructed by these methods, using relaxation theory. Finally, numerical examples are given.

Key-Words: - Optimal control, nonlinear elliptic systems, state constraints, classical penalized gradient projection method, relaxed penalized conditional descent method, relaxed controls.

1 Introduction

We consider an optimal control problem for systems governed by a highly nonlinear second order elliptic partial differential equation, with control and state constraints. The problem is formulated in the classical and in the relaxed form. Various necessary conditions for optimality are first given for both formulations. For the numerical solution of these problems, we then propose a penalized gradient projection method generating classical controls, and a penalized conditional descent method generating relaxed controls. Under appropriate assumptions, we prove that relaxed (resp. strong in L^2 classical) limits of subsequences (resp. sequences) constructed by the classical method are admissible and weakly extremal relaxed (resp. classical) for the relaxed (resp. classical) problem, and that relaxed limits of subsequences of controls constructed by the relaxed method are admissible and strongly extremal for the relaxed problem. Finally, numerical examples are given. For various classical and relaxed optimization and approximation methods applied to optimal control problems, see e.g. [2-15,17,19], and the references therein.

2 Classical and relaxed formulations

Let Ω be a bounded domain in $\mathbb{R}^d$, with Lipschitz boundary Γ. Consider the nonlinear elliptic state equation

(1) $Ay + f(x, y(x), w(x)) = 0$ in Ω,

(2) $y(x) = 0$ on Γ,

where A is the formal second order elliptic differential operator

(3) $Ay := -\sum_{j=1}^{d}\sum_{i=1}^{d} \frac{\partial}{\partial x_i}[a_{ij}(x)\frac{\partial y}{\partial x_j}].$

We denote by $(\cdot,\cdot)$ and $\|\cdot\|$ the inner product and norm in $L^2(\Omega)$, and by $(\cdot,\cdot)_1$ and $\|\cdot\|_1$ the inner product and norm in the Sobolev space $V := H_0^1(\Omega)$. The state equation will be interpreted in the following weak form

(4) $y \in V$ and

$$a(y,v) + \int_\Omega f(x, y(x), w(x))v(x)dx = 0,\ \forall v \in V,$$

where $a(\cdot,\cdot)$ denotes the usual bilinear form on $V \times V$ associated with A

(5) $a(y,v) := \sum_{i,j=1}^{d} \int_\Omega a_{ij}(x)\frac{\partial y}{\partial x_i}\frac{\partial v}{\partial x_j}dx$.

Define the set of *classical controls*

(6) $W := \{w : \Omega \to U \mid w \text{ measurable}\}$,

where $U \subset \mathbb{R}^\nu$ is a compact set, and the functionals

(7) $G_m(w) := \int_\Omega g_m(x, y(x), w(x))dx,\ \ m = 0,...,q$.

The classical optimal control Problem P is to minimize $G_0(w)$ subject to the control and state constraints

(8) $w \in W$,

(9) $G_m(w) = 0$, $m = 1,..., p$,

$G_m(w) \le 0$, $m = p+1,...,q$.

It is well known that the classical problem may have no solutions, even if the set U is convex. The existence of such a solution is usually proved under strong, often unrealistic for nonlinear systems, convexity assumptions (such as the Cesari property). Reformulated in the so-called relaxed form, the problem is convexified in some sense and has a solution in a larger space under weaker assumptions.

Next, define the set of *relaxed controls* (or Young measures; for the relevant theory, see [18,16])

(10) $R := \{r : \bar{\Omega} \to M_1(U) \,|\, r \text{ weakly measurable}\}$
$\subset L_w^\infty(\bar{\Omega}, M(U)) \equiv L^1(\bar{\Omega}, C(U))^*$,

where $M(U)$ (resp. $M_1(U)$) is the set of Radon (resp. probability) measures on U. The set R is endowed with the relative weak star topology, and R is convex, metrizable and compact. If each classical control $w(\cdot)$ is identified with its associated Dirac relaxed control $r(\cdot) := \delta_{w(\cdot)}$, then W may be regarded as a subset of R, and W is thus dense in R. For $\phi \in L^1(\bar{\Omega}; C(U))$ and $r \in R$, we shall use for simplicity the notation

(11) $\phi(x, r(x)) := \int_U \phi(x,u) r(x)(du).$

The relaxed optimal control Problem $\bar{P}$ is then defined by replacing w by r (with the above notation) and W by R in the classical problem.

We suppose that the coefficients a_{ij} satisfy the ellipticity condition

(12) $\sum_{i,j=1}^{d} a_{ij}(x) z_i z_j \geq \alpha_0 \sum_{i=1}^{d} z_i^2, \ \forall z_i, z_j \in \mathbb{R}, \ x \in \Omega,$

and that the functions f, f_y are defined on $\Omega \times \mathbb{R} \times U$, measurable for fixed y, u, continuous for fixed x, and satisfy the conditions

(13) $|f(x,0,u)| \leq \phi_0(x), \ \forall (x,u) \in \Omega \times U$,

where $\phi_0 \in L^s(\Omega)$, with $s \geq 2$, $s > d/2$ (e.g. $s = 2$, for $d \leq 3$), and

(14) $0 \leq f_y(x,y,u) \leq \phi_1(x) \eta_1(|y|)$,
$\forall (x,y,u) \in \Omega \times \mathbb{R} \times U,$

where η_1 is an increasing function from $\mathbb{R}^+$ to $\mathbb{R}^+$, and $\phi_1 \in L^s(\Omega)$.

Then, for every control $r \in R$, the state equation has a unique solution such that $y := y_r \in V \cap C^\alpha(\bar{\Omega})$, for some $\alpha \in (0,1)$ (see [1]).

We suppose now in addition that the functions g_m are defined on $\Omega \times \mathbb{R} \times U$, measurable for fixed y, u, continuous for fixed x, and satisfy

(15) $|g_m(x,y,u)| \leq \psi_{0m}(x)$,
$\forall (x,y,u) \in \Omega \times \mathbb{R} \times U$ with $|y| \leq C'$,

where $C' > C$, $\psi_{0m} \in L^1(\Omega)$.

Theorem 1 (i) (Assumptions on g_m omitted) The operator $r \mapsto y_r$ (resp. $w \mapsto y_w$), from R (resp. W with the relative topology of $L^2(\Omega)$) to V, and to $C_0(\bar{\Omega})$, is continuous.
(ii) The functionals $r \mapsto G_m(r)$ on R (resp. $w \mapsto G_m(w)$ on W with the L^2-topology) are continuous.
(iii) If the relaxed problem has an admissible control, then it has a solution.

Since $W \subset R$, we generally have

(16) $c_R := \min_{\text{constr. on } r} G_0(r) \leq \inf_{\text{constr. on } w} G_0(w) := c_W$,

where the equality holds, in particular, if there are no state constraints, as W is dense in R. Since usually approximation methods slightly violate the state constraints, approximating an optimal relaxed control by a relaxed or a classical one, hence the possibly lower relaxed optimal cost c_R, is not a drawback in practice (see [18], p. 259). Note also that approximating sequences of classical controls may converge to relaxed ones.

In order to state the optimality conditions, we suppose in addition that the functions f, f_u, g_m, g_{my}, g_{mu} are defined on $\Omega \times \mathbb{R} \times U'$, where U' is an open set containing U, measurable on Ω for fixed $(y,u) \in \mathbb{R} \times U$, continuous on $\mathbb{R} \times U$ for fixed $x \in \Omega$, and satisfy the conditions

(17) $|f_u(x,y,u)| \leq \phi_2(x), \ |g_{my}(x,y,u)| \leq \psi_{1m}(x),$

(18) $|g_{mu}(x,y,u)| \leq \psi_{2m}(x),$
$\forall (x,y,u) \in \Omega \times \mathbb{R} \times U$, with $|y| \leq C'$,

where $C' < C$, $\phi_2, \psi_{im} \in L^2(\Omega)$.

The following lemma and theorems can be proved by using the techniques of [8,18] (the weak relaxed minimum principle in Theorem 2 is proved similarly to Theorem 2.2 in [12]).

Lemma 1 With the derivatives in u omitted (resp. included) in our last assumptions, dropping the index m in g_m, G_m, the directional derivative of the functional G defined on R (resp. W, with U convex) is given, for $r, \bar{r} \in R$ (resp. $w, \bar{w} \in W$), by

(19) $DG(r, \bar{r} - r) = \lim_{\alpha \to 0^+} \dfrac{G(r + \alpha(\bar{r} - r)) - G(r)}{\alpha}$
$= \int_\Omega H(x, y(x), z(x), \bar{r}(x) - r(x)) dx$,

(resp.

(20) $DG(w,\bar{w}-w)=\lim_{\alpha\to 0^+}\dfrac{G(w+\alpha(\bar{w}-w))-G(w)}{\alpha}$
$=\int_\Omega H_u(x,y(x),z(x),w(x))(\bar{w}(x)-w(x))dx$),
where the Hamiltonian H is defined by
(21) $H(x,y,z,u):=-z\,f(x,y,u)+g(x,y,u)$,
and the adjoint state $z:=z_r\in V$ satisfies the linear adjoint equation
(22) $a(v,z)+(f_y(y,r)z,v)=(g_y(y,r),v)$,
(resp. $a(v,z)+(f_y(y,w)z,v)=(g_y(y,w),v)$),
$\forall v\in V$, with $y:=y_r$ (resp. $y:=y_w$).
Moreover, the operator $r\mapsto z_r$, from R to V (resp. $w\mapsto z_w$, from W to V), and the functional $(r,\bar{r})\mapsto DG(r,\bar{r}-r)$, defined on $R\times R$ (resp. $(w,\bar{w})\mapsto DG(w,\bar{w}-w)$, on $W\times W$), are continuous.

Theorem 2 (Necessary Conditions for Optimality) With the derivatives in u omitted (resp. included) in the assumptions, if $r\in R$ (resp. $w\in W$, with U convex) is optimal for Problem $\bar{P}$ or P (resp. Problem P), then r (resp. w) is *strongly extremal relaxed* (resp. *weakly extremal classical*), i.e. there exist multipliers $\lambda_m\in\mathbb{R}$, $m=0,...,q$, with $\lambda_0\ge 0$, $\lambda_m\ge 0$, $m=p+1,...,q$, $\sum_{m=0}^{q}|\lambda_m|=1$, such that
(23) $\sum_{m=0}^{q}\lambda_m DG_m(r,\bar{r}-r)\ge 0,\ \forall\bar{r}\in R$,
(24) $\lambda_m G_m(r)=0,\ m=p+1,...,q$,
(relaxed transversality conditions)
(resp.
(25) $\sum_{m=0}^{q}\lambda_m DG_m(w,\bar{w}-w)\ge 0,\ \forall\bar{w}\in W$,
(26) $\lambda_m G_m(w)=0,\ m=p+1,...,q$,
(classical transversality conditions)).
The global condition (23) is equivalent to the strong relaxed pointwise minimum principle
(27) $H(x,y(x),z(x),r(x))=\min_{u\in U}H(x,y(x),z(x),u)$,
a.e. in Ω,
where the complete Hamiltonian H and adjoint z are defined with $g:=\sum_{m=0}^{q}\lambda_m g_m$.
If U is convex, then (27) implies the weak relaxed pointwise minimum principle
(28) $H_u(x,y,z,r(x))r(x)$
$=\min_{\phi}H_u(x,y,z,r(x))\phi(x,r(x))$, a.e. in Ω,
where the minimum is taken over the set $B(\bar{\Omega},U;U)$ of Caratheodory functions (in the sense of Warga [18]), which in turn implies the global weak relaxed condition
(29) $\int_\Omega H_u(x,y,z,r(x))[\phi(x,r(x))-r(x)]dx\ge 0$,
$\forall\phi\in B(\bar{\Omega},U;U)$.

A control r satisfying (29) and (24) is called weakly extremal relaxed. The global condition (25) is equivalent to the weak classical pointwise minimum principle
(30) $H_u(x,y(x),z(x),w(x))w(x)$
$=\min_{u\in U}H_u(x,y(x),z(x),w(x))u$, a.e. in Ω.

3 Optimization methods

Let (M_m^l), $m=1,...,q$, be positive and increasing sequences such that $M_m^l\to\infty$ as $l\to\infty$, $\gamma\ge 0$, $b,c\in(0,1)$, and (β^l), (ζ_k) positive sequences, with (β^l) decreasing and converging to zero, and $\zeta_k\le 1$. We define the penalized functionals on W
(31) $G^l(w):=G_0(w)+\{\sum_{m=1}^{p}M_m^l[G_m(w)]^2$
$+\sum_{m=p+1}^{q}M_m^l[\max(0,G_m(w))]^2\}/2$.
The classical penalized gradient projection method is described by the following Algorithm, where U is assumed to be convex.

Algorithm 1
Step 1. Set $k:=0$, $l:=1$, and choose $w_0^1\in W$.
Step 2. Find $v_k^l\in W$ such that
(32) $e_k:=DG^l(w_k^l,v_k^l-w_k^l)+\frac{\gamma}{2}\|v_k^l-w_k^l\|^2$
$=\min_{\bar{v}\in W}[DG^l(w_k^l,\bar{v}-w_k^l)+\frac{\gamma}{2}\|\bar{v}-w_k^l\|^2]$,
and set $d_k:=DG^l(w_k^l,v_k^l-w_k^l)$.
Step 3. If $|d_k|\le\beta^l$, set $w^l:=w_k^l$, $v^l:=v_k^l$, $d^l:=d_k$, $e^l:=e_k$, $w_k^{l+1}:=w_k^l$, $l:=l+1$, and go to Step 2.
Step 4. (Modified Armijo Step Search) Find the lowest integer value $s\in\mathbb{Z}$, say $\bar{s}$, such that $\alpha(s)=c^s\zeta_k\in(0,1]$ and $\alpha(s)$ satisfies the inequality
(33) $G^l(w_k^l+\alpha(s)(v_k^l-w_k^l))-G^l(w_k^l)\le\alpha(s)bd_k$,
and then set $\alpha_k:=\alpha(\bar{s})$.
Step 5. Set $w_{k+1}^l:=w_k^l+\alpha_k(v_k^l-w_k^l)$, $k:=k+1$, and go to Step 2.

One can easily see by "completing the square" that Step 2 amounts to finding the projection v_k^l of the function

(34) $u_k^l(x) := w_k^l(x) - (1/\gamma) H_u(x, y_k^l, z_k^l, w_k^l)$,

onto W, which reduces to finding the corresponding pointwise projection onto U for a.a $x \in \Omega$. By the definition of the directional derivative and since $b, c \in (0,1)$, the Armijo step α_k in Step 4 can be found for every k. The parameter γ is chosen here experimentally to yield a good rate of convergence.

A (classical or relaxed) extremal (or weakly extremal) control is called *abnormal* if there exist multipliers as in the optimality conditions, with $\lambda_0 = 0$. A control is admissible *and* abnormal extremal in rather exceptional situations (see [18]).

With w^l as defined in Step 3, define the sequences of multipliers

(35) $\lambda_m^l := M_m^l G_m(w^l), \; m = 1, ..., p,$

(36) $\lambda_m^l := M_m^l \max(0, G_m(w^l)), \; m = p+1, ..., q.$

Theorem 3 We suppose that U is convex.
(i) In the presence of state constraints, if the whole sequence $(w_k^{l(k)})_{k \in \mathbb{N}}$ generated by Algorithm 1 converges to some $w \in W$ in L^2 strongly and the sequences (λ_m^l) are bounded, then w is admissible and weakly extremal for Problem P. In the absence of state constraints, if a subsequence $(w_k)_{k \in K}$ (no index l) converges to some $w \in W$ in L^2 strongly, then w is weakly extremal classical for Problem P.
(ii) In the presence of state constraints, if a subsequence $(w^l)_{l \in L}$ of the sequence generated by Algorithm 1 in Step 3, regarded as a sequence of relaxed controls, converges to some r in R, and the sequences $(\lambda_m^l)_{l \in L}$ are bounded, then r is admissible and weakly extremal relaxed for Problem $\bar{P}$. In the absence of state constraints, if a subsequence $(w_k)_{k \in K}$ (no index l) converges to some r in R, then r is weakly extremal relaxed for Problem $\bar{P}$.
(iii) In any of the convergences cases (i) or (ii) with state constraints, suppose that the classical, or the relaxed, problem has no admissible, abnormal extremal, controls. If the limit control is admissible, then the sequences of multipliers are bounded, and this control is also extremal as above.

Next, we define the penalized discrete functionals on R

(37) $G^l(r) := G_0(r) + \{\sum_{m=1}^{p} M_m^l [G_m(r)]^2$

$+ \sum_{m=p+1}^{q} M_m^l [\max(0, G_m(r))]^2\}/2$.

The relaxed penalized conditional descent method is described by the following Algorithm, where U is not necessarily convex.

Algorithm 2
Step 1. Set $k := 0$, $l := 1$, and choose $r_0^1 \in R$.
Step 2. Find $\bar{r}_k^l \in R$ such that

(38) $d_k := DG^l(r_k^l, \bar{r}_k^l - r_k^l) = \min_{r' \in R} DG^l(r_k^l, r' - r_k^l).$

Step 3. If $|d_k| \le \beta^l$, set $r^l := r_k^l$, $\bar{r}^l := \bar{r}_k^l$, $d^l := d_k$, $r_k^{l+1} := r_k^l$, $l := l+1$, and go to Step 2.
Step 4. (Modified Armijo Step Search) Find the lowest integer value $s \in \mathbb{Z}$, say $\bar{s}$, such that $\alpha(s) = c^s \zeta_k \in (0,1]$ and $\alpha(s)$ satisfies the inequality

(39) $G^l(r_k^l + \alpha(s)(\bar{r}_k^l - r_k^l)) - G^l(r_k^l) \le \alpha(s) b d_k$,

and then set $\alpha_k := \alpha(\bar{s})$.
Step 5. Choose any $r_{k+1}^l \in R$ such that

(40) $G^l(r_{k+1}^l) \le G^l(r_k^l + \alpha_k(\bar{r}_k^l - r_k^l)),$

set $k := k+1$, and go to Step 2.

With r^l as defined in Step 3, define the sequences of multipliers

(41) $\lambda_m^l := M_m^l G_m(r^l), \; m = 1, ..., p,$

(42) $\lambda_m^l := M_m^l \max(0, G_m(r^l)), \; m = p+1, ..., q.$

Theorem 4 We suppose that the derivatives in u are excluded in the last assumptions of Section 2.
(i) In the presence of state constraints, if a subsequence $(r^l)_{l \in L}$ of the sequence generated by Algorithm 2 in Step 3 converges to some $r \in R$ and the sequences (λ_m^l) are bounded, then r is admissible and strongly extremal relaxed for Problem $\bar{P}$. In the absence of state constraints, if a subsequence $(r_k)_{k \in K}$ (no index l) converges to some r in R, then r is strongly extremal relaxed for Problem $\bar{P}$.
(ii) In case (i) with state constraints, suppose that the relaxed problem has no admissible, abnormal extremal, controls. If r is admissible, then the sequences of multipliers are bounded and r is also strongly extremal relaxed for Problem $\bar{P}$.

For the implementation of relaxed algorithms similar to Algorithm 2, where Gamkrelidze controls are actually constructed, in the continuous and discrete cases, see [6] and [10].

In practice, by choosing in Algorithms 1 and 2 moderately growing sequences (M_m^l) and a sequence (β^l) relatively fast converging to zero, the resulting sequences of multipliers (λ_m^l) are often kept bounded. One can choose a fixed initial step $\zeta_k := \zeta \in (0,1]$ in Step 4; a usually faster and adaptive procedure is to set $\zeta_0 := 1$, and then $\zeta_k := \alpha_{k-1}$, for $k \geq 1$.

When directly applied to nonconvex problems whose solutions are non-classical relaxed controls, the above classical methods (Algorithm 1) may yield very slow convergence, due to highly oscillating involved extremal controls. If the constraint set U is convex, one can formulate the relaxed problem in the equivalent Gamkrelidze relaxed form, using convex combinations of Dirac controls involving a finite, usually small, number of classical controls. Algorithm 1 can then be applied to this extended classical control problem, with much better results (for details on this approach, see [11,12]). When U is not convex, one can use Algorithm 2 for solving such highly nonconvex problems.

Finally, Gamkrelidze relaxed controls (practically in discrete form) computed as above, or by Algorithm 2, can be approximated, and simulated, by classical controls using a standard procedure similar to [9].

4 Numerical examples

Example 1. Let $\Omega := (0,1)^2$. Define the functions

(43) $\bar{u}(x) := \bar{v}(x) = x_1 x_2,$

(44) $\bar{y}(x) := 8x_1x_2(1-x_1)(1-x_2),$

and consider the following optimal control problem, with state equation

(45) $-\Delta y + y^3/3 + (1+u-\bar{u}))y - \bar{y}^3/3 - \bar{y}$
$-16[x_1(1-x_1) + x_2(1-x_2)] - (v-\bar{v}) = 0$ in Ω,

(46) $y(x) = 0$ on Γ,

control constraints $(u(x), v(x)) \in U := [0,1]^2$, $x \in \Omega$, and cost functional to be minimized

(47) $G_0(u,v) = \dfrac{1}{2}\int_\Omega [(y-\bar{y})^2 + (u-\bar{u})^2 + (v-\bar{v})^2]dx.$

Clearly, the optimal controls are $\bar{u}$ and $\bar{v}$, the optimal state is $\bar{y}$, and the optimal cost is zero. Algorithm 1, without penalties, was applied to this problem using the finite element method with continuous piecewise linear basis functions on triangular elements (half squares of edge size $h = 0.01$) for solving the differential equations, with (not necessarily continuous) elementwise linear classical controls, with $\gamma = 0.5$, Armijo parameters $b = c = 0.5$. After 15 iterations, we obtained the following results

(48) $G_0(u_k, v_k) = 2.994 \cdot 10^{-9}$, $d_k = -6.211 \cdot 10^{-12}$,
$\varepsilon_k = 3.204 \cdot 10^{-5}$, $\eta_k = 3.052 \cdot 10^{-5}$,

where ε_k (resp. η_k) is the maximum error for the state (resp. controls) at the vertices of the triangles (resp. midpoints of the triangle edges).

Example 2. Introducing the state constraint

(49) $G_1(u,v) := \int_\Omega (y - 0.22)dx = 0,$

in Example 1, choosing $U := [0, 0.7]^2$, and applying here the penalized Algorithm 1, we obtained after 60 iterations in k the results

(50) $G_0(u_k^l, v_k^l) = 2.034848361 \cdot 10^{-3}$,
$G_1(u_k^l, v_k^l) = 8.247 \cdot 10^{-6}$, $d_k = -3.956 \cdot 10^{-5}$.

Example 3. Define the functions

(51) $\bar{w}(x) := \max(-1, 1 - 1.5(x_1 + x_2)),$

(52) $\bar{y}(x) := 8x_1x_2(1-x_1)(1-x_2),$

and consider the following problem, with state equation

(53) $-\Delta y + y^3/3 + (2 + w - \bar{w})y - \bar{y}^3/3 - 2\bar{y}$
$-16[x_1(1-x_1) + x_2(1-x_2)] = 0$ in Ω,

(54) $y(x) = 0$ on Γ,

nonconvex control constraint set $U := \{-1\} \cup [0.5, 1]$, and nonconvex cost functional

(55) $G_0(w) := \int_\Omega \{0.5(y-\bar{y})^2 - w^2 + 1\}dx.$

It can be easily verified that the unique optimal relaxed control r is given by

(56) $r(x)\{1\} = [\bar{w}(x) - (-1)]/2$
$\in \begin{cases} \{0\}, \text{ if } 1 - 1.5(x_1 + x_2) \leq -1 \\ (0,1], \text{ if } 1 - 1.5(x_1 + x_2) > -1 \end{cases}$

(57) $r(x)\{-1\} = 1 - r(x)\{1\},$

for $x \in \Omega$, with optimal state $\bar{y}$ and cost 0. The control r is concentrated at the two points 1 and -1; r is classical if $1 - 1.5(x_1 + x_2) \leq -1$, and non-classical otherwise. Note also that the optimal cost value 0 can be approximated as closely as desired by using a classical control (as W is dense in R), but clearly cannot be attained for such a control. Algorithm 2, without penalties, was applied to this problem using the finite element method of Example 1, here with elementwise constant relaxed controls. After 100 iterations in k, we obtained the results

(58) $G_0(r_k) = 9.152 \cdot 10^{-9}$, $d_k = -3.032 \cdot 10^{-8}$,
$\varepsilon_k = 3.543 \cdot 10^{-4}$,

where ε_k is the maximum state error at the vertices of the triangles.

Example 4. Introducing the state constraint

(59) $G_1(w) := \int_\Omega (y - 0.22)dx = 0,$

in Example 3 and applying the penalized Algorithm 2, we obtained after 200 iterations in k the results

(60) $G_0(r_k^l) = 3.087499887 \cdot 10^{-6},$

$G_1(r_k^l) = 7.187 \cdot 10^{-8}, \; d_k = -2.569 \cdot 10^{-7}.$

Finally, similar results were obtained after the approximation of the last computed relaxed controls by classical ones (see end of Section 3).

5 Conclusions

An optimal control problem involving highly nonlinear elliptic systems has been studied. Necessary conditions for optimality have been derived for the classical and relaxed formulations of the problem. A classical penalized gradient projection method and a relaxed penalized conditional descent method have been proposed. Using also relaxation theory, the behavior in the limit of sequences constructed by these methods has been analyzed.

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