Adaptive Model Reduction for Sensitivity Analysis

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Abstract: The bad accuracy of a simplified model can compromise the sensitivity analysis. We propose to build accurate simplified models for sensitivity analysis thanks to an adaptive model reduction method: the A Priori Hyper Reduction (APHR) method. The adaptivity allows to guaranty both the quality of the state estimation and a small number of shape functions involved in the reduced order model (ROM). This approach is very convenient for time dependent problems described by the finite element method. In case of non-linear problems, a reduction of integration point number named Hyper Reduction improves the efficiency of the simplified computations. This method can be thought of as an adaptive Snapshot POD. The ROM adaptations are based on iterative finite element computations. The initial guess of such iterative computations is obtained thanks to the current ROM. So the proposed method can also be interpreted as a convergence acceleration method based on model reduction. A non linear thermal example illustrates the capability of the APHR method. A numerical sensitivity analysis is solved to validate the efficiency of the adaptive strategy. The ROMs we proposed are time independent but slightly parameter dependent. The effect of the values of the parameters at the beginning of an inverse problem treatment can be forgotten, while the ROM is efficient for values of parameters in the vicinity of the optimal parameters.

Key–Words: Snapshot POD, hyper reduction, convergence acceleration, incremental learning

1 Introduction

Numerical sensitivity analysis are based on variations of model input parameters over a reasonable range in order to observe the relative change in the model response. Two interesting questions can be raised. How can be reused previous computational results in order to reduce the time needed to obtain an updated model response? And, how accurate should be the model to get a convenient sensitivity estimation? The relative change of model response should be around 5% in case of sensitive parameters. To catch a convenient estimation of such sensitivity the relative change in the model response must be greater than the variations of numerical defaults due to the variation of parameters. In practice, the model used for sensitivity analysis should be accurate enough. Two different kinds of approaches are available to reuse previous computational results: the model reduction methods and the convergence acceleration methods associated to an iterative solving technic [1]. A reduced order model (ROM) is deduced from the full model thanks to a set of shape functions that span a subspace large enough to approximate the response. The Snapshot POD [2] is a reduction method widely used in the frame work of non-linear time dependent analysis. The principle of convergence acceleration methods is to propose an initial guest to an iterative solver thanks to a subspace approximation. In case of time dependent problem the subspace used for convergence acceleration can be time dependent, despite the subspace spanned by the shape function of a ROM should be time independent. An accurate ROM, according to large variations of parameters, usually leads to a large number of shape functions and a large data base. With convergence acceleration methods, the size of the data base has no influence on the accuracy of the estimation of the model response. But it has an influence on the efficiency of the computation. Interesting approaches consist in mixing model reduction and convergence acceleration [3] [6]. To do so, a time independent subspace is obtained thanks to the Proper Orthogonal Decomposition of the state evolution obtained during the incremental computation. This paper is focused on the capability of the APHR method [3] to build ROM shape functions that are time independent but parameter dependent. Hence, during the treatment of an inverse problem, the ROM can be adapted to be efficient only in the vicinity of the optimal parameters.

To reduce the cost of the computation of a new model response we propose :

- a reduction of the number of state variables thanks

to a convenient choice of ROM shape functions;

- a reduction of the ROM integration points by an Hyper Reduction technique;

- an incremental adaptive strategy to avoid to check the quality of the estimated model response at all time steps;

- a selection of the most significant events as shape functions thanks to a POD [4] applied on reduced state variables (small eigenproblems);

- an expansion of the subspace spanned by the shape function thanks to an iterative solving of the high dimensional equations with an initial guess provided by the ROM.

The POD applied to reduced state variables warranties the time independence of the shape functions. Applied at the time instant t_j during the incremental computation, the POD provides a selection of the most significant events involved in the state evolution before t_j. This property is used to adapt the ROM during the incremental computation. Thus we avoid a too important size of ROM. But due to this procedure shape functions that are significant for time instants higher than t_j can be dramatically removed of the ROM. When a ROM is used to perform a sensitivity analysis, all the shape functions should be reused and eventually completed. No shape function should be removed. A learning strategy is proposed to master this memory effect. A single scalar parameter allows to gradually choose between forgetting or storing the significant events involved in a sequence of different model responses related to different parameter values. If we choose to select the significant part of all the computed states, the ROM shape functions are going to be time and parameter independent. But the size of the ROM tends to be very large. On the contrary, when solving an inverse problem, it's possible to forget the effect of the first values of the parameters during the iterative optimization of the parameter values. Then we obtain shape functions that are time independent but slightly parameter dependent.

The second section of this paper is a brief presentation of the APHR method. The Hyper Reduction is explained in case of a given ROM. The ROM adaptivity and the memory selection of the effect of a sequence of parameter values is presented as an incremental learning strategy. The third section provides numerical results obtained on a non-linear thermal problem.

2 The A Priori Hyper Reduction method

2.1 Formulation of the high dimensional model

The purpose of this section is to define the equations of the high dimensional model. This model is assumed to be obtained by the finite element method. In the framework of thermomechanical simulations, the state variable s could be a displacement field or a temperature field defined over the structure Ω. The Finite Element method allows to describe this state variable thanks to shape functions N_i and nodal degrees of freedom q_i such that:

$$s\left(x,\, t\right) = \sum_{i=1}^{i=\widetilde{n}} N_i\left(x\right)\, q_i\left(t\right) \quad \forall x \in \Omega\, \forall t \tag{1}$$

The state evolution is described by the value of the degrees of freedom at different time instants $t_j \in \{t_1,\, ...,\, t_m\}$ such that :

$$q_i\left(t\right) = q_i\left(t_j\right)\, \frac{t_j - t}{t_j - t_{j-1}} + q_i\left(t_j\right)\, \frac{t - t_{j-1}}{t_j - t_{j-1}} \tag{2}$$

A column $\underline{q}_j$ of state variables at the time instant t_j can be defined such that the ith componant of $\underline{q}_j$ is $q_i\left(t_j\right)$. Let's assume that a numerical scheme is used for the time integration of the balance equations. Then the following formulation of non linear governing equations is obtained :

$$\underline{q}_1 = q_{ini} \tag{3}$$

$$\underline{R}\left(\underline{q}_j,\, \underline{q}_{j-1},\, t_j\right) = 0 \quad \forall\, j = 2 \ldots m \tag{4}$$

$\underline{R}$ is the residual of the governing equations. $\underline{R}$ and $\underline{q}$ have the same size. Each line of the system of equations (4) corresponds to a local balance condition. A step by step approach is used. We assume that an iterative solver allows to forecast $\underline{q}_j$ for a known $\underline{q}_{j-1}$ thanks to an initial guest $\widetilde{\underline{q}}_j$. An estimated state evolution $\left(\widehat{\underline{q}}_j\right)_{j=2...m}$ is convenient if the following quality criteria is satisfied :

$$\left\|\underline{R}\left(\widehat{\underline{q}}_j, \widehat{\underline{q}}_{j-1},\, t_j\right)\right\| < \epsilon_R\, \left\|\underline{F}\right\| \quad \forall j = 2...m \tag{5}$$

where the norm $\|.\|$ is the euclidian norm such that $\left\|\underline{q}\right\|^2 = \underline{q}^T\, \underline{q}$. Such criteria depends on a column of a generalized force $\underline{F}$ corresponding to the significant forces induced by the state evolution over the entire time interval.

2.2 The Hyper Reduction

The Galerkin projection is the most popular formulation used to deduce the governing equations of the reduced state variables from the governing equations of the high dimensional problem. But in case of non linear problems a more efficient approach can be obtained by reducing the number of integration points used to obtain an estimation of the reduced state variables. Let's assume that a known ROM is defined by shape functions $(\phi_k)_{k=1...r}$ such that the state variable s is represented with reduced state variables $a_k(t)$:

$$s\,(x,\,t) = \sum_{k=1}^{k=r} \phi_k\,(x)\;\, a_k\,(t) \quad \forall x \in \Omega\, \forall t \tag{6}$$

The k^{th} shape function ϕ_k is a field of the same kind as s and it is defined by the k^{th} column of a reduction matrix $\underline{\underline{A}}$ such that :

$$\phi_k\,(x) = \sum_{i=1}^{i=\widetilde{n}} N_i\,(x)\;\, A_{i\,k} \quad \forall x \in \Omega \tag{7}$$

By introducing a column $\underline{a}_j$ of reduced state variables at the time instant t_j, such that the kth componant of $\underline{a}_j$ is $a_k\,(t_j)$, we obtain:

$$\underline{q}_j = \underline{\underline{A}}\,\underline{a}_j \quad \forall j = 2 \ldots m \tag{8}$$

The classical Galerkin procedure provides the following governing equations for the reduced state variables:

$$\underline{\underline{A}}^T\,\underline{R}\left(\underline{\underline{A}}\,\underline{a}_j,\;\underline{\underline{A}}\,\underline{a}_{j-1},\;t_j\right) = 0\,\forall\, j > 1 \tag{9}$$

Thus the number of governing equations and the number of reduced state variables are equal. In case of governing equations due to an energy minimization, the Galerkin procedure provides an optimal value of the reduced state variables. In case of non linear problem, if the convergence criteria (5) is satisfied, we are more interested in efficient computations than in the optimal value of the reduced state variables. To obtain a number of equations equal to the number of reduced state variables, we do not need to consider all the governing equations of the high dimensional problem. We propose to choose a selection of few governing equations of the high dimensional problem. This selection can be formally represented by a rectangular matrix $\underline{\underline{\Pi}}$ containing zeros and a one per line. If the value one is placed in column k, this means that the k^{th} governing equation is going to be selected. The formulation of the selected residuals is :

$$\underline{\underline{\Pi}}\,\underline{R}\left(\underline{\underline{A}}\,\underline{a}_j,\;\underline{\underline{A}}\,\underline{a}_{j-1},\;t_j\right)\;\forall\, j > 1 \tag{10}$$

The corresponding local values of the ROM shape functions are given by :

$$\underline{\underline{\Pi}}\,\underline{\underline{A}} \tag{11}$$

The same number of shape functions and reduced governing equations is obtained by the following Hyper Reduction formulation:

$$\underline{\underline{A}}^T\,\underline{\underline{\Pi}}^T\,\underline{\underline{\Pi}}\,\underline{R}\left(\underline{\underline{A}}\,\underline{a}_j,\;\underline{\underline{A}}\,\underline{a}_{j-1},\;t_j\right) = 0\,\forall\, j > 1 \tag{12}$$

To select the governing equations of the high dimensional problem, we consider a list of few loading nodes completed by the following procedure. With a loop on shape functions $(\phi_k)_{k=1...r}$ we add to the list of nodes two new nodes per shape function ϕ_k: one where the gradient of ϕ_k is maximum and one where the magnitude of ϕ_k is maximum. The selected governing equations are related to the degree of freedom of the selected list of nodes. These nodes are connected only to a few elements of the mesh. The part of the domain Ω covered by these elements is the reduced integration domain. It is sufficient to take into account these elements to compute $\underline{\underline{\Pi}}\,\underline{R}$.

A drawback of the Hyper Reduction formulation is an error amplification. To illustrate this procedure we can consider a simple 2D hyperbolic problem:

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 1\,\forall (x,y) \in \Omega \quad T_{\partial\Omega} = 0 \tag{13}$$

A regular grid of 11x11 nodes is used to mesh the domain $\Omega = \{x, y \mid x \in [0,1],\; y \in [0,1]\}$ with linear triangular elements. Thanks to the finite element method, we obtain the column of nodal values $\underline{q}_{FE}$. To study the error amplification due to the Hyper Reduction technic, we introduce a random perturbation $\underline{q}_{rand}$ of the finite element prediction to build an approximate shape function thanks to $\underline{A}$ such that:

$$\underline{A} = \underline{q}_{FE} + \underline{q}_{rand} \tag{14}$$

with :

$$\left\|\underline{q}_{rand}\right\| = 10^{-3}\;\left\|\underline{q}_{EF}\right\| \tag{15}$$

Then the reduced state variable a is computed thanks to the Hyper Reduction technic. We consider 1000 random trials. The mean value of $\left\|\underline{q}_{FE} - \underline{A}\,a\right\|$ provided with the Hyper Reduction is 3.38 higher than the default obtained with the classical Galerkin procedure. The maximum default reached by the Hyper Reduction method is 10.9 higher than the Galerkin one. Better results are obtained if the number of

shape functions is increased. For example, a second statistical analysis is performed by adding as ROM shape functions 4 eigenvectors corresponding to the lowest eigenvalues of the linear system. The mean value of the error becomes only 2.35 higher and the maximum value is only 4.9 higher when compared to the Galerkin procedure. Those results are sufficient to validate the Hyper Reduction technic in the frame work of an adaptive procedure involving quality control of the ROM and improvement of ROM shape functions.

2.3 Convergence accelerations by ROM predictions

All the state evolutions defined by the APHR method are described by a ROM. The APHR method is an a priori approach because no model response (over the entire time interval) is forecast by using the high dimensional model. The ROM evolves during the computation of the state estimation. We denote $\underline{\underline{A}}^{(n)}$ the reduction matrix of the n^{th} version of the ROM such that:

$$\underline{q}_j^{(n)} = \underline{\underline{A}}^{(n)} \, \underline{a}_j^{(n)} \quad \forall j = 2 \dots m \tag{16}$$

If no ROM is known, the initial reduction matrix $\underline{\underline{A}}^{(o)}$ can be chosen in order to represent linear polynomials. During the incremental computation of the state evolution, each time step begins by the computation of the reduce state variables $\underline{a}_j^{(n)}$ thanks to the Hyper Reduction technic. The residual of high dimensional problem allows to check the quality of $\underline{q}_j^{(n)}$. If the norm of the residual is too important ($\|\underline{R}\| > \epsilon_R \ \|\underline{F}\|$) then $\underline{q}_j^{(n)}$ is used as an initial guess of the iterative treatment of the equations of the high dimensional problem. Hence we obtain convenient state $\widehat{\underline{q}}_j$ thanks to a convergence acceleration technic. The difference $\Delta q = \widehat{\underline{q}}_j - \underline{q}_j^{(n)}$ corresponds to a missing shape function. To adapt the reduction matrix we add a new column such that :

$$\underline{\underline{A}}^{(n+1)} = \left[\underline{\underline{A}}^{(n)} \, , \ \frac{1}{\|\Delta q\|} \Delta q \right] \tag{17}$$

To update the column of reduced state variables over the time interval $[0, t_j]$ we just have to add a new line such that :

$$\underline{a}_k^{(n+1)} = \begin{bmatrix} \underline{a}_k^{(n)} \\ 0 \end{bmatrix} \quad \text{for } k < j \tag{18}$$

$$\underline{a}_j^{(n+1)} = \begin{bmatrix} \underline{a}_j^{(n)} \\ \|\Delta q\| \end{bmatrix} \tag{19}$$

Due to this updating procedure the new version of the ROM is defined over $[0, t_j]$. At the end of the incremental computation the ROM is therefore independent on time. Such adaptive procedure is not sufficient to obtain an efficient model reduction. One adaptation out of four is completed by a selection of the most significant events involved in the reduced state evolution. These selection is performed by a reduction matrix $\underline{\underline{V}}^{(n+1)}$ that contains the first eigenvectors of the Proper Orthogonal Decomposition of the state variables $\underline{a}_j^{(n+1)}$. These eigenvectors $\underline{V}_k^{(n+1)}$ maximize the following projection on the reduced state variables:

$$Q_j^{(n+1)} \left(\underline{V}_k^{(n+1)} \right) = \frac{\sum_{i=1}^{i=j} \left(\underline{a}_i^{(n+1)\,T} \, \underline{V}_k^{(n+1)} \right)^2}{\left\| \underline{V}_k^{(n+1)} \right\|^2} \tag{20}$$

Hence $\underline{V}_k^{(n+1)}$ is the eigenvector of the correlation matrix $\sum_{i=1}^{i=j} \underline{a}_i^{(n+1)} \, \underline{a}_i^{(n+1)\,T}$ related to the eigenvalue $\mu_k = Q_j^{(n+1)} \left(\underline{V}_k^{(n+1)} \right)$ $(\mu_1 \geq \mu_2 \geq ...)$. Obviously, we obtain orthogonal eigenvectors:

$$\underline{V}_k^{(n+1)\,T} \, \underline{V}_l^{(n+1)} = \delta_{kl} \tag{21}$$

where δ_{kl} is the Kronecker delta. The selection of significant events is performed according to the following criteria :

$$\underline{\underline{V}}^{(n+1)} = \left[\underline{V}_1^{(n+1)}, ..., \underline{V}_{\widehat{r}}^{(n+1)} \right] \tag{22}$$

$$\text{with } \widehat{r} = \max_{\mu_k > 10^{-8} \, \mu_1} k \tag{23}$$

To each empirical eigenvector $\underline{V}_k^{(n+1)}$ corresponds a vectors of the high dimensional model according to the reduction matrix $\underline{\underline{A}}^{(n+1)}$. Therefore the new shape functions of the ROM are defined by:

$$\underline{\underline{A}}^{(n+2)} = \underline{\underline{A}}^{(n+1)} \, \underline{\underline{V}}^{(n+1)} \tag{24}$$

After such adaptation, the minimization of $\left\| \underline{a}_i^{(n+1)} - \underline{\underline{V}}^{(n+1)} \, \underline{a}_i^{(n+2)} \right\|$ provides an updated state description such that:

$$\underline{a}_i^{(n+2)} = \underline{\underline{V}}^{(n+1)\,T} \, \underline{a}_i^{(n+1)} \quad \forall i \leq j \tag{25}$$

To reduce the number of residual checking different adaptive strategies can be considered. If the residuals are small enough after the ROM prediction of $\underline{q}_j^{(n)}$, we choose to perform the next time steps without checking the residuals. If $\|\underline{R}\| < 0.1 \, \epsilon_R \ \|\underline{F}\|$ two time steps are not checked. If $\|\underline{R}\| < 0.001 \, \epsilon_R \ \|\underline{F}\|$

ten time steps are not checked. But if the residuals computed over the reduced integration domain are large ($\left\|\underline{\underline{\Pi}}\,\underline{R}\right\| > 0.1\,\epsilon_R\,\left\|\underline{\underline{\Pi}}\,\underline{F}\right\|$) then the hyper reduction computation is stopped and the full residuals are checked.

In [5] the APHR method has been compared to the snapshot POD. It has been proved that the APHR method can provide the snapshot POD of an estimated state evolution. In such a case the selected snapshots are related to the time instant when the expansion of the ROM is performed.

2.4 Learning strategy

The adaptive procedure presented above is very interesting to build an adapted ROM when the initial ROM is very unadapted to represent the model response. But, if the initial ROM defined by $\underline{\underline{A}}^{(o)}$ allows to represent events at the end of the time interval despite it doesn't represent events at the beginning of the time interval, the adaptive procedure is going to erase the convenient shape function during the first time steps. This is due to the POD of state evolutions on a time interval when the convenient shape function are not significant. Then, during the last time steps, iterative treatments of the high dimensional problem will be necessary to add onto the ROM the removed shape functions. This is also a drawback of the algorithm proposed in [6].

To avoid this superfluous computations, we propose a learning strategy. The purpose of this strategy is the efficient treatment of a sequence of different estimations of the model response corresponding to different values of the model input parameters. In the frame work of numerical sensitivity analysis the initial ROM used to find new model response according to new parameter values should be very accurate to describe events on all the time interval. We must avoid to erase shape functions which were significant during the previous estimation of the model response and which don't have significant effect at the beginning of the incremental computation of the new model response. To do so, it is sufficient to introduce a virtual time axis. We consider that the sequence of the studied model responses belong to unique virtual time axis. This changes the POD selection of the most significant shape functions. Let's denote $\underline{\underline{C}}^{(n)}$ the covariance matrix of all the reduced state variables previously computed according to different time instants and different parameter values. During the ROM adaptation this covariance matrix must be updated in case of subspace expansion :

$$\underline{\underline{C}}^{(n+1)} = \begin{bmatrix} \underline{\underline{C}}^{(n)} & 0 \\ 0 & 0 \end{bmatrix} \tag{26}$$

and in case of POD selection :

$$\underline{\underline{C}}^{(n+2)} = \underline{\underline{V}}^{(n+1)\,T}\,\underline{\underline{C}}^{(n+1)}\,\underline{\underline{V}}^{(n+1)} \tag{27}$$

Thanks to the covariance matrix of all the reduced state variables previously computed we are able to keep in memory all the significant events involved in the previous computation and in the current computation. To do so we just have to consider the eigenvectors $\underline{V}_k^{(n+1)}$ related to the following covariance matrix :

$$\underline{\underline{C}}^{(n+1)} + \sum_{i=1}^{i=j} \underline{a}_i^{(n+1)}\,\underline{a}_i^{(n+1)\,T} \tag{28}$$

At the end of the computation of the new model response the covariance matrix $\underline{\underline{C}}^{(n)}$ must be updated before an other computation of the model response. During the treatment of an inverse problem, large variations of parameters are often considered. It is not necessary to keep in memory the effect of the parameter values that are not on the vicinity of the optimal parameters. Therefore, we propose to master the memory effect thanks to a scalar coefficient $\gamma \in [0, 1]$. The updated covariance matrix $\underline{\underline{\widetilde{C}}}^{(n)}$ depends on the memory coefficient γ :

$$\underline{\underline{\widetilde{C}}}^{(n)} = \gamma\,\underline{\underline{C}}^{(n)} + \sum_{i=1}^{i=m} \underline{a}_i^{(n)}\,\underline{a}_i^{(n)\,T} \tag{29}$$

If γ is equal to 1, the ROM shape function are parameter independent. During a numerical sensitivity analysis we suggest to choose γ equal to 1. If this sensitivity analysis is used to update the parameter values, we also suggest to choose γ equal to 1 to estimate the updated model response. But before performing an other sensitivity analysis on the vicinity of the updated parameter a choice $\gamma < 1$ will soften the influence of previous values of parameter on the ROM. Hence the ROM becomes slightly dependent on parameters.

3 Example of adaptive ROM for an inverse problem

3.1 A transient thermal problem

To illustrate the implementation of the APHR method, a transient thermal problem is studied. A 3D finite element model of an experimental thermal chock is the high dimensional model. The Figure 1 shows the experimental configuration. Hot water is going through 12 holes on an annular domain. The initial temperature of the domain is equal to -150 $^\circ C$. The aim of the numerical simulation is to identify the heat transfer coefficient between the water and the domain. For

this purpose, we exploit the temperature measurement on 8 points (θ_1 ... θ_8 Figure 1). We denote Γ_c the face of the domain in contact this the water. The water temperature is assumed to linearly decrease through the thickness from 90 $°C$ to 80 $°C$. The thermal capacity C and the conductivity k are dependent on the temperature. More details on this experiment can be found in [7].

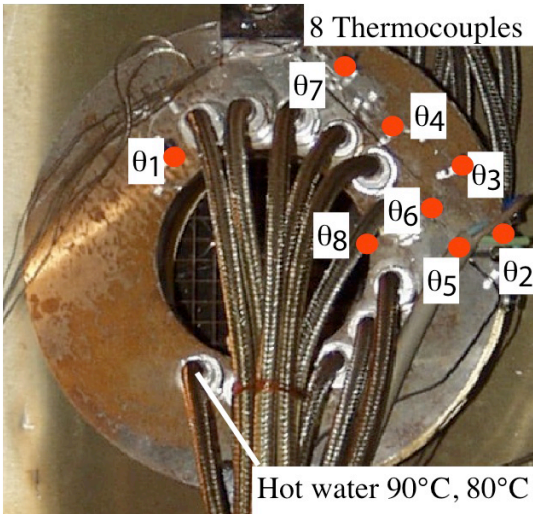

Figure 1: Experimental configuration.

The Figure 2 shows the finite element model (126115 tetrahedron and 24145 nodes). The time interval is regularly split into 49 time steps. The time integration scheme is the implicit forward Euler scheme. The column $\underline{q}$ contains the 24145 nodal temperatures of the considered mesh. Thanks to a classical Galerkin formulation we can define $\underline{R}$ such that :

$$\begin{aligned} \underline{q}^{*\,T} \quad & \underline{R}\left(\underline{q}_{j+1},\, \underline{q}_j,\, t_{j+1}\right) = \int_\Omega T^*\, \rho\, C(T)\, \dot{T}\, d\Omega \\ + \quad & \int_\Omega \mathbf{Grad}(T^*)\, k(T)\, \mathbf{Grad}(T)\, d\Omega \\ + \quad & \int_{\Gamma_c} T^*\, h\,(T)\ (T - T_c)\ d\Gamma \end{aligned} \tag{30}$$

with $t = t_{j+1}$, $T(x,t) = \sum_{i=1}^{i=n} N_i(x)\ q_i(t)$ and $T^*(x) = \sum_{i=1}^{i=n} N_i(x)\, q_i^*$. T_c is the water temperature.

The final temperature field shown on the Figure 2 has been obtained with a constant heat transfer coefficient h equal to 1000 $W.K^{-1}.m^{-2}$. This model leads to a correlation default on the temperatures equal to 65 $°C$.

3.2 The first estimation of the model response

A constant shape function is chosen to build the initial reduction matrix $\underline{\underline{A}}^{(o)}$. Obviously, this ROM is updated during the first time step. 15 adaptations are performed during the incremental computation of the first model response over the 49 time steps. The reduction matrix $\underline{\underline{A}}^{(15)}$ has 12 columns related to 12 shape functions and 12 reduced state variables. This matrix provides a ROM for all the time interval. The prescribed quality was $\epsilon_R = 10^{-3}$ and we obtained :

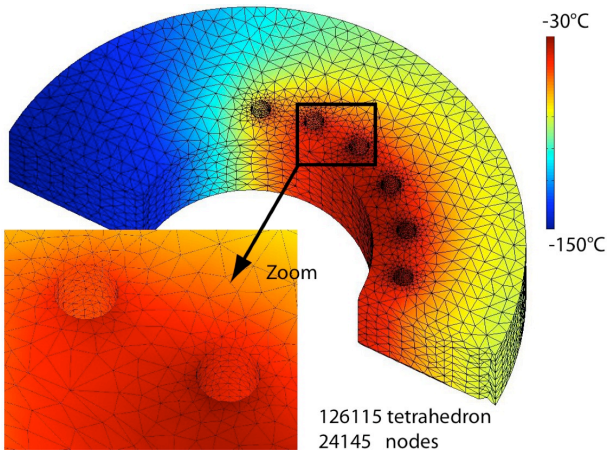

Figure 2: The high dimensional Finite Element model.

$$\max_j \left\| \underline{q}_{FE\,j} - \underline{\underline{A}}^{(15)}\, \underline{a}_j^{(15)} \right\|_\infty = 0.0013\ \max_j \left\| \underline{q}_{FE\,j} \right\|_\infty \tag{31}$$

The iterative treatment of the high dimensional non linear equations is performed by a Newton algorithm. The tangent matrix is updated only after 3 consecutive iterations and one iteration out of 3. Due to the convergence acceleration effect of the ROM estimation, the tangent matrix are updated only 6 times. During the classical treatment of the finite element equations the tangent matrix is updated 31 times. This is the main effect of the convergence acceleration technic. An other way to observe the convergence acceleration induced by the APHR method is to study the distance between the finite element state estimation an the initial guess for each enrichment stage of the ROM. The classical finite element initial guess is the state obtained at the end of the previous time step. On figure 3 we can see that the initial guess proposed by the APHR method is better than the classical one. We also can observe that most of the ROM adaptation append at the beginning of the time interval.

We can notice that the updating of the reduce state variables after each ROM adaptation is not an optimal one. A new hyper reduced computation of $\widetilde{\underline{a}}_j^{(15)}$ without any adaptation of the ROM provides a better state estimation:

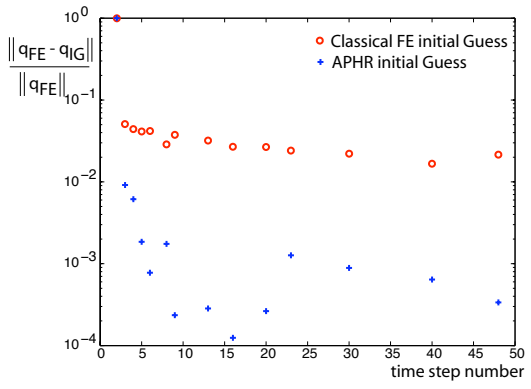

Figure 3: The reduced integration domain according to the 15^{th} version of the ROM.

$$\max_j \left\| \underline{q}_{FE\,j} - \underline{\underline{A}}^{(15)}\, \underline{\tilde{a}}_j^{(15)} \right\|_\infty = 0.0009 \max_j \left\| \underline{q}_{FE\,j} \right\|_\infty \quad (32)$$

This new computation is really fast. Thanks to the Hyper Reduction technic only 1292 elements are used instead of 126115. The reduced integration domain is shown on Figure 4. Such computation without adaptivity is interesting to know if a modification of parameter values has an influence on the model response. But if the influence exists, the adaptation is necessary to compute accurately the sensitivity of the model response to the modification of the parameters.

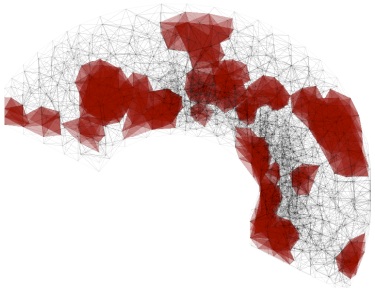

Figure 4: The reduced integration domain according to the 15^{th} version of the ROM.

3.3 Example of sensitivity analysis

Let's consider a numerical sensitivity analysis. We want to study a perturbation $\delta h(T)$ of the function $h(T)$ over the temperature interval $[-116\,°, -74\,°C]$ such that:

$$\begin{aligned} \forall \quad T \quad & \in [-150\,°C, -116\,°C] \cup [-74\,°C, 90\,°C] \\ & \delta h(T) = 0 \end{aligned} \quad (33)$$

$$\begin{aligned} \forall \quad T \quad & \in [-116\,°C,\ -95\,°C] \\ & \delta h(T) = \delta p\,(T + 116) \end{aligned} \quad (34)$$

$$\begin{aligned} \forall \quad T \quad & \in [-95\,°C,\ -74\,°C] \\ & \delta h(T) = \delta p\,(-74 - T) \end{aligned} \quad (35)$$

The magnitude of the modification is identified by Hyper Reduced computations (without any adaptation) until the value of δp produce a perturbation of the reduced state variables $\delta \underline{a}^{(15)}$ such that:

$$\max_j \left\| \delta \underline{a}_j^{(15)} \right\|_\infty \simeq 0.1 \max_j \left\| \underline{a}_j^{(15)} \right\|_\infty \quad (36)$$

When the magnitude of the perturbation is known ($\delta p = 200 W.K^{-1}.m^{-2}$) an adaptive computation of the modified response can be performed with an initial reduction matrix equal to $\underline{\underline{A}}^{(15)}$ and a covariance matrix of previous events $\underline{\underline{\tilde{C}}}^{(15)}$ such that:

$$\underline{\underline{\tilde{C}}}^{(15)} = \gamma \sum_{i=1}^{i=50} \underline{a}_i^{(15)}\, \underline{a}_i^{(15)\,T} \quad \text{with } \gamma = 1 \quad (37)$$

11 ROM adaptations are performed during the estimation of the new response of the model. Only two updates of the tangent matrix of high dimensional model are necessary (instead of 31 if a classical FE simulation is performed). No significant event is erased. The new reduction matrix $\underline{\underline{A}}^{(26)}$ is still convenient to represent accurately the first response of the model:

$$\begin{aligned} \max_j \quad & \left\| \underline{\underline{A}}^{(26)}\, \underline{a}_j^{(26)} - \underline{\underline{A}}^{(15)}\, \underline{a}_j^{(15)} \right\|_\infty \\ = \quad & 6.\,10^{-5} \max_j \left\| \underline{\underline{A}}^{(15)}\, \underline{a}_j^{(15)} \right\|_\infty \end{aligned} \quad (38)$$

The magnitude of the perturbation of the response is small, but greater than the accuracy of the ROM :

$$\max_j \left\| \underline{\underline{A}}^{(26)}\, \delta \underline{a}_j^{(26)} \right\|_\infty = 0.05 \max_j \left\| \underline{\underline{A}}^{(15)}\, \underline{a}_j^{(15)} \right\|_\infty \quad (39)$$

Some adaptations of the ROM are necessary because the spatial distribution of the perturbation on the model response is really different from the spatial distribution of the first model response. This is shown by a comparison of the first estimation of the temperatures θ_1, θ_2, ..., θ_8 on Figure 5 and the variation $\delta\theta_1$, $\delta\theta_2$, ..., $\delta\theta_8$ due to the perturbation δh on Figure 6.

To illustrate the learning strategy capability to forget previous events, the same perturbation is studied with the reduction matrix $\underline{\underline{A}}^{(15)}$ as initial data. But, the covariance matrix of previous events $\underline{\underline{\tilde{C}}}^{(15)}$ is build with $\gamma = 10^{-4}$. 17 adaptations (instead of 11) are necessary to forecast the modification of the model

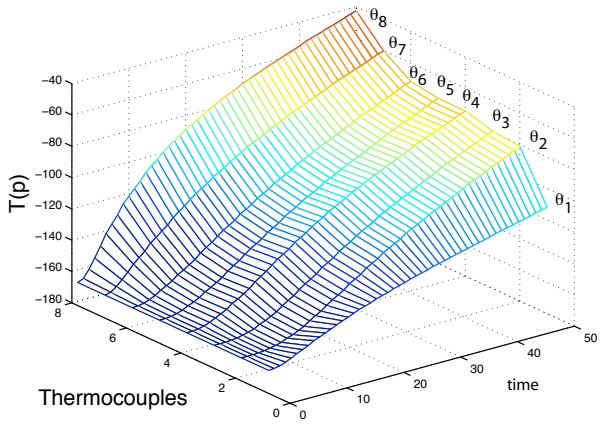

Figure 5: First prediction of θ_1, θ_2, ..., θ_8

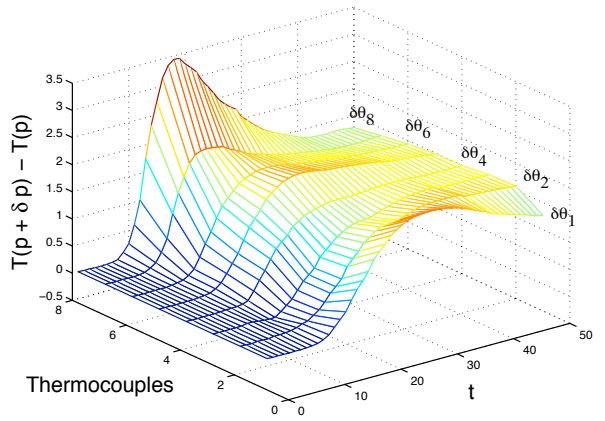

Figure 6: Variations $\delta\theta_1$, $\delta\theta_2$, ..., $\delta\theta_8$ due to the perturbation δh

response. And the last ROM ($n = 32$) is less efficient to represent the first model response:

$$\begin{aligned} \max_j \quad & \left\| \underline{\underline{A}}^{(32)} \, \underline{a}_j^{(32)} - \underline{\underline{A}}^{(15)} \, \underline{a}_j^{(15)} \right\|_\infty \\ = \quad & 14.\, 10^{-5} \, \max_j \left\| \underline{\underline{A}}^{(15)} \, \underline{a}_j^{(15)} \right\|_\infty \end{aligned} \tag{40}$$

The expected advantage of choosing γ lower than 1 is the lower number of ROM shape functions (17 instead of 18 in this case).

4 Conclusion

A learning strategy allows to build APHR ROM that are time independent and slightly dependent on parameters. Hence small ROMs can be obtained for the treatment of inverse problems. Despite the Hyper Reduction technic amplify the error of the simplified model, the accuracy of the APHR ROM is convenient to perform sensitivity analysis. In case of sensitive parameter adaptations of the ROM are necessary to catch the spatial distribution of the model response modification. Both state variables and integration points are reduced. Some resolution of high dimensional linear problems are performed to improve the ROM. But the number of such resolutions is very low compared to the one needed in case of classical solving of the high dimensional non linear equations. The APHR method can be applied to any non linear time dependent problem. It's efficiency depends on both hyper reduction technic and a convergence acceleration provided by a ROM prediction of the state variables.

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